MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **10(03)**(2022), 224–236.
http://doi.org/10.26637/mjm1003/005

A new fixed point result in bipolar controlled fuzzy metric spaces with application

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***R**eceived 12 November 2021; **A**ccepted 17 May 2022*

Abstract. In this paper, we introduce the notion of bipolar controlled fuzzy metric spaces which is an extension of the result of Sezen [20]. The paper concerns our sustained efforts for the materialization of controlled fuzzy metric spaces. Further, we establish a Banach-type fixed point theorem. We provide suitable examples with graphics which validate our result. We also employ an application to substantiate the utility of our established result to find the unique solution of an integral equation arising in automobile suspension system.

AMS Subject Classifications: 54H25, 47H10.

Keywords: Fixed point, Control function, Controlled fuzzy metric spaces, Bipolar controlled fuzzy metric spaces.

Contents

1. Introduction and Background

In 1922, S. Banach [8] provided an important result to fixed point theory. This topic has been studied, presented and generalized by many researchers in many different spaces. Firstly, the work of Bakhtin [7], Bourbaki [10] and Czerwik [11] expanded the theory of fixed points for b-metric spaces. Also, many authors proved some important fixed point theorems in b-metric spaces ([3], [4], [5]). Later, Abdeljawad et al. [1] proved some fixed point results in partial b-metric spaces. Nabil Mlaiki et al. [18] introduced controlled metric spaces and proved some fixed point theorems. Abdeljawad et al. [2] modified controlled metric spaces called double controlled metric spaces.

On the other hand, after producing the fuzzy subject of Zadeh [22], Kramosil and Michalek [16] introduced the concept of fuzzy metric spaces, which can be regarded as a generalization of the statistical metric spaces. Subsequently, M. Grabiec [13] defined G-complete fuzzy metric spaces and extended the complete fuzzy metric spaces. Following Grabiec's work, many authors introduced and generalized the different types of fuzzy contractive mappings and investigated some fixed point theorems in fuzzy metric spaces. George and Veeramani [12] modified the notion of M-complete fuzzy metric spaces with the help of continuous t-norms.

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Nădăban [19] introduced the concept of fuzzy b-metric spaces. Kim et al. [15] established some fixed point results in fuzzy b-metric spaces. Recently, Mehmood et al. [17] has defined a new concept called extended fuzzy b-metric spaces, which is the generalization of fuzzy b-metric spaces. Most recently Müzeyyen Sangurlu Sezen [20] introduced controlled fuzzy metric spaces, which is a generalization of extended fuzzy b-metric spaces.

In [9], Ayush Bartwal et. al. introduced new generalization of the fuzzy metric space called bipolar fuzzy metric space and proved some fixed point results in this space. The objective of this work is to prove a Banach type fixed point theorem in bipolar controlled fuzzy metric spaces. Our result generalizes many recent fixed point theorems in the literature ([15],[17],[19]). We furnish an example to validate our result. An application is also provided in support of our result.

2. Preliminaries

Now, we begin with some basic concepts, notations and definitions. Let $\mathbb{R}$ represent the set of real numbers, $\mathbb{R}_+$ represent the set of all non-negative real numbers and $\mathbb{N}$ represent the set of natural numbers.

We start with the following definitions of a fuzzy metric space. Schweizer and Sklar introduced the continuous t- norm as follows:

Definition 2.1. *[21]. A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous t- norm if for all $r_1, r_2, r_3 \in [0,1]$, the following conditions are hold:*
(T-1) $r_1 * r_2 = r_2 * r_1$ *and* $r_1 * (r_2 * r_3) = (r_1 * r_2) * r_3$,
(T-2) $r_1 * r_2 \leq r_3 * r_4$ *whenever* $r_1 \leq r_3$ *and* $r_2 \leq r_4$,
(T-3) $r_1 * 1 = r_1$,
(T-4) $*$ *is a continuous.*
The most commonly used t-norms are: $r_1 *_p r_2 = r_1 \Delta r_2$, $r_1 *_m r_2 = min\{r_1, r_2\}$ *and* $r_1 *_L r_2 = max\{r_1 + r_2 - 1, 0\}$ *which known as product, minimum and Lukasiewicz t-norms respectively.*

Kramosil and Michalek [16] introduced the notion of fuzzy metric space as follows:

Definition 2.2. *[16]. An ordered triple $(X, M, *)$ is called fuzzy metric space such that X is a nonempty set, $*$ defined a continuous t-norm and M is a fuzzy set on $X \times X \times (0, \infty)$, satisfying the following conditions, for all $x, y, z \in X, s, t > 0$,*
(KM-1) $M(x, y, 0) = 0$,
(KM-2) $M(x, y, t) = 1$ *iff* $x = y$,
(KM-3) $M(x, y, t) = M(y, x, t)$,
(KM-4) $(M(x, y, t) * M(y, z, s)) \leq M(x, z, t + s)$,
(KM-5) $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ *is left continuous.*

George and Veeramani[12] modified the definition of M-complete fuzzy metric spaces due to Kramosil and Michalek and the concept as follows:

Definition 2.3. *[12]. An ordered triple $(X, M, *)$ is called fuzzy metric space such that X is a nonempty set, $*$ defined a continuous t-norm and M is a fuzzy set on $X \times X \times (0, \infty)$, satisfying the following conditions:*
(FM-1) $M(x, y, t) > 0$,
(FM-2) $M(x, y, t) = 1$ *if and only if* $x = y$,
(FM-3) $M(x, y, t) = M(y, x, t)$,
(FM-4) $(M(x, y, t) * M(y, z, s)) \leq M(x, z, t + s)$,
(FM-5) $M(x, y, \cdot) : (0, \infty) \to [0, 1]$ *is left continuous,* $x, y, z \in X$ *and* $t, s > 0$.

