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Construction of Gabor frames in $l^2(\mathbb{Z})$ using Gabor frames in $L^2(\mathbb{R})$

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Abstract
In this paper we identified a collection of unitary operators which maps Gabor frames in $L^2(\mathbb{R})$ to Gabor frames in $l^2(\mathbb{Z})$. This is very important in construction of Gabor frames in $l^2(\mathbb{Z})$ from Gabor frames in $L^2(\mathbb{R})$ other than which obtained from Gabor frames in $L^2(\mathbb{R})$ through sampling.

Keywords
Weyl-Heisenberg frame, window function, orthonormal basis, unitary operator, window coefficient sequence.

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Contents

1. Introduction

Hilbert spaces and the accompanying notion of bases containing orthonormal vectors are of primary significance in signal processing, communications and information theory. Although, linear independency and orthonormality of the vectors in the basis urge limitations that sometimes make it demanding to have the basis elements assure additional useful properties. To overcome such limitations we use the theory of signal decompositions that is smooth enough to assist decompositions into perhaps, nonorthogonal and unnecessary signal sets. Here comes the concept of frames which gives basis like, but generally non unique representations of vectors in a Hilbert space.

Frame theory and related results for frames and bases has flourished quickly over the last few decades, specifically in the area of Gabor and wavelet systems. This period has proved that, very often, it is favourable to deal with frames in place of bases. In his paper *Theory of Communication*[8] by D.Gabor in the year 1946, initiated and developed a fundamental viewpoint to signal decompositions in label of simple signals. While studying nonharmonic Fourier series, Duffin and Schaeffer [6] introduced the concept of frames in the early fifties. In 1986, Grossmann [5] noticed and developed the relationship of frames to wavelets. For numerical calculations, it is easy to think of frames (or bases) with a simple layout. This is the inspiration behind Gabor frames and wavelets.

Fundamental Fourier analysis techniques were used to study more about Gabor and wavelet systems. However, in modern years additional abstract utensils like Group representations, algebra of operators and abstract harmonic analysis etc have been instituted. Various generalizations of frames like, frame of sub-spaces [1],[2], pseudo-frames [10], oblique frames [4] are there in literature. In which Gabor frames or Weyl-Heisenberg frames bagged a key position.

Frame operator corresponding to a given frame is a primitive instrument in frame theory (both in theoretical and application point of view). To construct a frame corresponding to a given positive and invertible operator have considerable practical significance. That is, it would be highly profitable to construct frames depending on the needs. Gabor frame operators have acquired remarkable recognition in the literature.

It is interesting to consider frames in the sequence space $l^2(\mathbb{Z})$ with a Gabor like nature without mentioning to frames in the function space $L^2(\mathbb{R})$. For these frames in the sequence space results are very similar to the Gabor theory in $L^2(\mathbb{R})$.

After a brief introduction, section 2 of this paper is just recalling of basics in general frame theory and basics of Gabor frames in the spaces $L^2(\mathbb{R})$ and $l^2(\mathbb{Z})$. In Section 3 we define a linear map from $L^2(\mathbb{R})$ to $l^2(\mathbb{Z})$ in a peculiar manner and its properties are discussed.

Through out this manuscript, we will denote a separable Hilbert space with $\mathscr{H}$ having inner product $\langle\cdot,\cdot\rangle$. We invite the readers to [3], [9] for more about frame theory and the proofs of the statements in this article.

2. Preliminaries

Let $\{\omega_k\}_{k=1}^{\infty}$ be a countable set of vectors in a Hilbert space $\mathscr{H}$. If this family $\{\omega_k\}_{k=1}^{\infty}$ satisfies the inequality

$$L\|y\|^2 \leq \sum_{k=1}^{\infty} |\langle y, \omega_k\rangle|^2 \leq M\|y\|^2 \tag{2.1}$$

for all $y \in \mathscr{H}$ with $L,\ M \in \mathbb{R}$ and $0 < L \leq M < \infty$, then we say that $\{\omega_k\}_{k=1}^{\infty}$ is a *frame* with *lower bound* L and *upper bound* M. If the above said sequence satisfies at least upper bound condition given in the inequality (2.1) then it is called a *Bessel sequence* and moreover, it is said to be a *frame sequence*, if it is a frame for $\overline{span}\{\omega_k\}_{k=1}^{\infty}$. A frame is said to be a *tight frame* if its frame bounds coincides; a tight frame with $L = M = 1$ is called a *normalized tight frame*. A frame is called an exact frame, if its frame sequence is minimal.

Following are some useful results which easily reflects from the definition of a frame.

1. For a frame $\{\omega_k\}_{k=1}^{\infty}$ in $\mathscr{H}$ with upper frame bound M, $\|\omega_k\| \leq \sqrt{M}$ for all k. Especially, $\|\omega_k\| \leq 1$ when $\omega_k\}_{k=1}^{\infty}$ is a normalized tight frame.

