Malaya Journal of Matematik, Vol. 8, No. 4, 2045-2049, 2020
https://doi.org/10.26637/MJM0804/0122

A quadratic functional equation stability in 2-Banach spaces

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Abstract
In this article,We discuss the analysis in 2-Banach Spaces of the stability problem of the quadratic functional equation

$$h(x+y)-h(x-y)=h(2x+y)-4h(x)-h(y).$$

Keywords
Quadratic functional equation, Intuitionistic fuzzy normed spaces, Hyers-Ulam stability.

AMS Subject Classification
47H10, 39B72, 34D20.

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Article History: Received **24** August **2020**; Accepted **09** November **2020**

Contents

1. Introduction and Preliminaries

The analysis of stability of a functional equation is related to a question of Ulam [18], while delivering his speech at the University of Wisconsin in 1940, concerning the stability of group homomorphisms. In Banach spaces,first positive answer to this equation of Ulam was provided by Hyers [10] in 1941.

D.H. Hyers provided the first result relating to the stability of functional equations. He has comprehensively answered the question of Ulam[10] for the case where $G1$ and $G2$ are Banach spaces . Subsequently, T. Aoki [1] attempted to weaken the condition for the bound of the norm of Cauchy difference and then generalised the theorem of Hyers for additive mappings and Th. M. Rassias [16] generalised the theorem of Hyers for linear mappings.
In this article, we obtain the stability of a quadratic - functional equation in 2-Banach spaces.

Definition 1.1. *Let A be a linear space over $\Re$ with $\dim A>1$ and let $||.,.||: A\times A\rightarrow\Re$ be a function satisfying the following properties:*

(i) $||x,y||=0$ *if and only if x and y are linearly dependents,*

(ii) $||x,y||=||y,x||$,

(iii) $||\lambda x,y||=|\lambda|||x,y||$,

(iv) $||x,y+z||\leq||x,y||+||x,z||$,

for all $x,y,z\in A$ and $\lambda\in\Re$. Then the function $||.,.||$ is called a norm on A and the pair $(A,||.,.||)$ is called a linear 2 - normed space. Sometimes the condition (iv) called the triangle inequality.
In 2011, W.G. Park [15] introduces a basic property of linear 2 - normed spaces as follows.

Lemma 1.2. *Let $(A,||.,.||)$ be a linear 2 - normed space. If $||x,y||=0$ for all $y\in A$, then $x=0$.*

In the year 1960's, A. White and S. Gahler [8, 9, 19, 20] the definition of 2- Banach Spaces was also implemented. The definitions of Cauchy sequences and in order to determine completeness,Converges are mandatory.

Definition 1.3. *A sequence $\{x_n\}$ in a linear 2 - normed space A is called a Cauchy sequence if there are two points $y,z\in A$ such that y and z are linearly independent, $\lim_{l,m\rightarrow\infty}||x_l-x_m,y||=0$ and $\lim_{l,m\rightarrow\infty}||x_l-x_m,z||=0$.*

Definition 1.4. *A sequence $\{x_n\}$ in a linear 2 - normed space A is called a convergent sequence if there is an $x \in A$ such that $\lim_{n\to\infty} ||x_n - x, y|| = 0$ for all $y \in A$. If $\{x_n\}$ converges to x, write $x_n \to x$ as $n \to \infty$ and call x the limit of $\{x_n\}$. In this case, we also write $\lim_{n\to\infty} x_n = x$.*

Lemma 1.5. *For a convergent sequence $\{x_n\}$ in a linear 2 - normed space A, $\lim_{n\to\infty} ||x_n, y|| = ||\lim_{n\to\infty} x_n, y||$ for all $y \in A$.*

Definition 1.6. *A linear 2-normed space in which every Cauchy sequence is a convergent sequence is called a 2 - Banach space.*

2. Stability of a Functional Equation Quadratic

Throughout this article, let A be a normed linear space and B be a 2 - Banach space.

