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$\overline{q}$-Inequalities on quantum integral

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Abstract
In this paper, we present $\overline{q}$-Young integral inequality, $\overline{q}$-Hölder integral inequality, $\overline{q}$-Minkowski integral inequality and $\overline{q}$-Ostrowski type integral inequalities for new definition of q-integral which is showed $\overline{q}$-integral.

Keywords
Ostrowski inequality, Young, Hölder and Minkowski integral inequalities, convex functions, $\overline{q}$-integrals.

AMS Subject Classification
26D15, 26A33, 26A51.

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Contents

1. Introduction

Quantum calculus is the modern name for the investigation of calculus without limits.The quantum calculus or q-calculus began with FH Jackson[14],[15] in the early twentieth century, but this kind of calculus had already been worked out by Euler and Jacobi. Recently it arose interest due to high demand of mathematics that models quantum computing. q-calculus appeared as a connection between mathematics and physics. It has a lot of applications in different mathematical areas such as number theory, combinatorics, orthogonal polynomials, basic hyper-geometric functions and other sciences quantum theory, mechanics and the theory of relativity.

There are many of the fundamental aspects of quantum calculus. It has been shown that quantum calculus is a subfield of the more general mathematical field of time scales calculus. Many mathematicians have done studies in q-calculus analysis in [5],[8],[9],[10],[16],[17],[18],[19].

Recently, N. Alp[2] proved the correct *q-Hermite Hadamard inequalities and* M. Kunt[4] obtained (p,q)*-Hermite–Hadamard inequalities.*

In 2017, Alp and Sarikaya [3] gave a new definition of q-integral which is showed $\overline{q}$-integral.

The aim of this paper present some well-known integral inequalities on $\overline{q}$-integral. In second section we give notations and preliminaries for q-analogue. In third section we obtain some auxiliary results. In fourth section we establish $\overline{q}$-Young, $\overline{q}$-Hölder and $\overline{q}$-Minkowski integral inequalities and in finally section we obtain $\overline{q}$-Ostrowski type integral inequalities on $\overline{q}$-integral. Let remember following integral inequalities on classical analysis.

In 1938, Ostrowski [7] proved the following integral inequality:

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$, be a differentiable mapping in $\overset{\circ}{I}$ the interior of I and $a,b \in \overset{\circ}{I}$ with $a<b$. If $|f'(x)| \leq M$ for all $x \in [a,b]$, then we have the inequality

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq (b-a)M\left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right] \tag{1.1}$$

for $\forall x \in [a,b]$.The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller constant.

For $a>0$, $b>0$ and $\frac{1}{p}+\frac{1}{r}=1$ with $p>1$ the following inequality is well-known Young inequality[13]:

$$a.b \leq \frac{a^p}{p} + \frac{b^r}{r}.$$

For $\frac{1}{p}+\frac{1}{r}=1$ with $p>1$ the following inequality is well-known Hölder inequality[6]:

$$\int_a^b |f(t)\,g(t)|\,dt \le \|f\|_p \|g\|_r .$$

For $p > 1$ the following inequality is well-known Minkowski inequality[11]:

$$\|f+g\|_p \le \|f\|_p + \|g\|_p .$$

2. Notations and Preliminaries

In this section, first we give definition and notations of q-analogue with q-derivatives then definition and properties of $\overline{q}$-integral. Although some different notations were used in various articles, they have the same meaning. We put together them below:

For $0 < q < 1$ here and further we use the following notations(see[1])

$$[n]_q = \frac{1-q^n}{1-q} = 1+q+q^2+\dots+q^{n-1}. \tag{2.1}$$

The definition of q-binomial coefficients:

$$\left[\begin{array}{c} n \\ i \end{array}\right]_q = \frac{[n]_q!}{[n-i]_q!\,[i]_q!} = \left[\begin{array}{c} n \\ n-i \end{array}\right]_q . \tag{2.2}$$

Also the q-binomial multiplication is as follows:

$$\begin{aligned} &(x-a)_q^n \quad (2.3)\\ =\ & \prod_{i=0}^{n-1}\left(x-q^i a\right) \\ =\ & (x-a)(x-qa)\left(x-q^2a\right)\dots\left(x-q^{n-1}a\right), \ \text{ if } n\in\mathbb{Z}^+ \end{aligned}$$

and by using (2.2) and (2.3) the following formula is called Gauss's binomial formula:

$$(x-a)_q^n = \sum_{i=0}^{n} (-1)^i \left[\begin{array}{c} n \\ i \end{array}\right]_q q^{\frac{i(i-1)}{2}} x^{n-i} a^i. \tag{2.4}$$

On the other hand, the following notations will be used throughout the study:

$$(a:q)_0 = 1;$$

$$(1-a)_q^n = (a:q)_n = \prod_{i=0}^{n}\left(1-q^i a\right);$$

$$(1-a)_q^\infty = (a:q)_\infty = \prod_{i=0}^{\infty}\left(1-q^i a\right) \tag{2.5}$$

and

$$(1-a)_q^n = \frac{(1-a)_q^\infty}{(1-q^n a)_q^\infty} = \frac{(a:q)_\infty}{(q^n a:q)_\infty}, \quad \text{if } n\in\mathbb{C}. \tag{2.6}$$

Notice that, under our assumptions on q, the infinite product (2.5) is convergent. Moreover, the definitions (2.3) and (2.6) are consistent.