2. If $\{\omega_k\}_{k=1}^{\infty}$ is a frame and if A is bounded and invertible operator on $\mathscr{H}$, then $\{A\omega_k\}_{k=1}^{\infty}$ is a frame for $\mathscr{H}$. If $\{\omega_k\}_{k=1}^{\infty}$ is exact so is $\{A\omega_k\}_{k=1}^{\infty}$. If A is unitary, then $\{A\omega_k\}_{k=1}^{\infty}$ has the same frame bounds as $\{\omega_k\}_{k=1}^{\infty}$.

3. If $\mathscr{H}$ is a Hilbert space and $\{\omega_k\}_{k=1}^{\infty}$ is a frame in $\mathscr{H}$, then $\{\omega_k\}_{k=1}^{\infty}$ spans a subspace which is dense in $\mathscr{H}$.

4. If $\mathscr{H}$ has a frame, then it is separable.

5. If $\{\omega_k\}_{k=1}^{\infty}$ is a frame for a Hilbert space $\mathscr{H}$, then $\{\omega_k\}_{k=1}^{\infty}$ is complete in $\mathscr{H}$.

Now we give few simple examples of frames in a separable Hilbert space $\mathscr{H}$. Let us consider an orthonormal basis $\{v_k\}_{k=1}^{\infty}$ in $\mathscr{H}$.

- By repeating each element in $\{v_k\}_{k=1}^{\infty}$ twice we obtain $\{\omega_k\}_{k=1}^{\infty} = \{v_1, v_1, v_2, v_2, ...\}$, which is a tight frame with frame bound $L = 2$. If only v_1 is repeated twice we obtain $\{\omega_k\}_{k=1}^{\infty} = \{v_1, v_1, v_2, v_3, ...\}$, which is a frame with bounds $L = 1,\ M = 2$
- Let $\{\omega_k\}_{k=1}^{\infty} = \{v_1, \frac{v_2}{\sqrt{2}}, \frac{v_2}{\sqrt{2}}, \frac{v_3}{\sqrt{3}}, \frac{v_3}{\sqrt{3}}, \frac{v_3}{\sqrt{3}}, ...\}$, that is $\{\omega_k\}_{k=1}^{\infty}$ is the sequence where each vector $\frac{v_k}{\sqrt{k}}$ is duplicated k times. Then for each $y \in \mathscr{H}$, we have $\sum_{k=1}^{\infty} |\langle y, \omega_k\rangle|^2 = \sum_{k=1}^{\infty} k|\langle y, \frac{v_k}{\sqrt{k}}\rangle|^2 = \|y\|^2$
  So $\{\omega_k\}_{k=1}^{\infty}$ is a normalised tight frame for $\mathscr{H}$.

Since a frame $\{\omega_k\}_{k=1}^{\infty}$ is a Bessel sequence, the operator $T : l^2(\mathbb{N}) \to \mathscr{H}$ defined by $T\{c_k\}_{k=1}^{\infty} = \sum_{k=1}^{\infty} c_k\omega_k$ is bounded and T is known as the *synthesis operator* or *pre-frame operator*. The adjoint operator of T is the operator $T^* : \mathscr{H} \to l^2(\mathbb{N})$ given by, $T^*y = \{\langle y, \omega_k\rangle\}_{k=1}^{\infty}$ and is said to be the *analysis operator*. By composing T and T^* we obtain the *frame operator*

$$S : \mathscr{H} \to \mathscr{H},\ \ Sy = TT^*y = \sum_{k=1}^{\infty} \langle y, \omega_k\rangle\omega_k$$

The convergence of the series defining S is guaranteed for all $y \in \mathscr{H}$ from the fact that $\{\omega_k\}_{k=1}^{\infty}$ is a Bessel sequence in $\mathscr{H}$. It is interesting to note that, scalar multiple of the identity operator will be the frame operator of a tight frame and identity operator is the frame operator for a normalized tight frame.

Let $\{\omega_k\}_{k=1}^{\infty}$ be a frame in a Hilbert space $\mathscr{H}$ with frame operator S and frame bounds $L,\ M$. Then the following hold.

1. S is bounded, invertible, self-adjoint and positive. In fact $LI \leq S \leq MI$.

2. $\{S^{-1}\omega_k\}_{k=1}^{\infty}$ is a frame with frame bounds M^{-1}, L^{-1}; if L and M are the optimal frame bounds for $\{\omega_k\}_{k=1}^{\infty}$, then the bounds M^{-1}, L^{-1} are the optimal frame bounds for $\{S^{-1}\omega_k\}_{k=1}^{\infty}$.

3. The frame operator for $\{S^{-1}\omega_k\}_{k=1}^{\infty}$ is S^{-1}, furthermore, $M^{-1}I \leq S^{-1} \leq L^{-1}I$

4. $\{S^{-1/2}\omega_k\}_{k=1}^{\infty}$ is a normalized tight frame.