Theorem 2.1. *Let $\mu \in [0,\infty)$, $p,q \in (0,2)$. If $h : A \to B$ is a function such that it is*

$$||h(x+y) - h(x-y) - h(2x+y) + 4h(x) + h(y), z|| \leq \mu(||x||^p + ||y||^q), \forall x,y \in A, z \in B. \quad (2.1)$$

Then a unique Quadratic mapping occurs $Q : A \to B$ such that it is

$$||h(x) - Q(x) - h(0), y|| \leq \frac{\mu||x||^p}{4-2^p}, \quad \forall x \in A, y \in B. \quad (2.2)$$

Proof. Let $f : A \to B$ be a function defined $f(x) = h(x) - h(0)$, for each $x \in A$. Then $f(0) = 0$.
Also
$||f(x+y) - f(x-y) - f(2x+y) + 4f(x) + f(y), z|| \leq \mu(||x||^p + ||y||^q), \quad \forall x \in A, z \in B.$
Putting $y = 0$ in eq. (2.1), we are getting

$$||4f(x) - f(2x), z|| \leq \mu||x||^p, \quad \forall x \in A, z \in B \quad (2.3)$$
$$||f(x) - \frac{1}{4}f(2x), z|| \leq \frac{\mu}{4}||x||^p, \quad \forall x \in A, z \in B \quad (2.4)$$

Replacing x by $2x$ and dividing by 2^2 in eq. (2.4) we are getting

$$||\frac{1}{4}f(2x) - \frac{1}{4^2}f(2^2x), z|| \leq \frac{\mu}{4^2}||2x||^p$$
$$||\frac{1}{4}f(2x) - \frac{1}{4^2}f(2^2x), z|| \leq \frac{\mu 2^p}{4^2}||x||^p, \quad \forall x \in A, z \in B \quad (2.5)$$

Using eq. (2.4) and eq. (2.5),we are getting

$$\begin{aligned}
&||f(x) - \frac{1}{4^2}f(2^2x), z|| \\
&\leq ||f(x) - \frac{1}{4}f(2x), z|| + ||\frac{1}{4}f(2x) - \frac{1}{4^2}f(2^2x), z|| \\
&\leq \frac{\mu}{4}||x||^p + \frac{\mu 2^p}{4^2}||x||^p \\
&= \frac{\mu}{4}||x||^p \left[1 + \frac{2^p}{4}\right] \\
&= \frac{\mu}{4}||x||^p \left[1 + 2^{p-2}\right], \quad \forall x \in A, z \in B.
\end{aligned}$$

Through the use of induction on n, we are getting

$$\begin{aligned}
&||f(x) - \frac{1}{4^n}f(2^nx), z|| \\
&\leq \frac{\mu}{4}||x||^p \sum_{i=0}^{n-1}\left[2^{(p-2)i}\right] \\
&\leq \frac{\mu}{4}||x||^p \frac{1-2^{(p-2)n}}{1-2^{p-2}} \\
&\leq \frac{\mu||x||^p}{4(1-2^{p-2})}\left(1-2^{(p-2)n}\right) \\
&\leq \frac{\mu||x||^p}{4-2^p}\left[1-2^{(p-2)n}\right], \quad \forall x \in A, z \in B. \quad (2.6)
\end{aligned}$$

Replacing x by $a^m x$ and dividing a^{3m} in eq. (2.6)

$$\begin{aligned}
&||\frac{f(a^mx)}{4^m} - \frac{1}{4^{m+n}}f(2^{m+n}x), z|| \\
&= \frac{1}{4^m}||f(2^mx) - \frac{1}{4^n}f(2^m.2^nx), z|| \\
&= \frac{1}{4^m}\frac{\mu}{4}||2^mx||^p \sum_{i=0}^{n-1} 2^{(p-2)i} \\
&= \frac{2^{mp}}{4^m}\frac{\mu}{4}||x||^p \sum_{i=0}^{n-1} 2^{(p-2)i} \\
&= 2^{mp-2m}\frac{\mu}{4}||x||^p \sum_{i=0}^{n-1} 2^{(p-2)i} \\
&= \frac{\mu}{4}||x||^p \sum_{i=0}^{n-1} 2^{(p-2)(m+i)} \\
&= \frac{\mu}{4}||x||^p 2^{(p-2)m} \sum_{i=0}^{n-1} 2^{(p-2)i} \\
&= \frac{\mu}{4}||x||^p 2^{(p-2)m}\left[\frac{1-2^{(p-2)n}}{1-2^{p-2}}\right] \\
&= \mu||x||^p\left[\frac{2^{(p-2)m} - 2^{(p-2)m}.2^{(p-2)n}}{4-2^p}\right] \\
&= \mu||x||^p\left[\frac{2^{(p-2)m} - 2^{(p-2)(m+n)}}{4-2^p}\right] \\
&\to 0 \quad as \quad m,n \to \infty, \quad \forall x \in A, z \in B.
\end{aligned}$$