Definition 2.1. *In [15], For f has $D_q^n f(a)$, Jackson introduced the following q-counterpart of Taylor series:*

$$f(x) = \sum_{n=0}^{\infty} \frac{(1-q)^n}{(q;q)_n} D_q^n f(a)\,(x-a)_q^n = \sum_{n=0}^{\infty} \frac{D_q^n f(a)\,(x-a)_q^n}{[n]_q!} \tag{2.7}$$

where D_q is the q-difference operator.

Let give definition q-derivates. Let $J := [a,b] \subset \mathbb{R}$, $J^\circ := (a,b)$ be interval and $0 < q < 1$ be a constant. The definition of q-derivative of a function $f : J \to \mathbb{R}$ at a point $x \in J$ on $[a,b]$ as follows:

Definition 2.2. *[12]Assume $f : J \to \mathbb{R}$ is a continuous function and let $x \in J$. Then the expression*

$$\begin{aligned} & {}_aD_q\ f(x) \quad (2.8)\\ &= \frac{f(x) - f(qx+(1-q)a)}{(1-q)(x-a)}, \quad x \ne a, \\ {}_aD_q\ f(a) &= \lim_{x\to a} {}_aD_q\ f(x) \end{aligned}$$

is called the q-derivative on J of function f at x.

We say that f is q -differentiable on J provided ${}_aD_q\ f(x)$ exists for all $x \in J$. Note that if $a = 0$ in (2.8), then ${}_0D_q\ f = D_q f$, where D_q is the well-known q-derivative of the function $f(x)$ defined by

$$D_q\ f(x) = \frac{f(x) - f(qx)}{(1-q)\,x}.$$

For more details, see [1].

Lemma 2.3. *[12] Let $\alpha \in \mathbb{R}$, then we have*

$${}_aD_q\ (x-a)^\alpha = [\alpha]_q (x-a)^{\alpha-1}. \tag{2.9}$$

The following definitions and theorems with respect to $\overline{q}$-integral were referred in [3]:

Definition 2.4. *Let $f : J \to \mathbb{R}$ is continuous function. For $0 < q < 1$*

$$\begin{aligned} &\int_a^b f(s)\ {}_ad_{\overline{q}}s \quad (2.10)\\ =\ & \frac{(1-q)(b-a)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n f(q^n b + (1-q^n)a) - f(b)\right] \end{aligned}$$

which second sense quantum integral definition that call $\overline{q}$-integral for $x \in J$.

Moreover, if $c \in (a,x)$ then the definite $\overline{q}$-integral on J is defined by

$$\int_c^x f(s)\ {}_a d_{\overline{q}} s \tag{2.11}$$
$$= \int_a^x f(s)\ {}_a d_{\overline{q}} s - \int_a^c f(s)\ {}_a d_{\overline{q}} s$$
$$= \frac{(1-q)(x-a)}{2q} \times \left[(1+q)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a) - f(x)\right] - \frac{(1-q)(c-a)}{2q} \times \left[(1+q)\sum_{n=0}^{\infty} q^n f(q^n c + (1-q^n)a) - f(c)\right].$$

The following theorem is the correction of the Theorem 1 in [3]:

Theorem 2.5. *Let $f : J \to \mathbb{R}$ be a continuous function. Then we have*

$$ {}_aD_q \int_a^x f(s)\ {}_a d_{\overline{q}} s = \frac{f(x) + f(qx + (1-q)a)}{2}. \tag{2.12}$$

Proof. From definition of $\overline{q}$-integral, we have

$$\int_a^x f(s)\ {}_a d_{\overline{q}} s = \frac{(1-q)(x-a)}{2q} \times \left[(1+q)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a) - f(x)\right]$$

and take q-derivative of above equality write that

$$ {}_aD_q \int_a^x f(s)\ {}_a d_{\overline{q}} s = {}_aD_q \left\{ \frac{(1-q)(x-a)}{2q} \times \left[(1+q)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a) - f(x)\right]\right\}$$
$$= \frac{1}{(1-q)(x-a)} \left\{ \frac{(1-q)(x-a)}{2q} \times \left[(1+q)\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a) - f(x)\right] - \frac{(1-q)(x-a)q}{2q} \times \left[(1+q)\sum_{n=0}^{\infty} q^n f\left(q^{n+1} x + \left(1-q^{n+1}\right)a\right) - f(qx + (1-q)a)\right]\right\}$$
$$= \frac{1}{2q}\left[(1+q)\left(\sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a) - \sum_{n=0}^{\infty} q^{n+1} f\left(q^{n+1}x + \left(1-q^{n+1}\right)a\right)\right) + qf(qx+(1-q)a) - f(x)\right]$$
$$= \frac{qf(x) + qf(qx+(1-q)a)}{2q}$$
$$= \frac{f(x) + f(qx+(1-q)a)}{2}.$$