The frame $\{S^{-1}\omega_k\}_{k=1}^{\infty}$ is called the *canonical dual* frame of $\{\omega_k\}_{k=1}^{\infty}$ because it plays the same role in the frame theory as the dual of a basis.

Next we state a prime result in frame theory namely the frame decomposition. It shows that if $\{\omega_k\}_{k=1}^{\infty}$ is a frame for $\mathscr{H}$, then every element in $\mathscr{H}$ has a representation as a superposition of the frame elements. Further, theorem says that the sequence $\{\langle y, S^{-1}\omega_k\rangle\}_{k=1}^{\infty}$ contains all information about each $y \in \mathscr{H}$. The constants $\langle y, S^{-1}\omega_k\rangle$ are called the *frame coefficients*.

Theorem 2.1. *Let $\{\omega_k\}_{k=1}^{\infty}$ be a frame in a Hilbert space $\mathscr{H}$ with a frame operator S. Then for all $y \in \mathscr{H}$,*
$y = \sum_{k=1}^{\infty} \langle y, S^{-1}\omega_k\rangle\omega_k$ *and* $y = \sum_{k=1}^{\infty} \langle y, \omega_k\rangle S^{-1}\omega_k.$
Convergence of both series ensured for all $y \in \mathscr{H}$.

Among several classes of frames in frame theory and related topics, Gabor frames or Weyl-Heisenberg frames have attained remarkable consideration as they are generated by a single element. We now come out with some basics of Gabor frame analysis.

2.1 Gabor frames in $L^2(\mathbb{R})$

Here we consider those frames in $L^2(\mathbb{R})$ which are generated from a single vector from $L^2(\mathbb{R})$ through translations and modulations. Such type of frames are called Weyl–Heisenberg frames (or Gabor frames). For positive real numbers a, b, and for an element $g \in L^2(\mathbb{R})$, E_{mb} and T_{na} are, respectively, the modulation operators and translation operators given by $E_{mb}f(x) = e^{2\pi imbx}f(x),\ x \in \mathbb{R}$ and $T_{na}f(x) = f(x-na),\ x \in \mathbb{R}$. It is clear that both these maps are unitary maps acting on $L^2(\mathbb{R})$. The main concern of Gabor Analysis is the representation of signals (vectors) as an infinite series whose terms are the translated and modulated version of a single vector in the space. It is interesting to note that, if $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a Gabor frame in $L^2(\mathbb{R})$, then

$$f = \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}h\rangle E_{mb}T_{na}g, \quad \forall f \in L^2(\mathbb{R}).$$

for some $h \in L^2(\mathbb{R})$. The elegant choice of h is $S^{-1}g$, where S is the frame operator. The function $h = S^{-1}g$ is called the *canonical dual generator*.

The Gabor system $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is actually involves only the time frequency shifts of a single vector g along the lattice $\{(na, mb)\}_{m,n\in\mathbb{Z}}$. This type of frames are called regular Gabor frames. Given a Gabor frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$, the number $(ab)^{-1}$ is called the *redundancy*. One of the fundamental result says that the product ab decides whether it is possible for $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ to be a frame for $L^2(\mathbb{R})$ for some choice of $g \in L^2(\mathbb{R})$. Note that ,if (g, a, b) is a Gabor frame with frame operator S, then $ab = \|S^{-1/2}g\|_2^2$.

We can observe that not all frame operators on $L^2(\mathbb{R})$ can be Gabor frame operators [9]. As we are aware, every positive and invertible bounded linear operators on a separable Hilbert space $\mathscr{H}$ is a frame operator of some frame in $\mathscr{H}$. However, this characterization is inadequate for Gabor frame operators. That is a frame operator on $L^2(\mathbb{R})$ need not be a Gabor frame operator. Following proposition guarantees the existence of a normalized tight frame and then we record a characterization of Gabor frame operators given in [7].

Proposition 2.2. *Consider two positive constants a, b, the translation and modulation parameters, satisfying the condition $0 \leq ab \leq 1$, we can find an element $g \in L^2(\mathbb{R})$ such a way that (g, a, b) is a normalized tight Gabor frame which, therefore has the identity operator as its frame operator.*

Theorem 2.3. *A bounded linear operator on $L^2(\mathbb{R})$ is the frame operator of some Gabor frame if and only if it is positive, invertible and commutes with a translation operator T_a and a modulation operator E_b, where a and b are positive real numbers with $ab \leq 1$.*

We now record some basics of Gabor frame analysis in the sequence space $l^2(\mathbb{Z})$.

All computational calculation with elements in $L^2(\mathbb{R})$ will require a discrete structure, where all estimations are carried out with (finite) sequences in $l^2(\mathbb{Z})$. Hence it is necessary to know that special set-up on a Gabor frame $\{E_{mb}T_{na}g :\ m,n \in \mathbb{Z}\}$ in $L^2(\mathbb{R})$ in fact imply that we can create a frame for $l^2(\mathbb{Z})$ with Gabor like nature.