In 2017, Nădăban [19] introduced the idea of a fuzzy b-metric space to generalize the notion of a fuzzy metric spaces introduced by Kramosil and Michalek [16].

Definition 2.4. *[19]. Let X is a non-empty set and $k \geq 1$ be a given real number and $*$ be a continuous t-norm. A fuzzy set M in $X^2 \times (0, \infty)$ is called fuzzy b-metric on X if for all $x, y, z \in X$, the following conditions hold.*
(bM-1) $M(x, y, 0) = 0$,
(bM-2) $[M(x, y, t) = 1, (\forall)t > 0]$ *if and only if* $x = y$,
(bM-3) $M(x, y, t) = M(y, x, t), (\forall)t > 0$,
(bM-4) $M(x, z, k(t + s)) \geq M(x, y, t) * M(y, z, s)$, *(∀)t,s ¿ 0,*
$(bM-5)$*M(x, y, ·)* $: [0, \infty) \to [0, 1]$ *is left continuous and* $lim_{t\to\infty} M(x, y, t) = 1$.
The quadruple $(X, M, *, k)$ is said to be a fuzzy b-metric space.

Mehmood et al. [17] introduced the notion of an extended fuzzy b-metric space following the approach of Grabiec [13].

Definition 2.5. *[17]. Let X be a non-empty set, $\alpha : X \times X \to [1, \infty)$, $*$ is a continuous t-norm and M_α is a fuzzy set on $X^2 \times (0, \infty)$, is called extended fuzzy b-metric on X if for all $x, y, z \in X$ and $s, t > 0$, satisfying the following conditions.*
$(FbM_\alpha 1)$ $M_\alpha(x, y, 0) = 0$,
$(FbM_\alpha 2)$ $M_\alpha(x, y, t) = 1$ *iff* $x = y$,
$(FbM_\alpha 3)$ $M_\alpha(x, y, t) = M_\alpha(y, x, t)$,
$(FbM_\alpha 4)$ $M_\alpha(x, z, \alpha(x, z)(t + s)) \geq M_\alpha(x, y, t) * M_\alpha(y, z, s)$,
$(FbM_\alpha 5)$ $M_\alpha(x, y, \cdot) : (0, \infty) \to [0, 1]$ *is continuous.*
Then $(X, M_\alpha, *, \alpha(x, y))$ is an extended fuzzy b-metric space.

In [20], Sezen introduced the controlled fuzzy metric spaces, which is a generalization of extended fuzzy b-metric spaces.

Definition 2.6. *[20]. Let X be a non-empty set, $\lambda : X \times X \to [1, \infty)$, $*$ is a continuous t-norm and M_λ is a fuzzy set on $X^2 \times (0, \infty)$, satisfying the following conditions, for all $a, c, d \in X, s, t > 0$:*
(FM-1) $M_\lambda(a, c, 0) = 0$,
(FM-2) $M_\lambda(a, c, t) = 1$ *iff* $a = c$,
(FM-3) $M_\lambda(a, c, t) = M_\lambda(c, a, t)$,
(FM-4) $M_\lambda(a, d, t + s) \geq M_\lambda(a, c, \frac{t}{\lambda(a,c)}) * M_\lambda(c, d, \frac{s}{\lambda(c,d)})$,
(FM-5) $M_\lambda(a, c, \cdot) : [0, \infty) \to [0, 1]$ *is continuous,*
Then, the triple $(X, M_\lambda, *)$ is called a controlled fuzzy metric space on X.

In [9], Ayush Bartwal et. al. introduced new generalization of the fuzzy metric space called fuzzy bipolar metric space and prove some fixed point results in this space.

Definition 2.7. *[9]. Let X and Y be two non-empty sets. A quadruple $(X, Y, M_b, *)$ is said to be fuzzy bipolar metric space (FB-space), where $*$ is continuous t-norm and M_b is a fuzzy set on $X \times Y \times (0, \infty)$, satisfying the following conditions for all $t, s, r > 0$:*
(FB-1) $M_b(x, y, t) > 0$ *for all* $(x, y) \in X \times Y$,
(FB-2) $M_b(x, y, t) = 1$ *if and only if* $x = y$ *for* $x \in X$ *and* $y \in Y$,
(FB-3) $M_b(x, y, t) = M_b(y, x, t)$ *for all* $x, y \in X \cap Y$,
(FB-4) $M_b(x_1, y_2, t + s + r) > M_b(x_1, y_1, t) * M_b(x_2, y_1, s) * M_b(x_2, y_2, r)$
for all $x_1, x_2 \in X$ *and* $y_1, y_2 \in Y$,
(FB-5) $M_b(x, y, .) : [0, \infty) \to [0, 1]$ *is left continuous,*
(FB-6) $M_b(x, y, .)$ *is non-decreasing for all* $x \in X$ *and* $y \in Y$.

3. Main Results

In this section, we introduce some new definitions and establish a fixed point theorem in bipolar controlled fuzzy metric spaces.