The proof is completed. □

Theorem 2.6 (Change of Variables Property)**.** *Let $f : J \to \mathbb{R}$ be a function and $0 < q < 1$. Then we have*

$$\int_0^1 f(sb + (1-s)a)\ {}_0 d_{\overline{q}} s = \frac{1}{b-a}\int_a^b f(t)\ {}_a d_{\overline{q}} t \ . \tag{2.13}$$

Theorem 2.7. *Let $f : J \to \mathbb{R}$ be a continuous function. Then, for $c \in (a,x)$ we have*

$$\int_c^x {}_aD_q\ f(s)\ {}_a d_{\overline{q}} s \tag{2.14}$$
$$= \frac{qf(x) + f(qx+(1-q)a) - qf(c) - f(qc+(1-q)a)}{2q}.$$

Theorem 2.8. *Assume $f, g : J \to \mathbb{R}$ are continuous functions. Then, for $x \in J$,*

$$\int_a^x [f(s) + g(s)]\ {}_a d_{\overline{q}} s = \int_a^x f(s)\ {}_a d_{\overline{q}} s + \int_a^x g(s)\ {}_a d_{\overline{q}} s\ . \tag{2.15}$$

Theorem 2.9. *Assume $f, g : J \to \mathbb{R}$ are continuous functions, $\alpha \in \mathbb{R}$. Then, for $x \in J$,*

$$\int_a^x (\alpha f)(s)\ {}_a d_{\overline{q}} s = \alpha \int_a^x f(s)\ {}_a d_{\overline{q}} s\ . \tag{2.16}$$

Theorem 2.10. *Assume $f,g : J \to \mathbb{R}$ are continuous functions. Then, for $x \in J$*

$$\begin{aligned} &\int_c^x f(s) \ {}_aD_q \ g(s) \ {}_ad_{\overline{q}}s \qquad (2.17) \\ = \ & \left. \frac{qf(s)g(s)+f(qs+(1-q)a)g(qs+(1-q)a)}{2q} \right|_c^x \\ & -\int_c^x g(qs+(1-q)a) \ {}_aD_q \ f(s) \ {}_ad_{\overline{q}}s \ . \end{aligned}$$

Theorem 2.11. *For $\alpha \in \mathbb{R}\backslash\{-1\}$, the following formula holds:*

$$\int_a^x (s-a)^{\alpha} \ {}_ad_{\overline{q}}s = \frac{1+q^{\alpha}}{2[\alpha+1]_q}(x-a)^{\alpha+1}. \qquad (2.18)$$

3. Auxiliary Results

In this section, we present some auxiliary results which are used throughout this article.

Lemma 3.1. *Let $f : [a,b] \to \mathbb{R}$ be a convex q-differentiable function on (a,b) and $0 < q < 1$. Then we have*

$$\begin{aligned} &\frac{1}{b-a}\left[\int_a^x (t-a) \ {}_aD_q \ f(t) \ {}_ad_{\overline{q}}t \right. \qquad (3.1) \\ &\left. +\int_x^b (t-b) \ {}_aD_q \ f(t) \ {}_ad_{\overline{q}}t \right] \\ = \ & \frac{1}{2q}[qf(x)+f(qx+(1-q)a) \\ &-(1-q)f(qb+(1-q)a)] \\ &-\frac{1}{b-a}\int_a^b f(qt+(1-q)a) \ {}_ad_{\overline{q}}t \ . \end{aligned}$$

Proof. By using (2.17), then we have

$$\begin{aligned} &\int_a^x (t-a) \ {}_aD_q \ f(t) \ {}_ad_{\overline{q}}t \qquad (3.2) \\ &+\int_x^b (t-b) \ {}_aD_q \ f(t) \ {}_ad_{\overline{q}}t \\ = \ & \left. \frac{q(t-a)f(t)+(qt+(1-q)a-a)f(qt+(1-q)a)}{2q} \right|_a^x \\ &-\int_a^x f(qt+(1-q)a) \ {}_aD_q \ (t-a) \ {}_ad_{\overline{q}}t \\ &+ \left. \frac{q(t-b)f(t)+(qt+(1-q)a-b)f(qt+(1-q)a)}{2q} \right|_x^b \\ &-\int_x^b f(qt+(1-q)a) \ {}_aD_q \ (t-b) \ {}_ad_{\overline{q}}t \\ = \ & \frac{q(b-a)f(x)+(b-a)f(qx+(1-q)a)-(1-q)(b-a)f(qb+(1-q)a)}{2q} \\ &-\int_a^b f(qt+(1-q)a) \ {}_ad_{\overline{q}}t \\ = \ & \frac{b-a}{2q}[qf(x)+f(qx+(1-q)a) \\ &-(1-q)f(qb+(1-q)a)] \\ &-\int_a^b f(qt+(1-q)a) \ {}_ad_{\overline{q}}t \end{aligned}$$

divide by $(b-a)$ equation (3.2) and the proof is completed. □

Remark 3.2. *In Lemma 3.1, by using change of variables property, then we have*

$$\begin{aligned} &\int_a^x (t-a) \ {}_aD_q \ f(t) \ {}_ad_{\overline{q}}t \qquad (3.3) \\ &+\int_x^b (t-b) \ {}_aD_q \ f(t) \ {}_ad_{\overline{q}}t \\ = \ & (x-a)^2\int_0^1 t \ {}_aD_q \ f(xt+(1-t)a) \ {}_0d_{\overline{q}}t \\ &+(b-x)^2\int_0^1 (t-1) \ {}_aD_q \ f(bt+(1-t)x) \ {}_0d_{\overline{q}}t \ . \end{aligned}$$