For each $b \in \mathbb{R}$, the modulation operator $\hat{E}_b : l^2(\mathbb{Z}) \longrightarrow l^2(\mathbb{Z})$ is defined by, $\hat{E}_b g(j) = e^{2\pi ibj}g(j)$,
for all $g = (...., g(-1), g(0), g(1), ...) \in l^2(\mathbb{Z})$, where the j^{th} coordinate of g is denoted by $g(j)$. Similarly for each $n \in \mathbb{Z}$ the translation operator $\hat{T}_n : l^2(\mathbb{Z}) \longrightarrow l^2(\mathbb{Z})$ is defined by $\hat{T}_n g(j) = g(j-n)$
for all $g = (...., g(-1), g(0), g(1), ...) \in l^2(\mathbb{Z})$.

The definition of $\hat{E}_b$ is valid for all $b \in \mathbb{R}$, still we are interested in modulations of the form $\hat{E}_{m/M}$, where $M \in \mathbb{N}$ is firmed and $m \in \mathbb{Z}$. The scalar $1/M$ represents the modulation parameter for Gabor systems in $L^2(\mathbb{R})$. But these two settings are entirely different. In $L^2(\mathbb{R})$, modulation operator with distinct parameters are automatically distinct, but this will not be true in sequence space. Actually, with the above definition $\hat{E}_{\frac{m}{M}} = \hat{E}_{\frac{m}{M}+k}$, for all $k \in \mathbb{Z}$. Therefore $\{\hat{E}_{m/M}g :\}_{m\in\mathbb{Z}}$ can't be a Bessel sequence in $l^2(\mathbb{Z})$ unless $g = 0$. Because of this we will consider modulations of the form $\hat{E}_{m/M}$ with $m = 0, 1, 2, ...M-1$. Gabor system in the sequence space $l^2(\mathbb{Z})$, generated by an element $g \in l^2(\mathbb{Z})$ with the modulation parameter $\frac{1}{M}$ and translation parameter $N, (M, N \in \mathbb{N})$ is now defined as the family of sequences $\{\hat{E}_{m/M}\hat{T}_{nN}g : m = 0, 1, ...M-1, n \in \mathbb{Z}\}$. Precisely, $\hat{E}_{m/M}\hat{T}_{nN}g$ is the sequence in $l^2(\mathbb{Z})$ whose j^{th} coordinate is $\hat{E}_{m/M}\hat{T}_{nN}g(j) = e^{2\pi i\frac{m}{M}(j-nN)}g(j-nN)$. For more detailed results about frame theory in the spaces $L^2(\mathbb{R})$ and $l^2(\mathbb{Z})$ one can go through [3].

3. Construction of Gabor frames in $l^2(\mathbb{Z})$ from Gabor frames in $L^2(\mathbb{R})$

From Proposition 2.2 we can find a vector $g \in L^2(\mathbb{R})$ such that the collection $\{E_{\frac{m}{M}}T_{nN}g : m,n \in \mathbb{Z}\}$ is a Gabor frame in $L^2(\mathbb{R})$ for any two natural numbers M, N with $\frac{N}{M} \leq 1$. In this section we focused on the construction of a Gabor frame $\{\hat{E}_{\frac{m}{M}}\hat{T}_{nN}g : m = 0, 1, 2...M-1, n \in \mathbb{Z}\}$ in $l^2(\mathbb{Z})$ from a Gabor frame $\{E_{\frac{m}{M}}T_{nN}g : m,n \in \mathbb{Z}\}$ in $L^2(\mathbb{R})$ for any two positive integers M, N with $\frac{N}{M} \leq 1$.

The following lemma and theorem which is available in [3] is very useful in our discussion and it guarantees the existence of pseudo inverse of a bounded linear operator on a Hilbert space with closed range.

Lemma 3.1. *Let U be a bounded linear operator from a Hilbert space $\mathscr{H}$ to a Hilbert space $\mathscr{K}$ with its range set R_U is closed. Then there exists a bounded operator $U^\dagger$ from $\mathscr{K}$ to $\mathscr{H}$ such that $UU^\dagger f = f$ for all $f \in R_U$. Moreover $UU^\dagger$ is* *the orthogonal projection of $\mathscr{H}$ onto R_U .*

Theorem 3.2. *Let $\{\omega_k\}_{k=1}^{\infty}$ be a frame in $\mathscr{K}$ with bounds L and M and let $U : \mathscr{K} \longrightarrow \mathscr{H}$ a bounded linear map with non trivial closed range. Then $\{U\omega_k\}_{k=1}^{\infty}$ is a frame sequence with bounds $L \parallel U^\dagger \parallel^{-2}$ and $M \parallel U \parallel^2$.*

Here we state a remark, which directly follows from the theorem.