Definition 3.1. *Let X and Y be two non-empty sets, A quadruple $(X, Y, M_{b\lambda}, *)$ is said to be bipolar controlled fuzzy metric space, where $*$ is continuous t-norm defined as $a * b = min\{a, b\}$ and $\lambda : X \times X \to [1, \infty)$, and $M_{b\lambda}$ is a fuzzy set on $X^2 \times (0, \infty)$. If for all $x \in X,\ y \in Y$ and $s, t, r > 0$. $M_{b\lambda}$ satisfying the following conditions:*
($FM_{b\lambda}$-1) $M_{b\lambda}(x, y, 0) = 0$.
($FM_{b\lambda}$-2) $M_{b\lambda}(x, y, t) = 1$ iff $x = y$. for $x \in X$ and $y \in Y$.
($FM_{b\lambda}$-3) $M_{b\lambda}(x, y, t) = M_{b\lambda,\mu}(y, x, t)$. for all $x, y \in X \cap Y$.
($FM_{b\lambda}$-4) $M_{b\lambda}(x_1, y_2, t + s + r) \geq M_{b\lambda}(x_1, y_1, \frac{t}{\lambda(x_1,y_1)}) * M_{b\lambda}(x_2, y_1, \frac{s}{\lambda(x_2,y_1)}) * M_{b\lambda}(x_2, y_2, \frac{r}{\lambda(x_2,y_2)})$. for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$.
($FM_{b\lambda}$-5) $M_{b\lambda}(x, y, \cdot) : [0, \infty) \to [0, 1]$ is continuous.
($FM_{b\lambda}$-6) $M_{b\lambda}(x, y, \cdot)$ is non decreasing for all $x \in X,\ y \in Y$.

Definition 3.2. *Let $(X, Y, M_{b\lambda}, *)$ be a fuzzy bipolar controlled metric spaces. Then*

1. *A sequence $\{x_n, y_n\}$ in $X \times Y$ is said to be BPC-convergent sequence to x in $X \times Y$, if both x_n and y_n converge to same point.*
2. *A sequence $\{x_n, y_n\}$ in $X \times Y$ said to be BPC-Cauchy sequence if and only if $\lim_{n\to\infty} M_\lambda(x_n, y_m, t) = 1$ for any $n, m > 0$ and for all $t > 0$.*
3. *The fuzzy bipolar controlled metric space $(X, Y, M_{b\lambda}, *)$ is called BPC-complete if every BPC- Cauchy sequence is convergent in it.*

Now, we display an example to verify our definition 3.1.

Example 3.3. *Let $X = (0, 2],\ \ Y = [2, \infty)$. Define $M_{b\lambda}$ is a fuzzy set on $X^2 \times (0, \infty)$, as*

$$M_{b\lambda}(x, y, t) = \begin{cases} 1 & \text{if } x = y \\ txy & \text{if } x \neq y \text{ and } t \geq 0\ , \end{cases}$$

With the continuous t-norm $*$ such that $t_1 * t_2 = min\{t_1, t_2\}$. Define $\lambda : X \times X \to [1, \infty)$, as

$$\lambda(x, y) = \begin{cases} 1 & \text{if } x \in X \text{ and } y \in Y \\ 1 + \frac{1}{a} & \text{otherwise,} \end{cases}$$

Let us show that $(X, Y, M_{b\lambda}, *)$ is a bipolar controlled fuzzy metric spaces. It is easy to prove conditions ($FM_{b\lambda}$-1), ($FM_{b\lambda}$-2) and ($FM_{b\lambda}$-3). We have to examine the following case to show that condition ($FM_{b\lambda}$-4) holds.
For $x \neq y$ **and** $t \geq 0$*: By assuming $x_1 = 2,\ \ y_1 = 3,\ \ x_2 = 1$ and $y_2 = 4$, we obtain a non-trivial sequence as $(x_n, y_n) = \{(2, 3), (1, 4), (\frac{1}{2}, 5), \cdots\}$ and taking $t = 1,\ s = 2, r = 3$.*

$$\begin{aligned} M_{b\lambda}(x_1, y_2, t + s + r) &= M_\lambda(2, 4, 6) = 48 \\ &\geq M_{b\lambda}(2, 3, \frac{1}{\lambda(2, 3)}) * M_{b\lambda}(1, 3, \frac{2}{\lambda(1, 3)}) * M_{b\lambda}(1, 4, \frac{3}{\lambda(1, 4)}) = 6 \\ &= M_{b\lambda}(x_1, y_1, \frac{t}{\lambda(x_1, y_1)}) * M_{b\lambda}(x_2, y_1, \frac{s}{\lambda(x_2, y_1)}) * M_{b\lambda}(x_2, y_2, \frac{r}{\lambda(x_2, y_2)}) \end{aligned}$$

Which satisfies condition of bipolar controlled fuzzy metric spaces. But, if we take $x_1 = \frac{1}{2},\ x_2 = 2,\ \ x_3 = \frac{1}{3}$ and $t = 1,\ s = 2$ for all $x_1, x_2, x_3 \in X$ and $t, s > 0$ in the definition [20], we have

$$\begin{aligned} M_\lambda(x_1, x_3, t + s) &= M_\lambda(\frac{1}{2}, \frac{1}{3}, 2) = 0.33 \\ &\leq M_\lambda(\frac{1}{2}, 2, \frac{1}{\lambda(\frac{1}{2}, 2)}) * M_\lambda(2, \frac{1}{3}, \frac{2}{\lambda(2, \frac{1}{3})}) = 0.8, \\ &= M_\lambda(x_1, x_2, \frac{t}{\lambda(x_1, x_2)}) * M_\lambda(x_2, x_3, \frac{s}{\lambda(x_2, x_3)}), \end{aligned}$$

which not satisfies the condition [20] of controlled fuzzy metric spaces.