Remark 3.3. *In Lemma 3.1 if we take $q \to 1^{-}$, we recapture* *the following equation for convex function.*

$$\frac{1}{b-a}\left[\int_a^x (t-a)f'(t)dt+\int_x^b (t-b)f'(t)dt\right] = f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt.$$

Lemma 3.4. *Let $p>1$, and $(a-t)_q^p$ is q-binomial,then we have*

$$\int_0^x (a-t)_q^p \ {}_0d_{\overline{q}}t = \frac{1}{2[p+1]_q}\left(1+q-q\left(a-\frac{x}{q}\right)_q^{p+1}-(a-x)_q^{p+1}\right). \tag{3.4}$$

Proof. By using q-binomial formula and (2.18) we obtain the following result

$$\begin{aligned}
&\int_0^x (a-t)_q^p \ {}_ad_{\overline{q}}t \\
=& \int_0^x \sum_{n=0}^{p}(-1)^n q^{\frac{n(n-1)}{2}}\left[\begin{array}{c}p\\ n\end{array}\right]_q a^{p-n}t^n \ {}_0d_{\overline{q}}t \\
=& \sum_{n=0}^{p}(-1)^n q^{\frac{n(n-1)}{2}}\left[\begin{array}{c}p\\ n\end{array}\right]_q a^{p-n}\int_0^x t^n \ {}_0d_{\overline{q}}t \\
=& \sum_{n=0}^{p}(-1)^n q^{\frac{n(n-1)}{2}}\left[\begin{array}{c}p\\ n\end{array}\right]_q a^{p-n}\frac{1+q^n}{2[n+1]_q}x^{n+1} \\
=& \frac{1}{2}\sum_{n=0}^{p}(-1)^n \frac{q^{\frac{n(n-1)}{2}}[p]_q!}{[p-n]_q![n+1]_q!}a^{p-n}x^{n+1} \\
&+\frac{1}{2}\sum_{n=0}^{p}\frac{(-1)^n q^{\frac{n(n-1)}{2}}[p]_q!}{[p-n]_q![n+1]_q!}q^n a^{p-n}x^{n+1} \\
=& \frac{1}{2}\sum_{n=1}^{p+1}\frac{(-1)^{n-1} q^{\frac{(n-1)(n-2)}{2}}[p]_q!}{[p-n+1]_q![n]_q!}a^{p-n+1}x^n \\
&+\frac{1}{2}\sum_{n=1}^{p+1}\frac{(-1)^{n-1} q^{\frac{(n-1)(n-2)}{2}}[p]_q!}{[p-n+1]_q![n]_q!}q^{n-1}a^{p-n+1}x^n \\
=& -\frac{1}{2[p+1]_q}\times \\
&\left(q\sum_{n=1}^{p+1}(-1)^n q^{\frac{n(n-1)}{2}}\left[\begin{array}{c}p+1\\ n\end{array}\right]_q a^{p+1-n}\left(\frac{x}{q}\right)^n\right. \\
&\left.+\sum_{n=1}^{p+1}(-1)^n q^{\frac{n(n-1)}{2}}\left[\begin{array}{c}p+1\\ n\end{array}\right]_q a^{p+1-n}x^n\right) \\
=& -\frac{q}{2[p+1]_q}\left[\left(a-\frac{x}{q}\right)_q^{p+1}-1\right] \\
&-\frac{1}{2[p+1]_q}\left[(a-x)_q^{p+1}-1\right] \\
=& \frac{1+q}{2[p+1]_q}-\frac{q\left(a-\frac{x}{q}\right)_q^{p+1}+(a-x)_q^{p+1}}{2[p+1]_q} \\
=& \frac{1}{2[p+1]_q}\left(1+q-q\left(a-\frac{x}{q}\right)_q^{p+1}-(a-x)_q^{p+1}\right)
\end{aligned}$$

and the proof is completed. □

Example 3.5. *Let $p>1$, and $(a-t)_q^p$ is q-binomial,then we have*

$$\int_0^1 (1-t)_q^p \ {}_0d_{\overline{q}}t = \frac{1+q}{2[p+1]_q}. \tag{3.5}$$

Proof. By using (3.4) with choose $a=1$ and $x=1$ then the proof is completed as follows

$$\begin{aligned}
&\int_0^1 (1-t)_q^p \ {}_0d_{\overline{q}}t \\
=& \frac{1}{2[p+1]_q}\left(1+q-q\left(1-\frac{1}{q}\right)_q^{p+1}-(1-1)_q^{p+1}\right)
\end{aligned}$$

by using $\left(1-\frac{1}{q}\right)_q^{p+1}=\left(1-\frac{1}{q}\right)\left(1-\frac{q}{q}\right)\left(1-\frac{q^2}{q}\right)..=0$ and we obtain

$$\int_0^1 (1-t)_q^p \ {}_0d_{\overline{q}}t = \frac{1+q}{2[p+1]_q}$$

which is desired. □

4. $\overline{q}$-Young, $\overline{q}$-Hölder's and $\overline{q}$-Minkowski Inequalities