Remark 3.3. *Let $\{\omega_k\}_{k=1}^{\infty}$ be a frame in $\mathscr{K}$ with bounds L and M and $U : \mathscr{K} \longrightarrow \mathscr{H}$ a bounded linear surjective operator. Then $\{U\omega_k\}_{k=1}^{\infty}$ is a frame in $\mathscr{H}$ with frame bounds $L \parallel U^\dagger \parallel^{-2}$ and $M \parallel U \parallel^2$.*

Theorem 3.4. *Let M, N are natural numbers with $\frac{N}{M} \leq 1$. Suppose that the collection $\{E_{\frac{m}{M}} T_{nN} g : m,\ n \in \mathbb{Z}\}$ is a Gabor frame in $L^2(\mathbb{R})$ for some $g \in L^2(\mathbb{R})$. Then for any surjective bounded linear operator $U : L^2(\mathbb{R}) \to l^2(\mathbb{Z})$ with the property that $UE_{\frac{m}{M}} T_{nN} = \hat{E}_{\frac{m}{M}} \hat{T}_{nN} U$ for $m,\ n \in \mathbb{Z}$, the sequence $\{\hat{E}_{\frac{m}{M}} \hat{T}_{nN} Ug : m = 0,1,2.....,M-1, n \in \mathbb{Z}\}$ is a Gabor frame in $l^2(\mathbb{Z})$.*

Proof. Let $g \in L^2(\mathbb{R})$ and M, N are two natural numbers such that $\frac{N}{M} \leq 1$, assume that the collection $\{E_{\frac{m}{M}} T_{nN} g : m,n \in \mathbb{Z}\}$ is a Gabor frame in $L^2(\mathbb{R})$ and U be a surjective bounded linear operator from $L^2(\mathbb{R})$ to $l^2(\mathbb{Z})$ with the property that $UE_{\frac{m}{M}} T_{nN} = \hat{E}_{\frac{m}{M}} \hat{T}_{nN} U$ for $m,\ n \in \mathbb{Z}$. Then by Remark 3.3 $\{U(E_{\frac{m}{M}} T_{nN} g) : m,n \in \mathbb{Z}\}$ is a frame in $l^2(\mathbb{Z})$. Since U satisfies $UE_{\frac{m}{M}} T_{nN} = \hat{E}_{\frac{m}{M}} \hat{T}_{nN} U$ for $m,n \in \mathbb{Z}$, and $\hat{E}_{\frac{m}{M}+k} \hat{T}_{nN} g = \hat{E}_{\frac{m}{M}} \hat{T}_{nN} g$ for any $k \in \mathbb{Z}$, we see that $\{UE_{\frac{m}{M}} T_{nN} g : m,\ n \in \mathbb{Z}\} = \{\hat{E}_{\frac{m}{M}} \hat{T}_{nN} Ug : m,\ n \in \mathbb{Z}\} = \{\hat{E}_{\frac{m}{M}} \hat{T}_{nN} Ug : m = 0,\ 1,\ 2,\ \ldots,\ M-1,\ n \in \mathbb{Z}\}$ and hence the frame $\{\hat{E}_{\frac{m}{M}} \hat{T}_{nN} Ug : m = 0,\ 1,\ 2,\ \ldots,\ M-1, n \in \mathbb{Z}\}$ is a Gabor frame in $l^2(\mathbb{Z})$.

□

It is worthwhile to know that, what kind of restrictions on a Gabor frame $\{E_{\frac{m}{M}} T_{nN} g : m,n \in \mathbb{Z}\}$ really imply that we have a frame for $l^2(\mathbb{Z})$ having a Gabor like nature. The appropriate conditions were discovered by Janssen [11]. He proved that there is a canonical way to get discrete Gabor frames *via* Gabor frames for function space $L^2(\mathbb{R})$ through sampling. A detailed discussion of these theories are available in [3]. We consider a Gabor system for $L^2(\mathbb{R})$ of the form $\{E_{\frac{m}{M}} T_{nN} g : m,n \in \mathbb{Z}\}$, where $g \in L^2(\mathbb{R})$ is the *window function* or *generating function* and $M,\ N\ \in \mathbb{N}$. In searching a Gabor like system in $l^2(\mathbb{Z})$ the natural question arising is, "which type of linear transformations maps a Gabor frame in $L^2(\mathbb{R})$ to a Gabor like frame in $l^2(\mathbb{Z})$".

Let us consider the characteristic function on the interval $[0,1]$,that is, $h = \chi_{[0,1]}$. Then the collection $\{E_k T_j h : k,\ j \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{R})$ [3].
For $g \in L^2(\mathbb{R})$ we can express, $g = \sum_{k,j \in \mathbb{Z}} \alpha_{kj} E_k T_j h$, where $\alpha_{kj} = \langle g, E_k T_j h \rangle$.