Example 3.4. *Let* $X = [0,1),\ \ Y = [1,\infty)$. *Define* $M_{b\lambda}$ *is a fuzzy set on* $X^2 \times (0,\infty)$, *as*

$$M_{b\lambda}(x,y,t) = \frac{t}{t + d(x,y)}$$

With the continuous t-norm $*$ *such that* $t_1 * t_2 = min\{t_1, t_2\}$. *Define* $\lambda : X \times X \to [1,\infty)$, *as*

$$\lambda(x,y) = \begin{cases} 1 & \text{if } x \in X \text{ and } y \in Y \\ max\{x,y\} & \text{otherwise,} \end{cases}$$

Then $(X, Y, M_{b\lambda}, *)$ *is bipolar controlled fuzzy metric space.*

Now, we present our main result as follows:

Theorem 3.5. *Let* $(X, Y, M_{b\lambda}, *)$ *be a Bipolar controlled fuzzy metric spaces with* $\lambda : X \times X \to [1,\infty)$ *and suppose that*

$$lim_{n\to\infty} M_{b\lambda}(x,y,t) = 1, \tag{3.1}$$

for all $x \in X,\ \ y \in Y$ *and* $t > 0$. *If* $T : X \cup Y \to X \cup Y$ *satisfies that:*

1. $T(X) \subseteq X$ *and* $T(Y) \subseteq Y$,

2. $$M_{b\lambda}(Tx, Ty, kt) \geq M_{b\lambda}(x,y,t), \tag{3.2}$$

   where $k \in (0,1)$. *Also, we assume that for every* $x_n \in X$,

$$lim_{n\to\infty} \lambda(x_n, y) \tag{3.3}$$

exist and are finite. Then T *has a unique fixed point.*

Proof. Let $x_0 \in X$ and $y_0 \in Y$ and define (x_n, y_n) as a sequence by $x_n = Tx_{n-1}$ and $y_n = Ty_{n-1}$ for all $n \in \mathbb{N}$ on bipolar controlled fuzzy metric space $(X, Y, M_{b\lambda}, *)$. If $x_n = x_{n-1}$ then x_n is a fixed point of T. Suppose that $x_n \neq x_{n-1}$ for all $t > 0$ and $n \in \mathbb{N}$. Successively applying inequality (3.2), we get

$$\begin{aligned} M_{b\lambda}(x_n, y_{n+1}, t) &= M_{b\lambda}(Tx_{n-1}, Ty_n, t) \\ &\geq M_{b\lambda}(x_{n-2}, y_{n-1}, \frac{t}{k}) \\ &\vdots \\ &\geq M_{b\lambda}(x_0, x_1, \frac{t}{k^{n-1}}). \end{aligned} \tag{3.4}$$

Now, using the condition ($FM_{b\lambda}$-4), we have

$$
\begin{aligned}
M_{b\lambda}(x_n, y_{n+m}, t) &\geq M_{b\lambda}(x_n, y_{n+1}, \frac{t}{3\lambda(x_n, y_{n+1})}) * M_{b\lambda}(x_{n+1}, y_{n+2}, \frac{t}{3\lambda(x_{n+1}, y_{n+2})}) \\
&\quad * M_{b\lambda}(x_{n+2}, y_{n+m}, \frac{t}{3\lambda(x_{n+2}, y_{n+m})}) \\
&\geq M_{b\lambda}(x_n, y_{n+1}, \frac{t}{3\lambda(x_n, y_{n+1})}) \\
&\quad * M_{b\lambda}(x_{n+1}, y_{n+2}, \frac{t}{3\lambda(x_{n+1}, y_{n+2})} \\
&\quad * M_{b\lambda}(x_{n+2}, y_{n+3}, \frac{t}{(3)^2\lambda(x_{n+2}, y_{n+m})\lambda(x_{n+2}, y_{n+3})}) \\
&\quad * M_{b\lambda}(x_{n+3}, y_{n+4}, \frac{t}{(3)^2\lambda(x_{n+2}, y_{n+m})\lambda(x_{n+3}, y_{n+4})}) \\
&\quad * M_{b\lambda}(x_{n+4}, y_{n+m}, \frac{t}{(3)^2\lambda(x_{n+2}, y_{n+m})\lambda(x_{n+4}, y_{n+m})}) \\
&\geq M_{b\lambda}(x_n, y_{n+1}, \frac{t}{3\lambda(x_n, y_{n+1})}) \\
&\quad * M_{b\lambda}(x_{n+1}, y_{n+2}, \frac{t}{3\lambda(x_{n+1}, y_{n+2})} \\
&\quad * M_{b\lambda}(x_{n+2}, y_{n+3}, \frac{t}{(3)^2\lambda(x_{n+2}, y_{n+m})\lambda(x_{n+2}, y_{n+3})}) \\
&\quad * M_{b\lambda}(x_{n+3}, y_{n+4}, \frac{t}{(3)^2\lambda(x_{n+2}, y_{n+m})\lambda(x_{n+3}, y_{n+4})}) \\
&\quad * M_{b\lambda}(x_{n+4}, y_{n+5}, \frac{t}{(3)^3\lambda(x_{n+2}, y_{n+m})\lambda(x_{n+4}, y_{n+5})}) \\
&\quad * M_{b\lambda}(x_{n+5}, y_{n+6}, \frac{t}{(3)^3\lambda(x_{n+2}, y_{n+m})\lambda(x_{n+4}, y_{n+6})}) \\
&\quad * M_{b\lambda}(x_{n+6}, y_{n+7}, \frac{t}{(3)^3\lambda(x_{n+2}, y_{n+m})\lambda(x_{n+6}, y_{n+7})}) \\
&\vdots \\
&\geq M_{b\lambda}(x_0, x_1, \frac{t}{3k^{n-1}\lambda(x_n, x_{n+1})}) \\
&\quad * [*_{i=n+1}^{n+m-2} M_{b\lambda}(x_0, y_1, \frac{t}{(3)^{m-1}k^{i-1}(\prod_{j=n+1}^{i}\lambda(x_j, y_{n+m}))\lambda(x_i, y_{i+1})})] \\
&\quad * [M_{b\lambda}(x_0, y_1, \frac{t}{(3)^{m-1}k^{n+m-1}(\prod_{i=n+1}^{n+m-1}\lambda(x_i, y_{n+m}))})].
\end{aligned} \tag{3.5}
$$