In this section, we obtained $\overline{q}$-Young, $\overline{q}$-Hölder's and $\overline{q}$-Minkowski Inequalities on $\overline{q}$-integral.

Theorem 4.1 ($\overline{q}$-Young Inequality)**.** *$a>0$, $b>0$ and $\frac{1}{p}+\frac{1}{r}=1$ with $p>1$*

$$a.b \le \frac{1+q^{p-1}}{2[p]_q}a^p+\frac{1+q^{r-1}}{2[r]_q}b^r. \tag{4.1}$$

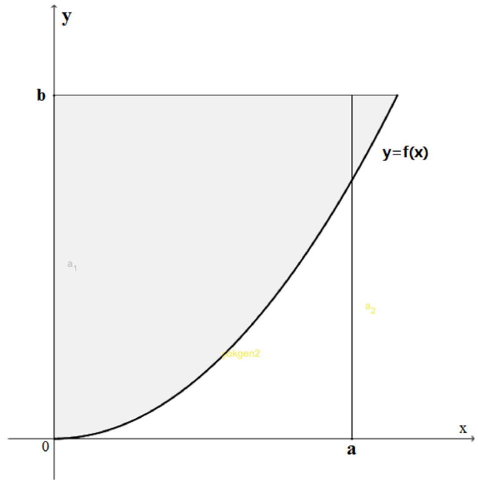

Figure 1. $y = x^{p-1}$

Proof. Choose $y = x^{p-1}$ functions for $p > 1$ and $\frac{1}{p} + \frac{1}{r} = 1$ with $a > 0$, $b > 0$. Let draw the graph of $y = x^{p-1}$

$$\begin{aligned} s_1 &= \int_0^a x^{p-1} \quad {}_a d_{\overline{q}} x = \frac{1-q}{1-q^p} \frac{1+q^{p-1}}{2} a^p, \\ s_2 &= \int_0^b y^{\frac{1}{p-1}} \quad {}_a d_{\overline{q}} y = \frac{1-q}{1-q^{\frac{p}{p-1}}} \frac{1+q^{\frac{1}{p-1}}}{2} b^{\frac{p}{p-1}} \\ &= \frac{1-q}{1-q^r} \frac{1+q^{r-1}}{2} b^r. \end{aligned}$$

According to the graph of $y = x^{p-1}$ we have

$$a.b \leq s_1 + s_2 = \frac{1-q}{2} \left[\frac{1+q^{p-1}}{1-q^p} a^p + \frac{1+q^{r-1}}{1-q^r} b^r \right]$$

$$a.b \leq \frac{1+q^{p-1}}{2[p]_q} a^p + \frac{1+q^{r-1}}{2[r]_q} b^r$$

which is completed the proof. □

Remark 4.2. *In Theorem 4.1, if we take $q \to 1^-$, we recapture Young inequality in [13].*

Theorem 4.3 ($\overline{q}$-Hölder's Inequality)**.** *Let $\frac{1}{p} + \frac{1}{r} = 1$ with $p > 1$. Then the following inequality holds*

$$\int_a^b |f(t)g(t)| \quad {}_a d_{\overline{q}} t \leq \left[\frac{1+q^{p-1}}{2[p]_q} + \frac{1+q^{r-1}}{2[r]_q} \right] \|f\|_p \|g\|_r \tag{4.2}$$

where

$$\|f\|_p = \left(\int_a^b |f(t)|^p \quad {}_a d_{\overline{q}} t \right)^{\frac{1}{p}}.$$

Proof. Choose $a = \frac{|f(t)|}{\|f\|_p}$, $b = \frac{|g(t)|}{\|g\|_r}$ and by using $\overline{q}$-Young inequality, we write

$$\frac{|f(t)|}{\|f\|_p} \frac{|g(t)|}{\|g\|_r} \leq \frac{1+q^{p-1}}{2[p]_q} \frac{|f(t)|^p}{\|f\|_p^p} + \frac{1+q^{r-1}}{2[r]_q} \frac{|g(t)|^r}{\|g\|_r^r}. \tag{4.3}$$