Therefore $\{\frac{1}{4^n}h(2^n x)\}$ is a 2- cauchy sequence in B for each $x \in A$.
Since B is a 2- Banach space $\{\frac{1}{4^n}h(2^n x)\}$ 2- converges for each $x \in A$.
Define the function $Q : A \to B$ as

$$Q(x) = \lim_{n\to\infty} \frac{1}{4^n} h(2^n x), \quad \forall x \in A.$$

Now from eq. (2.6)

$$\lim_{n\to\infty} ||f(x) - \frac{1}{4^n} f(2^n x), z|| \leq \mu ||x||^p \frac{1}{4-2^p}, \quad \forall x \in A, z \in B.$$

Furthermore

$$||h(x) - Q(x) - h(0), z|| \leq \frac{\mu ||x^p||}{4-2^p}, \quad \forall x \in A, z \in B.$$

Next, we illustrate that Q is satisfying

$$h(x+y) - h(x-y) = h(2x+y) - 4h(x) - h(y) \tag{2.7}$$

For $x \in X$

$$\begin{aligned}
&||Q(x+y) - Q(x-y) - Q(2x+y) + 4Q(x) + Q(y), z|| \\
&= \lim_{n\to\infty} \frac{\mu}{4^n} (||2^n x||^p + ||2^n y||^q) \\
&= \lim_{n\to\infty} \frac{\mu}{4^n} [2^{np} ||x||^p + 2^{nq} ||y||^p] \\
&= \lim_{n\to\infty} \mu \left[2^{n(p-2)} ||x||^p + 2^{n(q-2)} ||y||^q \right] \\
&= 0, \quad \forall z \in B.
\end{aligned}$$

Therefore

$$||Q(x+y) - Q(x-y) - Q(2x+y) + 4Q(x) + Q(y), z|| = 0$$

So

$$Q(x+y) - Q(x-y) - Q(2x+y) + 4Q(x) + Q(y) = 0$$

To prove Uniqueness of Q:
Let Q^* be another quadratic equation satisfying function eq. (2.7) and eq. (2.2)
Provided that Q and Q^* are quadratic,

$$Q(2^n x) = 4^n Q(x), Q^*(2^n x) = 4^n Q^*(x), \forall x \in A$$

Now, for $x \in A$

$$\begin{aligned}
&||Q(x) - Q^*(x), z|| \\
&= \frac{1}{4^n} ||Q(2^n x) - Q^*(2^n x), z|| \\
&\leq \frac{1}{4^n} [||Q(2^n x) - f(2^n x), z|| + ||f(2^n x) - Q^*(2^n x), z||] \\
&\leq \frac{2\mu}{4^n} \frac{||2^n x||^p}{4-2^p} \\
&= \frac{2.2^{np}\mu}{4^n} \frac{||x||^p}{4-2^p} \\
&= 2\mu\, 2^{np-2n} \frac{||x||^p}{4-2^p} \\
&= 2\mu\, 2^{n(p-2)} \frac{||x||^p}{4-2^p} \\
&\to 0 \quad as\, n \to \infty, \forall z \in B.
\end{aligned}$$

Therefore

$$||Q(x) - Q^*(x), z|| = 0, \quad \forall z \in B.$$
$$Q(x) = Q^*(x), \quad \forall z \in B.$$

■

Theorem 2.2. *Let $\mu \in [0,\infty)$, $p,q \in (0,2)$. If $f : A \to B$ is a function such that it is $||h(x+y) - h(x-y) - h(2x+y) + 4h(x) - h(y), z|| \leq \mu (||x||^p + ||y||^q)$ for all $x, y \in A$ and $z \in B$. Then a unique Quadratic mapping occurs $Q : A \to B$ such that it is*

$$||h(x) - Q(x) - h(0), y|| \leq \frac{\mu ||x||^p}{2^p - 4}, \quad \forall x \in A, y \in B. \tag{2.8}$$

Proof. Using above Theorem theorem 2.1

$$||4f(x) - f(2x), z|| \leq \mu ||x||^p, \quad \forall x \in A, y \in B. \tag{2.9}$$