Therefore, by taking limit as $n \to \infty$ in (3.5), from (3.4) together with (3.1) we have

$$\lim_{n\to\infty} M_{b\lambda}(x_n, y_{n+m}, t) \geq 1 * 1 * \cdots * 1 = 1 \;\; for\ all \;\; t > 0, \; n < m\ and\ n, m \in \mathbb{N}.$$

Thus, (x_n, y_n) is a BPC-Cauchy sequence in X. From the completeness of $(X, Y, M_{b\lambda}, *)$, there exists $u \in X \cap Y$ which is a limit of the both sequences $\{x_n\}$ and $\{y_n\}$ such that

$$\lim_{n\to\infty} M_{b\lambda}(Tu, u, t) = 1, \tag{3.6}$$

for all $t > 0$. Now we show that u is a fixed point of T. For any $t > 0$ and from the condition ($FM_{b\lambda}$-4), we have

$$M_{b\lambda}(Tu, u, t) \geq M_{b\lambda}(Tu, Ty_n, \frac{t}{3\lambda(Tu, Ty_n)}) * M_{b\lambda}(Tx_n, Ty_{n+1}, \frac{t}{3\lambda(Tx_n, Ty_{n+1})}) * M_{b\lambda}(Tx_{n+1}, u, \frac{t}{3\lambda(Tx_{n+1}, u)}) \tag{3.7}$$

Letting $n \to \infty$ in (3.7) and using (3.6), we get $M_{b\lambda}(Tu, u, t) = 1$ for all $t > 0$, that is, $Tu = u$.
For uniqueness, let $w \in X \cap Y$ is another fixed point of T and there exists $t > 0$ such that $M_{b\lambda}(u, w, t) \neq 1$, then it follows from (3.2) that

$$\begin{aligned} M_{b\lambda}(u, w, t) &= M_{b\lambda}(Tu, Tw, t) \\ &\geq M_{b\lambda}(u, w, \frac{t}{k}) \\ &\geq M_{b\lambda}(u, w, \frac{t}{k^2}) \\ &\vdots \\ &\geq M_{b\lambda}(u, w, \frac{t}{k^n}), \end{aligned} \tag{3.8}$$

for all $n \in \mathbb{N}$. By taking limit as $n \to \infty$ in (3.8), $M_{b\lambda}(u, w, t) = 1$ for all $t > 0$, that is, $u = w$. This completes the proof. ■

Now we furnish an example to validate Theorem 3.5.

Example 3.6. *Let $X = [0, 2)$ and $Y = [2, \infty)$. Define $M_{b\lambda} : X \times X \times [0, \infty) \to [0, 1]$ as*

$$M_{b\lambda}(x, y, t) = \begin{cases} 1 & \text{if } x = y \\ \frac{t}{(t+\frac{2}{y})} & \text{if } x \in X \text{ and } y \in Y \\ \frac{t}{(t+\frac{2}{x})} & \text{if } x \in Y \text{ and } y \in X \\ \frac{1}{(t+1)} & \text{otherwise.} \end{cases}$$

With the continuous t-norm $*$ such that $t_1 * t_2 = min\{t_1, t_2\}$. Define $\lambda : X \times Y \to [1, \infty)$, as

$$\lambda(x, y) = \begin{cases} 1 & \text{if } x, y \in X \\ max\{x, y\} & \text{otherwise.} \end{cases}$$

Clearly $(X, Y, M_{b\lambda}, *)$ is a bipolar controlled fuzzy metric space. Consider $T : X \cup Y \to X \cup Y$ by

$$T(u) = \begin{cases} u & \text{if } u \in X \\ u^2 + 1 & \text{if } u \in Y, \end{cases}$$

for all $x \in X,\ \ y \in Y$ and $k = 0.5$. We have to examine the inequality (3.2) for all the four cases given below.
***Case I**. If $x = y$ then we have $Tx = Ty$. In this case:*

$$M_{b\lambda}(Tx, Ty, kt) = 1 = M_{b\lambda}(x, y, t). \tag{3.9}$$

Case II. *Let $x \in X$ and $y \in Y$, then we have $Tx \in X$ and $Ty \in Y$. In this case:*

$$\begin{aligned} M_{b\lambda}(Tx, Ty, kt) &= \frac{kt}{(kt + \frac{2}{Ty})} \\ &= \frac{0.5t}{(0.5t + \frac{2}{y^2+1})} \\ &\geq \frac{t}{(t + \frac{2}{y})} \\ &= M_{b\lambda}(x, y, t). \end{aligned} \tag{3.10}$$

Figure $1(a)$ *shows the illustration of above case on* $2D$ *view, in which the variation of* $M_{b\lambda}(Tx, Ty, kt)$ *as a*