Now take $\overline{q}$-integral inequality (4.3) on $[a,b]$, we get

$$\begin{aligned} &\frac{1}{\|f\|_p \|g\|_r} \int_a^b |f(t)g(t)| \quad {}_a d_{\overline{q}} t \\ \leq\ & \frac{1+q^{p-1}}{2[p]_q} \frac{1}{\|f\|_p^p} \int_a^b |f(t)|^p \quad {}_a d_{\overline{q}} t \\ &+ \frac{1+q^{r-1}}{2[r]_q} \frac{1}{\|g\|_r^r} \int_a^b |g(t)|^r \quad {}_a d_{\overline{q}} t \\ \leq\ & \frac{1+q^{p-1}}{2[p]_q} + \frac{1+q^{r-1}}{2[r]_q} \end{aligned}$$

and thus,

$$\int_a^b |f(t)g(t)| \quad {}_a d_{\overline{q}} t \leq \left[\frac{1+q^{p-1}}{2[p]_q} + \frac{1+q^{r-1}}{2[r]_q} \right] \|f\|_p \|g\|_r$$

which is completed the proof. □

Remark 4.4. *In Theorem 4.2, if we take $q \to 1^-$, we recapture classical Holder's inequality.*

Theorem 4.5 ($\overline{q}$-Minkowski Inequality)**.** *For $p > 1$*

$$\|f+g\|_p \leq \left[\frac{1+q^{p-1}}{2[p]_q} + \frac{1+q^{r-1}}{2[r]_q} \right] \left(\|f\|_p + \|g\|_p \right). \tag{4.4}$$

Proof. From $\overline{q}$-Hölder's inequality, we get

$$\begin{aligned} &\|f+g\|_p^p \\ \leq\ & \int_a^b |f(t)| \, |f(t)+g(t)|^{p-1} \quad {}_a d_{\overline{q}} t \\ &+ \int_a^b |g(t)| \, |f(t)+g(t)|^{p-1} \quad {}_a d_{\overline{q}} t \\ \leq\ & \left(\frac{1+q^{p-1}}{2[p]_q} + \frac{1+q^{r-1}}{2[r]_q} \right) \\ &\times \left(\int_a^b |f(t)+g(t)|^{r(p-1)} \quad {}_a d_{\overline{q}} t \right)^{\frac{1}{r}} \\ &\times \left(\left[\int_a^b |f(t)|^p \quad {}_a d_{\overline{q}} t \right]^{\frac{1}{p}} + \left[\int_a^b |g(t)|^p \quad {}_a d_{\overline{q}} t \right]^{\frac{1}{p}} \right). \end{aligned}$$

By using $r(p-1)=p$, it follows that

$$\|f+g\|_p \leq \left[\frac{1+q^{p-1}}{2[p]_q}+\frac{1+q^{r-1}}{2[r]_q}\right]\left(\|f\|_p+\|g\|_p\right)$$

and the proof is completed. □

Remark 4.6. *In Theorem 4.4, if we take $q\to 1^-$, we recapture classical Minkowski inequality.*

5. Ostrowski Type Inequalities

In this section, we obtained $\overline{q}$-Ostrowski type inequalities for $\overline{q}$-integral as follows:

Theorem 5.1. *Let $f:[a,b]\to\mathbb{R}$ be a convex q-differentiable function on (a,b) and $\left|{}_aD_q\ f(x)\right|\leq M$ for all $x\in[a,b]$. Then, we have*

$$\begin{aligned}&\left|\frac{1}{2q}\{qf(x)+f(qx+(1-q)a)\right. \\ &-(1-q)f(qb+(1-q)a)\} \\ &\left.-\frac{1}{b-a}\int_a^b f(qt+(1-q)a)\ {}_ad_{\overline{q}}t\right| \\ \leq\ & (b-a)M\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right]\end{aligned} \tag{5.1}$$

for all $x\in[a,b]$ and $0<q<1$.

Proof. By using equation (3.1), we have

$$\begin{aligned}&\left|\frac{(b-a)}{2q}\{qf(x)+f(qx+(1-q)a)\right. \\ &-(1-q)f(qb+(1-q)a)\} \\ &\left.-\int_a^b f(qt+(1-q)a)\ {}_ad_{\overline{q}}t\right| \\ \leq\ & \int_a^x |t-a|\left|{}_aD_q\ f(t)\right|\ {}_ad_{\overline{q}}t \\ &+\int_x^b |t-b|\left|{}_aD_q\ f(t)\right|\ {}_ad_{\overline{q}}t \\ \leq\ & M\int_a^x (t-a)\ {}_ad_{\overline{q}}t\ +M\int_x^b (b-t)\ {}_ad_{\overline{q}}t \\ =\ & M\left.\frac{(1+q)(t-a)^2}{2[2]_q}\right|_a^x \\ &-M\left[\left.\frac{(1+q)(t-a)^2}{2[2]_q}+(a-b)(t-a)\right|_x^b\right] \\ =\ & M\left[(x-a)^2+(b-a)\left(\frac{a+b}{2}-x\right)\right] \\ =\ & (b-a)^2M\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right]\end{aligned} \tag{5.2}$$

with divide by $(b-a)$ inequality (5.2), and the proof is completed. □

Remark 5.2. *In Theorem 5.1, if we take $q\to 1^-$, we recapture inequality (1.1).*