To find an example of $U : L^2(\mathbb{R}) \to l^2(\mathbb{Z})$ satisfying the condition in the theorem 3.4, consider the orthonormal basis $\{E_k T_j h : k,\ j \in \mathbb{Z}\}$ for $L^2(\mathbb{R})$

For each $g \in L^2(\mathbb{R})$, we can write, $g = \sum_{l,k \in \mathbb{Z}} \langle g, E_l T_k h \rangle E_l T_k h$

Take the coefficients as $\alpha_{lk} = \langle g, E_l T_k h \rangle$, so that $g = \sum_{l,k \in \mathbb{Z}} \alpha_{lk} E_l T_k h$
Let us define $U : L^2(\mathbb{R}) \to l^2(\mathbb{Z})$ by, $U(E_l T_k \chi_{[0,1]}) = \hat{E}_l \hat{T}_k e_1$ where $\{e_t\}_{t \in \mathbb{Z}}$ is the standard orthonormal basis for $l^2(\mathbb{Z})$. It is clear that, $\{\hat{E}_l \hat{T}_k e_1 : l,k \in \mathbb{Z}\}$ is a frame in $l^2(\mathbb{Z})$. We now extend U linearly to $L^2(\mathbb{R})$ by
$U(\sum_{r,s \in \mathbb{Z}} \beta_{rs} E_r T_s \chi_{[0,1]}) = \sum_{r,s \in \mathbb{Z}} \beta_{rs} \hat{E}_r \hat{T}_s e_1,$
for any element $f = \sum_{r,s \in \mathbb{Z}} \beta_{rs} E_r T_s \chi_{[0,1]}$ in $L^2(\mathbb{R})$.

Therefore, $U(g) = \sum_{j,k \in \mathbb{Z}} \alpha_{jk} \hat{E}_j \hat{T}_k e_1$

hence, $\hat{E}_{\frac{m}{M}} \hat{T}_{nN} U(g) = \hat{E}_{\frac{m}{M}} \hat{T}_{nN} (\sum_{j,k \in \mathbb{Z}} \alpha_{jk} \hat{E}_j \hat{T}_k e_1)$

note that $U(g) = 0$ implies, $\sum_{j \in \mathbb{Z}} \alpha_{jk} \hat{E}_j e_k = 0$ for all k

$$\Rightarrow \quad \sum_{j \in \mathbb{Z}} \alpha_{jk} e^{2\pi ijk} = 0 \quad \text{for all } k$$
$$\Rightarrow \quad \sum_{j \in \mathbb{Z}} \alpha_{jk} = 0 \quad \text{for all } k$$

Let $x \in [0,1]$, then $g(x) = \sum_{l,k \in \mathbb{Z}} \alpha_{lk} E_l T_k \chi_{[0,1]}(x)$

when $x = 0$ or $x = 1$, $g(x) = \sum_{l,0 \in \mathbb{Z}} \alpha_{l0} e^{2\pi ijx}$

$$= \sum_{l,0 \in \mathbb{Z}} \alpha_{l0} = 0$$

Hence $g(0) = g(1) = 0$
Thus $g = 0$, on all integers and hence g is not injective.
Let $f \in l^2(\mathbb{Z})$, then $f = \sum_{r,s \in \mathbb{Z}} \beta_{rs} \hat{E}_r \hat{T}_s e_1$

Then define $h = \sum_{r,s \in \mathbb{Z}} \beta_{rs} \hat{E}_r \hat{T}_s \chi_{[0,1]}$

so that $h \in L^2(\mathbb{R})$ and $U(h) = f$.
Thus if we define a linear operator $U : L^2(\mathbb{R}) \to l^2(\mathbb{Z})$ by, $U(E_l T_k \chi_{[0,1]}) = \hat{E}_l \hat{T}_k e_1$ for all $k,l \in \mathbb{Z}$, U maps Gabor frame in $L^2(\mathbb{R})$ to Gabor frame in $l^2(\mathbb{Z})$.

Note that $U(E_{\frac{m}{M}} T_{nN} g) = \hat{E}_{\frac{m}{M}} U(T_{nN} g)$

$$= \hat{E}_{\frac{m}{M}} \sum_{j,k \in \mathbb{Z}} \alpha_{jk} \hat{E}_j \hat{T}_{nN+k} e_1$$
$$= \hat{E}_{\frac{m}{M}} \hat{T}_{nN} (\sum_{j,k \in \mathbb{Z}} \alpha_{jk} \hat{E}_j \hat{T}_k e_1)$$
$$= \hat{E}_{\frac{m}{M}} \hat{T}_{nN} U(g), \text{ and this is}$$

generated by $U(g)$.

Definition 3.5. *Let $g \in L^2(\mathbb{R})$, then for each $m,n \in \mathbb{Z}$ and for each pair of positive integers M and N, there is a sequence of* *complex numbers $\{\eta_{r,s}\}$ such that $E_{\frac{m}{M}}T_{nN}g = \sum_{r,s\in\mathbb{Z}} \eta_{r,s}E_rT_sh$.*

This sequence is called window coefficient sequence of g with respect to the quadruple (m,n,M,N) and each term of this sequence are called window coefficients.