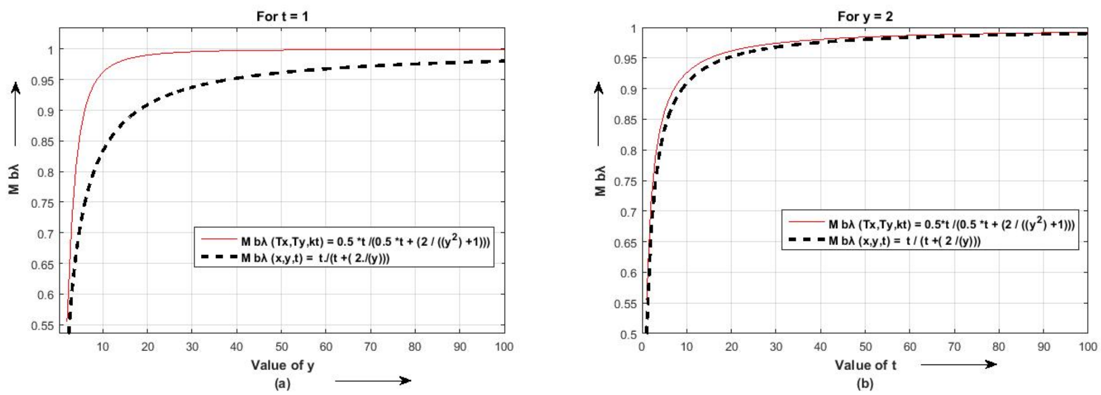

Figure 1: Variation of $M_{b\lambda}(Tx, Ty, kt)$ with $M_{b\lambda}(x, y, t)$ of Example 3.6, case-II on $2D$ view, for:
(a)$M_{b\lambda}(Tx, Ty, kt)$ vs $M_{b\lambda}(x, y, t)$ at $t = 1$ and $y \in (2, 100)$.
(b)$M_{b\lambda}(Tx, Ty, kt)$ vs $M_{b\lambda}(x, y, t)$ at $t \in (1, 100)$ and $y = 2$.

function of y *with fixed values of* t*, is shown as a red colored curve. A dotted curved line represents the variation of* $M_{b\lambda}(x, y, t)$ *as a function of* y *relative to* t*, the variation of this curve is similar to the red colored line with little smaller values of* $M_{b\lambda}(x, y, t)$.
Figure $1(b)$ *shows the variation of* $M_{b\lambda}(Tx, Ty, kt)$ *as a function of* t *with fixed values of* y*, is shown as a red colored curve. A dotted curved line represents the variation of* $M_{b\lambda}(x, y, t)$ *as a function of* t *fixed to* y*, the variation of this curve is similar to the red colored line with little smaller values of* $M_{b\lambda}(x, y, t)$.

Figure $2(a)$ *shows the illustration of case II of Example* 3.6 *on* $3D$ *view, in which the variation of* $M_{b\lambda}(Tx, Ty, kt)$ *as a function of* y *with different values of* t*, is shown as a red-yellow surface and a blue-black surface represents the variation of* $M_{b\lambda}(x, y, t)$ *as a function of* y *relative to* t*, the variation of this curve is similar to the red-yellow surface with little smaller values of* $M_{b\lambda}(x, y, t)$.

Figure $2(b)$ *is similar to the variation of* $M_{b\lambda}(Tx, Ty, kt)$ *as a function of* y *with different values of* t*, is shown as a yellow surface curve and a green surface represents the variation of* $M_{b\lambda}(x, y, t)$ *as a function of* y *relative to* t.

Table 1 *and* 2 *show the variation between* $M_{b\lambda}(Tx, Ty, kt)$ *and* $M_{b\lambda}(x, y, t)$ *as a function of* y *with relative to* t*, this table justifies inequality* (3.10)*, which observed in both the curves for the value of* t *is a higher than* 50 *as a function of* y*. At* $t = 50$, $M_{b\lambda}(Tx, Ty, kt)$ *becomes* 1 *and after higher value of* t*, it remains constant* $(= 1)$*.* $M_{b\lambda}(x, y, t)$ *doesn't become to* 1 *till* $t = 100$*, but it approached nearby to* 1.

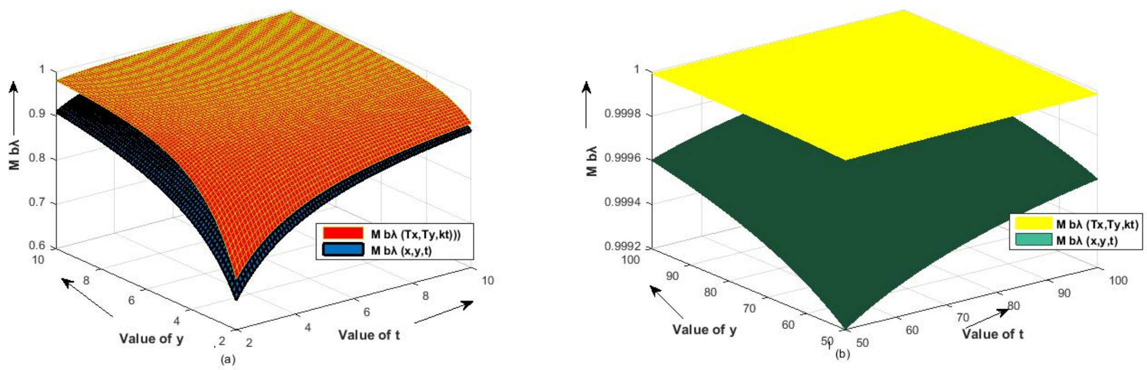

Figure 2: Variation of $M_{b\lambda}(Tx, Ty, kt)$ with $M_{b\lambda}(x, y, t)$ of Example 3.6, case-II on $3D$ view, for:
(a)$M_{b\lambda}(Tx, Ty, kt)$ vs $M_{b\lambda}(x, y, t)$ at $t \in (1, 10)$ and $y \in (2, 10)$.
(b)$M_{b\lambda}(Tx, Ty, kt)$ vs $M_{b\lambda}(x, y, t)$ at $t \in (50, 100)$ and $y \in (50, 100)$.