Theorem 5.3. *Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a convex q-differentiable function on I° and $\left|{}_aD_q\ f(x)\right|$ is convex for all $x\in I^\circ$ and $a,b\in I$. Then the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{2q}\{qf(x)+f(qx+(1-q)a)\right. \\ &-(1-q)f(qb+(1-q)a)\} \\ &\left.-\frac{1}{b-a}\int_a^b f(qt+(1-q)a)\ {}_ad_{\overline{q}}t\right| \\ \leq\ & \frac{q}{2[3]_q}\frac{(x-a)^2\left|{}_aD_q\ f(a)\right|+(b-x)^2\left|{}_aD_q\ f(b)\right|}{b-a} \\ &+\frac{1+q^2}{2[3]_q}\frac{(x-a)^2+(b-x)^2}{b-a}\left|{}_aD_q\ f(x)\right|.\end{aligned} \tag{5.3}$$

Proof. By using (3.1),(3.3) and convexty of $\left|{}_aD_q\ f(x)\right|$, then we have

$$\begin{aligned}&\left|\frac{1}{2q}\{qf(x)+f(qx+(1-q)a)\right. \\ &-(1-q)f(qb+(1-q)a)\} \\ &\left.-\frac{1}{b-a}\int_a^b f(qt+(1-q)a)\ {}_ad_{\overline{q}}t\right| \\ =\ & \frac{1}{b-a}\left|\int_a^x (t-a)\ {}_aD_q\ f(t)\ {}_ad_{\overline{q}}t\right. \\ &\left.+\int_x^b (t-b)\ {}_aD_q\ f(t)\ {}_ad_{\overline{q}}t\right| \\ =\ & \left|\frac{(x-a)^2}{b-a}\int_0^1 t\ {}_aD_q\ f(xt+(1-t)a)\ {}_0d_{\overline{q}}t\right. \\ &\left.+\frac{(b-x)^2}{b-a}\int_0^1 (t-1)\ {}_aD_q\ f(bt+(1-t)x)\ {}_0d_{\overline{q}}t\right| \\ \leq\ & \frac{(x-a)^2}{b-a}\int_0^1 t\begin{pmatrix} t\left|{}_aD_q\ f(x)\right| \\ +(1-t)\left|{}_aD_q\ f(a)\right|\end{pmatrix}{}_0d_{\overline{q}}t \\ &+\frac{(b-x)^2}{b-a}\int_0^1 (1-t)\begin{pmatrix} t\left|{}_aD_q\ f(b)\right| \\ +(1-t)\left|{}_aD_q\ f(x)\right|\end{pmatrix}{}_0d_{\overline{q}}t\end{aligned}$$

$$
\begin{aligned}
&= \frac{(x-a)^2}{b-a}\left[\frac{1+q^2}{2[3]_q}\left|\ {}_aD_q\ f(x)\right| + \left(\frac{1}{2}-\frac{1+q^2}{2[3]_q}\right)\left|\ {}_aD_q\ f(a)\right|\right] \\
&\quad + \frac{(b-x)^2}{b-a}\left[\left(\frac{1}{2}-\frac{1+q^2}{2[3]_q}\right)\left|\ {}_aD_q\ f(b)\right| + \frac{1+q^2}{2[3]_q}\left|\ {}_aD_q\ f(x)\right|\right] \\
&= \frac{q}{2[3]_q}\cdot\frac{(x-a)^2\left|\ {}_aD_q\ f(a)\right| + (b-x)^2\left|\ {}_aD_q\ f(b)\right|}{b-a} \\
&\quad + \frac{1+q^2}{2[3]_q}\cdot\frac{(x-a)^2+(b-x)^2}{b-a}\left|\ {}_aD_q\ f(x)\right|
\end{aligned}
$$

where $\frac{1}{2}-\frac{1+q^2}{2[3]_q}=\frac{q}{2[3]_q}$ and the proof is completed. □

Corollary 5.4. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a convex differentiable function on I° and $|f'(x)|$ is convex for all $x\in I^\circ$ and $a,b\in I$. Then the following inequality holds:*

$$
\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \qquad (5.4)\\
&\le \frac{(x-a)^2|f'(a)|+(b-x)^2|f'(b)|}{6(b-a)} + \frac{(x-a)^2+(b-x)^2}{3(b-a)}\left|f'(x)\right|.
\end{aligned}
$$

Proof. In (5.3) if we take $q\to 1^-$, we recapture (5.4) and the proof is completed. □