Proposition 3.6. *Let $\{\eta_{r,s}\}$ be the window coefficient sequence of $g \in L^2(\mathbb{R})$ with respect to the quadruple (m,n,M,N) where $m,n \in \mathbb{Z}$ and M,N are positive integers, then*

$$\eta_{r,s} = \sum_{k\in\mathbb{Z}} \alpha_{k,j-nN} e^{2\pi i \frac{m}{M} s} \beta_{r-k,0}$$

where $\alpha_{kj} = \langle g, E_kT_j\chi_{[0,1]}\rangle$ and $\beta_{rs} = \langle E_{\frac{m}{M}}\chi_{[0,1]}, E_rT_s\chi_{[0,1]}\rangle$

Proof. Let $g \in L^2(\mathbb{R})$. Then g is of the form, $g = \sum_{j,k\in\mathbb{Z}} \langle g, E_kT_j\chi_{[0,1]}\rangle E_kT_j\chi_{[0,1]}$ and $\beta_{rs} = \langle E_{\frac{m}{M}}\chi_{[0,1]}, E_rT_s\chi_{[0,1]}\rangle$.
For each $m,n \in \mathbb{Z}$, the elements $E_{\frac{m}{M}}T_{nN}g$ in the Gabor frame $\{E_{\frac{m}{M}}T_{nN}g : m,\ n \in \mathbb{Z}\}$ for $L^2(\mathbb{R})$ takes the form

$$\begin{aligned}
E_{\frac{m}{M}}T_{nN}g =\ & E_{\frac{m}{M}}T_{nN}\Big(\sum_{j,k\in\mathbb{Z}} \alpha_{jk}E_kT_jh\Big)\\
&= \sum_{j,k\in\mathbb{Z}} \alpha_{jk}E_{\frac{m}{M}}T_{nN}(E_kT_j)h\\
&= \sum_{j,k\in\mathbb{Z}} \alpha_{jk}e^{2\pi i\frac{m}{M}nN}T_{nN}E_kE_{\frac{m}{M}}T_jh\\
&= \sum_{j,k\in\mathbb{Z}} \alpha_{jk}e^{2\pi i\frac{m}{M}nN}T_{nN}E_ke^{2\pi i\frac{m}{M}j}T_jE_{\frac{m}{M}}h\\
&= \sum_{j,k\in\mathbb{Z}} \alpha_{jk}e^{2\pi i\frac{m}{M}(nN+j)}e^{2\pi inNk}E_kT_{nN+j}(E_{\frac{m}{M}}h)\\
&= \sum_{j,k\in\mathbb{Z}} \alpha_{jk}e^{2\pi i\frac{m}{M}(nN+j)}E_kT_{nN+j}(E_{\frac{m}{M}}h)
\end{aligned}$$

Now let , $E_{\frac{m}{M}}h = \sum_{k,j\in\mathbb{Z}} \beta_{kj}E_kT_jh$

where,
$$\begin{aligned}
\beta_{kj} &= \langle E_{\frac{m}{M}}h, E_kT_jh\rangle\\
&= \int E_{\frac{m}{M}}h\overline{E_kT_jh}dx\\
&= \int e^{2\pi i\frac{m}{M}x}\chi_{[0,1]}(x)e^{-2\pi ik(x-j)}\chi_{[0,1]}(x-j)dx
\end{aligned}$$

$$= \begin{cases} \int e^{2\pi i\frac{m}{M}x}e^{-2\pi ikx}dx & \text{if} \quad j=0\\ 0 & \text{if} \quad j\neq 0\end{cases}$$

$$= \begin{cases} \int e^{2\pi i(\frac{m}{M}-k)x}dx & \text{if} \quad j=0\\ 0 & \text{if} \quad j\neq 0\end{cases}$$

$$= \begin{cases} \dfrac{1}{(\frac{m}{M}-k)2\pi i} & \text{if} \quad j=0, m\in M\mathbb{Z}\\ \dfrac{M}{(m-kM)2\pi i}[e^{2\pi i(\frac{m}{M}-k)}-1] & \text{if} \quad m\neq 0, j=0\\ 0 & \text{if} \quad j\neq 0\\ 1 & \text{if} \quad m=0,\ k=0\end{cases}$$

therefore,

$$\begin{aligned}
E_{\frac{m}{M}}T_{nN}g =\ & \textstyle\sum_{k,j\in\mathbb{Z}} \alpha_{k,j}e^{2\pi i\frac{m}{M}(nN+j)}E_kT_{nN+j}(\sum_{p\in\mathbb{Z}}\beta_{p,0}E_ph)\\
&= \sum_{k,j\in\mathbb{Z}} \alpha_{k,j}e^{2\pi i\frac{m}{M}(nN+j)}\Big(\sum_{p\in\mathbb{Z}}\beta_{p,0}E_{k+p}T_{nN+j}h\Big)\\
&= \sum_{k,j\in\mathbb{Z}}\Big(\sum_{p\in\mathbb{Z}} \alpha_{k,j}e^{2\pi i\frac{m}{M}(nN+j)}\beta_{p,0}E_{k+p}T_{nN+j}h\Big)
\end{aligned}$$