Value of t	Value of y	$M_{b\lambda}(Tx, Ty, kt)$	$M_{b\lambda}(x, y, t)$
1	2	0.5556	0.5000
	20	0.9901	0.9091
	50	0.9984	0.9615
	100	0.9996	0.9804
50	2	0.9843	0.9804
	20	0.9998	0.9980
	50	1.0000	0.9992
	100	1.0000	0.9996

Table 1: Variation of $M_{b\lambda}(Tx, Ty, kt)$ with $M_{b\lambda}(x, y, t)$ of inequality (3.10), as a function of y with fixed value of $t = 1$ and $t = 50$.

Value of y	Value of t	$M_{b\lambda}(Tx, Ty, kt)$	$M_{b\lambda}(x, y, t)$
2	1	0.7143	0.6667
	20	0.9615	0.9524
	50	0.9843	0.9804
	100	0.9921	0.9901
50	1	0.9984	0.9615
	20	0.9999	0.9980
	50	1.0000	0.9992
	100	1.0000	0.9996

Table 2: Variation of $M_{b\lambda}(Tx, Ty, kt)$ with $M_{b\lambda}(x, y, t)$ of inequality (3.10) as a function of t with fixed value of $y = 2$ and $y = 50$.

Case III. *Let $x \in Y$ and $y \in X$, then we have $Tx \in Y$ and $Ty \in X$. In this case:*

$$\begin{aligned} M_{b\lambda}(Tx, Ty, kt) &= \frac{kt}{(kt + \frac{2}{Tx})} \\ &= \frac{0.5t}{(0.5t + \frac{2}{x^2+1})} \\ &\geq \frac{t}{(t + \frac{2}{x})} \\ &= M_{b\lambda}(x, y, t). \end{aligned} \tag{3.11}$$

Figure $3(a)$ Shows the illustration of case III of example 3.6, on $2D$ view, in which the variation of $M_{b\lambda}(Tx, Ty, kt)$ as a function of x with fixed values of t, is shown as red colored dotted curve and the variation of $M_{b\lambda}(x, y, t)$ as a function of x fixed to t shown as a blue colored curve.
Figure $3(b)$ is the variation of $M_{b\lambda}(Tx, Ty, kt)$ as a function of t with fixed values of x, is shown as red colored dotted curve and the variation of $M_{b\lambda}(x, y, t)$ as a function of t fixed to x shown as a blue colored curve.

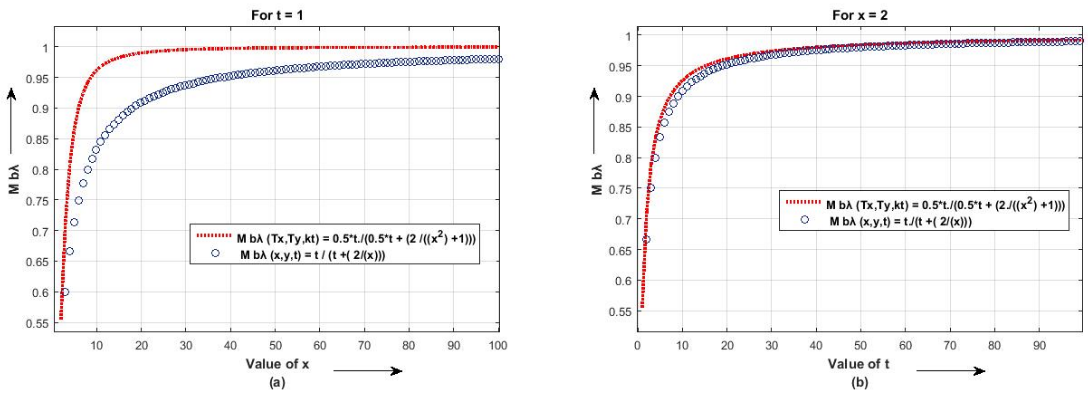

Figure 3: Variation of $M_{b\lambda}(Tx, Ty, kt)$ with $M_{b\lambda}(x, y, t)$ of Example 3.6, Case III on $2D$ view, for:
(a)$M_{b\lambda}(Tx, Ty, kt)$ vs $M_{b\lambda}(x, y, t)$ at $t = 1$ and $x \in (3, 50)$.
(b)$M_{b\lambda}(Tx, Ty, kt)$ vs $M_{b\lambda}(x, y, t)$ at $t \in (1, 50)$ and $x = 2$.

Case IV. *For other states of $x,\ y$ and similarly $Tx,\ Ty$. $M_{b\lambda}(Tx, Ty, kt)$ and $M_{b\lambda}(x, y, t)$ depends on only t. we have,*

$$\begin{aligned} M_{b\lambda}(Tx, Ty, kt) &= \frac{1}{(kt+1)} \\ &= \frac{1}{(0.5t+1)} \\ &\geq \frac{1}{(t+1)} \\ &= M_{b\lambda}(x, y, t). \end{aligned} \tag{3.12}$$

Therefore, all the conditions of Theorem 3.5 hold and T has a unique fixed point $x = 1$.