Theorem 5.5. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ and $\left|\ {}_aD_q\ f(x)\right|^r$ be convex q-differentiable function on I° for $\frac{1}{p}+\frac{1}{r}=1$ with $p>1$ and $a,b\in I$. Then the following inequality holds:*

$$
\begin{aligned}
&\left|\frac{1}{2q}\{qf(x)+f(qx+(1-q)a) - (1-q)f(qb+(1-q)a)\} - \frac{1}{b-a}\int_a^b f(qt+(1-q)a)\ {}_ad_{\overline{q}}t\right| \qquad (5.5)\\
&\le \left[\frac{1+q^{p-1}}{4[p]_q}+\frac{1+q^{r-1}}{4[r]_q}\right] \\
&\quad\times\left\{\frac{(x-a)^2}{b-a}\left(\frac{1+q^p}{[p+1]_q}\right)^{\frac{1}{p}} \times\left(\left|\ {}_aD_q\ f(x)\right|^r+\left|\ {}_aD_q\ f(a)\right|^r\right)^{\frac{1}{r}}\right. \\
&\quad+\frac{(b-x)^2}{b-a}\left(\frac{1+q}{[p+1]_q}\right)^{\frac{1}{p}} \\
&\quad\left.\times\left(\left|\ {}_aD_q\ f(b)\right|^r+\left|\ {}_aD_q\ f(x)\right|^r\right)^{\frac{1}{r}}\right\}.
\end{aligned}
$$

Proof. By using Lemma (3.1) and (3.3), then we have

$$
\begin{aligned}
N &= \left|\frac{(b-a)}{2q}\{qf(x)+f(qx+(1-q)a) - (1-q)f(qb+(1-q)a)\} - \int_a^b f(qt+(1-q)a)\ {}_ad_{\overline{q}}t\right| \\
&\le (x-a)^2\int_0^1 t\left|\ {}_aD_q\ f(xt+(1-t)a)\right|\ {}_0d_{\overline{q}}t \\
&\quad+(b-x)^2\int_0^1(1-t)\left|\ {}_aD_q\ f(bt+(1-t)x)\right|\ {}_0d_{\overline{q}}t
\end{aligned}
$$

and by using $\overline{q}$-Hölder's inequality then we have

$$
\begin{aligned}
N &\le \left[\frac{1+q^{p-1}}{2[p]_q}+\frac{1+q^{r-1}}{2[r]_q}\right] \\
&\quad\times\left\{(x-a)^2\left(\int_0^1 t^p\right)^{\frac{1}{p}} \times\left(\int_0^1\left\{\begin{array}{c} t\left|\ {}_aD_q\ f(x)\right|^r \\ +(1-t)\left|\ {}_aD_q\ f(a)\right|^r\end{array}\right\}\ {}_0d_{\overline{q}}t\right)^{\frac{1}{r}}\right. \\
&\quad+(b-x)^2\left(\int_0^1(1-t)_q^p\ \ {}_0d_{\overline{q}}t\right)^{\frac{1}{p}} \\
&\quad\left.\times\left(\int_0^1\left\{\begin{array}{c} t\left|\ {}_aD_q\ f(b)\right|^r \\ +(1-t)\left|\ {}_aD_q\ f(x)\right|^r\end{array}\right\}\ {}_0d_{\overline{q}}t\right)^{\frac{1}{r}}\right\}.
\end{aligned}
$$

Now by using equality (3.5) we have

$$N \leq \left[\frac{1+q^{p-1}}{2[p]_q}+\frac{1+q^{r-1}}{2[r]_q}\right] \times\left\{(x-a)^2\left(\frac{1+q^p}{2[p+1]_q}\right)^{\frac{1}{p}}\times\left(\frac{\left|\,{}_aD_q\ f(x)\right|^r}{2}+\frac{\left|\,{}_aD_q\ f(a)\right|^r}{2}\right)^{\frac{1}{r}}+(b-x)^2\left(\frac{1+q}{2[p+1]_q}\right)^{\frac{1}{p}}\times\left(\frac{\left|\,{}_aD_q\ f(b)\right|^r}{2}+\frac{\left|\,{}_aD_q\ f(x)\right|^r}{2}\right)^{\frac{1}{r}}\right\}$$

$$= \left[\frac{1+q^{p-1}}{4[p]_q}+\frac{1+q^{r-1}}{4[r]_q}\right] \times\left\{(x-a)^2\left(\frac{1+q^p}{[p+1]_q}\right)^{\frac{1}{p}}\times\left(\left|\,{}_aD_q\ f(x)\right|^r+\left|\,{}_aD_q\ f(a)\right|^r\right)^{\frac{1}{r}}+(b-x)^2\left(\frac{1+q}{[p+1]_q}\right)^{\frac{1}{p}}\times\left(\left|\,{}_aD_q\ f(b)\right|^r+\left|\,{}_aD_q\ f(x)\right|^r\right)^{\frac{1}{r}}\right\}$$

with divide by $(b-a)$ last inequality the proof is completed. □

Corollary 5.6. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ and $|f'(x)|^r$ be convex differentiable function on I° for $\frac{1}{p}+\frac{1}{r}=1$ with $p>1$ and $a,b\in I$. Then the following inequality holds:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{5.6}$$
$$\leq \frac{1}{2^{\frac{1}{r}}(p+1)^{\frac{1}{p}}}\times\left\{\frac{(x-a)^2}{b-a}\left(|f'(x)|^r+|f'(a)|^r\right)^{\frac{1}{r}}+\frac{(b-x)^2}{b-a}\left(|f'(b)|^r+|f'(x)|^r\right)^{\frac{1}{r}}\right\}.$$

Proof. In (5.5) if we take $q\to 1^-$, we recapture (5.4) and the proof is completed. □

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