taking , $r = k+p,\ s = nN+j$

$$\begin{aligned}
&= \sum_{r,s\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}} \alpha_{k,j-nN}e^{2\pi i\frac{m}{M}s}\beta_{r-k,0}\Big)E_rT_sh\\
&= \sum_{r,s\in\mathbb{Z}} \eta_{r,s}E_rT_sh
\end{aligned}$$

where, $\eta_{r,s} = \sum_{k\in\mathbb{Z}} \alpha_{k,j-nN}e^{2\pi i\frac{m}{M}s}\beta_{r-k,0}$

□

The following theorem gives a sufficient condition for a unitary operator U from $L^2(\mathbb{R})$ to $l^2(\mathbb{Z})$ which maps a Gabor frame $(g, \frac{1}{M},\ N)$ in $L^2(\mathbb{R})$ to a Gabor frame $(Ug, \frac{1}{M},\ N)$ in $l^2(\mathbb{Z})$.

Theorem 3.7. *Let $g \in L^2(\mathbb{R})$ and N,M are positive integers such that $\frac{N}{M} \leq 1$, $\{E_{\frac{m}{M}}T_{nN}g : m,n \in \mathbb{Z}\}$ is a Gabor frame in $L^2(\mathbb{R})$. Assume that if the window coefficient sequence of g with respect to the quadruple (m,n,M,N) given by $\{\eta_{r,s}\}$ satisfies $\eta_{r,s} = \alpha_{r,s-nN}e^{2\pi i\frac{m}{M}s}$ for all $r,s \in \mathbb{Z}$ where $\alpha_{kj} = \langle g, E_kT_jh\rangle$, $h = \chi_{[0,1]}$, then there is a unitary operator $U : L^2(\mathbb{R}) \to l^2(\mathbb{Z})$ so that $\{\hat{E}_{\frac{m}{M}}\hat{T}_{nN}Ug : m = 0,1,2...M-1, n \in \mathbb{Z}\}$ is a Gabor frame in $l^2(\mathbb{Z})$.*

Proof. Let us define $U : L^2(\mathbb{R}) \to l^2(\mathbb{Z})$ by, $U(E_lT_k\chi_{[0,1]}) = \hat{E}_l\hat{T}_ke_1$ where $\{e_j\}_{j\in\mathbb{Z}}$ is the standard orthonormal basis for $l^2(\mathbb{Z})$.
$g = \sum_{j,k\in\mathbb{Z}} \alpha_{jk}E_kT_jh$, where $h = \chi_{[0,1]}$

$$U(g) = \sum_{k,j\in\mathbb{Z}} \alpha_{k,j}\hat{E}_k\hat{T}_je_1$$

$$\hat{T}_{nN}U(g) = \sum_{k,j\in\mathbb{Z}} \alpha_{k,j}\hat{T}_{nN+j}e_1$$

$$\begin{aligned}
\hat{E}_{\frac{m}{M}}\hat{T}_{nN}U(g) &= \sum_{k,j\in\mathbb{Z}} \alpha_{k,j}\hat{E}_{\frac{m}{M}}\hat{T}_{nN+j}e_1\\
&= \sum_{k,j\in\mathbb{Z}} \alpha_{k,j}e^{2\pi i\frac{m}{M}(nN+j)}\hat{T}_{nN+j}e_1\\
&= \sum_{r,s\in\mathbb{Z}} \alpha_{r,s-nN}e^{2\pi i\frac{m}{M}s}\hat{T}_se_1 \qquad (1.1)
\end{aligned}$$

Now by definition of the window coefficient sequence $\{\eta_{r,s}\}$, $E_{\frac{m}{M}}T_{nN}g = \sum_{r,s\in\mathbb{Z}} \eta_{r,s}E_rT_sh$.

$$\begin{aligned}
\text{Therefore, } U(E_{\frac{m}{M}}T_{nN}g) &= U\Big(\sum_{r,s\in\mathbb{Z}} \eta_{r,s}E_rT_sh\Big)\\
&= \sum_{r,s\in\mathbb{Z}} \eta_{r,s}\hat{T}_se_1 \qquad (1.2)
\end{aligned}$$

Thus from equations (1.1) and (1.2), $UE_{\frac{m}{M}}T_{nN} = \hat{E}_{\frac{m}{M}}\hat{T}_{nN}U$ satisfies only when $\eta_{r,s} = \alpha_{r,s-nN}e^{2\pi i \frac{m}{M}s}$.

Hence by Theorem 3.4 $\{\hat{E}_{\frac{m}{M}}\hat{T}_{nN}Ug : m = 0,1,2...M-1, n \in \mathbb{Z}\}$ is a Gabor frame in $l^2(\mathbb{Z})$. □

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