Remark 3.7. *By taking $\lambda = 1$, in Theorem 3.5 we infer the Theorem 2 in [20].*

Theorem 3.8. *Let $(X, Y, M_{b\lambda}, *)$ be bipolar controlled fuzzy metric spaces and $T : X \cup Y \to X \cup Y$ be a mapping satisfying $lim_{n\to\infty} M_{b\lambda}(x, y, t) = 1$. Suppose there exists a constant $k \in (0, 1)$ such that*

$$\int_0^{M_{b\lambda}(Tx,Ty,kt)} \varphi(t)dt \geq \int_0^{M_{b\lambda}(x,y,t)} \varphi(t)dt, \tag{3.13}$$

for all $x \in X$ and $y \in Y$. Then T has a fixed point.

Proof. By taking $\varphi(t) = 1$ in equation (3.13), we obtain Theorem 3.5. ■

4. Application

It is well known that the realistic application for the spring mass system in engineering difficulties is an automobile suspension system. Consider the motion of a car's spring as it travels down a rough and pitted road, where the forcing term is the rough road and the damping is provided by shock absorbers. Gravity, ground vibrations, earthquakes, tension force, and other external forces may work on the system.

The critically damped motion of this system subjected to the external force F is governed by the following initial value problem, Let m be the mass of the spring and F be the external force acting on it [14].

$$\begin{cases} m\frac{d^2u}{dt^2} + l\frac{du}{dt} - mF(t, u(t)) = 0, \\ u(0) = 0, \\ u'(0) = 0, \end{cases} \tag{4.1}$$

where $l > 0$ is the damping constant and is a continuous function. It is easy to show that the problem (4.1) is equivalent to the integral equation:

$$u(t) = \int_0^T \zeta(t, r)F(t, r, u(r))dr. \tag{4.2}$$

where $\zeta(t, r)$ is Green's function given by

$$\zeta(t, r) = \begin{cases} \frac{1-e^{\mu(t-r)}}{\mu} & \text{if } 0 \leq r \leq t \leq T. \\ 0 & \text{if } 0 \leq t \leq r \leq T. \end{cases} \tag{4.3}$$

where $\mu = l/m$ is a constant. In this section, by using Theorem 3.5, we will show the existence of a solution to the integral equation:

$$u(t) = \int_0^T G(t, r, u(r))dr. \tag{4.4}$$

Let $X = C([0, T])$ be the set of real continuous functions defined on $[0, T]$. For $k \in (0, 1)$ we define

$$M_{b\lambda}(x, y, t) = sup_{t\in[0,T]} \frac{t}{t + (|x(t) - y(t)|)}. \tag{4.5}$$

for all $x \in X$ and $y \in Y$. Define $\lambda : X \times X \to [1, \infty)$, as

$$\lambda(x, y) = \begin{cases} 1 & \text{if } x \in X \text{ and } y \in Y \\ max\{x, y\} & \text{otherwise,} \end{cases}$$

It is easy to prove that $(X, Y, M_{b\lambda}, *)$ is a bipolar controlled fuzzy metric spaces. Consider the mapping $S : X \cup Y \to X \cup Y$ defined by

$$fx(t) = \int_0^T G(t, r, x(r))dr. \tag{4.6}$$

Suppose that

1. there exist a continuous function $\zeta : [0, T] \times [0, T] \to \mathbb{R}^+$ such that

$$sup_{t\in[0,T]} \int_0^T \zeta(t, r)dr \leq 1, \tag{4.7}$$

2. $G : [0, T] \times [0, T] \times \mathbb{R} \to \mathbb{R}^+$ is continuous such that

$$|G(t, r, x(r)) - G(t, r, y(r))| \geq |x(r) - y(r)|, \tag{4.8}$$

for all $k \in (0,1)$. Then, the integral equation (4.4) has a unique solution. **Proof** Let $x \in X$ and $y \in Y$, by using of assumptions $(1)-(2)$, we have

$$\begin{aligned} M_{b\lambda}(Sx,Sy,kt) &= sup_{t\in[0,T]}\frac{kt}{kt+(|Sx(t)-Sy(t)|)} \\ &= sup_{t\in[0,T]}\frac{kt}{kt+(|\int_0^T G(t,r,x(r))dr-\int_0^T G(t,r,y(r))dr|)} \\ &= sup_{t\in[0,T]}\frac{kt}{kt+(\int_0^T |G(t,r,x(r))dr-G(t,r,y(r))|dr)} \\ &\geq sup_{t\in[0,T]}\frac{kt}{kt+(\int_0^T |x(r)-y(r)|dr)} \\ &\geq sup_{t\in[0,T]}\frac{t}{t+(\int_0^T |x(r)-y(r)|dr)} \\ &\geq M_{b\lambda}(x,y,t). \end{aligned} \tag{4.9}$$

Therefore all the condition of Theorem 3.5 are satisfied. As a result, the mapping S has a unique fixed point $x \in X \cap Y$, which is a solution of the integral equation (4.4).

Conclusion

In this article, we extend the controlled fuzzy metric spaces of Sezen [20] by introducing bipolar controlled fuzzy metric spaces. We prove a Banach-type fixed point theorem. Our investigations and results obtained were supported by suitable examples with graphics. We also provide an application of our result to the existence of solution to an integral equation. This work provides a new path for researchers in the concerned field.

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