Replacing x by $\frac{x}{2}$ in eq. (2.9)

$$||4f(\frac{x}{2}) - f(x), z|| \leq \frac{\mu}{2^p} ||x||^p \tag{2.10}$$

Again Replacing x by $\frac{x}{2}$ in eq. (2.10)

$$||4f(\frac{x}{4}) - f(\frac{x}{2}), z|| \leq \frac{\mu}{2^{2p}} ||x||^p, \quad \forall x \in A, z \in B. \tag{2.11}$$

Using eq. (2.9) and eq. (2.10), we are getting

$$\begin{aligned}
&||16f(\frac{x}{4}) - f(x)|| \\
&\leq ||16f(\frac{x}{4}) - 4f(\frac{x}{2}), z|| + ||4f(\frac{x}{2}) - f(x), z|| \\
&\leq \frac{4\mu}{2^{2p}} ||x||^p + \frac{\mu}{2^p} ||x||^p \\
&= \mu ||x||^p \left[\frac{4}{2^{2p}} + \frac{1}{2^p} \right] \\
&= \mu ||x||^p \left[\frac{4+2^p}{2^{2p}} \right] \\
&= \mu ||x||^p \left[4.2^{-2p} + 2^{-p} \right], \quad \forall x \in A, z \in B.
\end{aligned}$$

Through the use of induction on n, we are getting

$$
\begin{aligned}
&||4^n f(\frac{x}{2^n})-f(x),z|| \\
&\le \mu||x||^p \sum_{i=0}^{n-1} 4^i 2^{-p(i+1)} \\
&\le \mu||x||^p \sum_{i=0}^{n-1} 2^{2i-pi-i} \\
&\le \mu||x||^p \sum_{i=0}^{n-1} 2^{(-i(p-2)-p)} \\
&\le \frac{\mu||x||^p}{2^p} \sum_{i=0}^{n-1} 2^{-i(p-2)} \\
&= \frac{\mu||x||^p}{2^p} \frac{1-2^{-n(p-2)}}{1-2^{-(p-2)}}, \quad \forall x \in A, z \in B. \qquad (2.12)
\end{aligned}
$$

For $m,n \in N$ and for $x \in A$

$$
\begin{aligned}
&||4^n f\left(\frac{x}{2^n}\right)-f(x),z|| \\
&\le \frac{\mu}{2^p}||x||^p + \frac{4\mu}{2^{2p}}||x||^p + \frac{4^2\mu||x||^p}{2^{3p}} \cdots + \frac{4^{n-1}\mu||x||^p}{2^{np}} \\
&= \frac{\mu||x||^p}{2^p}\left(\frac{1-\left(\frac{4}{2^p}\right)^n}{1-\frac{4}{2^p}}\right) \\
&= \frac{\mu||x||^p}{2^p}\left(\frac{1-2^{(2-p)n}}{2^p-4}\right).2^p \\
&= \mu||x||^p\left(\frac{1-2^{(2-p)n}}{2^p-4}\right)
\end{aligned}
$$

For $m,n \in N$

$$
\begin{aligned}
&\left\|4^{m+n} f\left(\frac{x}{2^{n+m}}\right)-4^m f\left(\frac{x}{2^m}\right),z\right\| \\
&= 4^m\left\|4^n f\left(\frac{x}{2^{n+m}}\right)-f\left(\frac{x}{2^m}\right),z\right\| \\
&\le 4^m\mu\left\|\frac{x}{2^m}\right\|^p\left(\frac{1-2^{(2-p)n}}{2^p-4}\right) \\
&\le 4^m\mu||x||^p\frac{1}{2^{mp}}\left(\frac{1-2^{(2-p)n}}{2^p-4}\right) \\
&= \mu||x||^p 2^{2m-mp}\left(\frac{1-2^{(2-p)n}}{2^p-4}\right) \\
&= \mu||x||^p\left(\frac{2^{(2-p)m}-2^{(2-p)n+(2-p)m}}{2^p-4}\right) \\
&= \mu||x||^p\left[\frac{2^{(2-p)m}-2^{(2-p)(m+n)}}{2^p-4}\right] \\
&\to 0 \quad as\, n,m \to \infty, \forall z \in B.
\end{aligned}
$$

Therefore $\{4^n h(\frac{x}{2^n})\}$ is a 2- cauchy sequence m for each $x \in A$.

Since B is a 2- Banach space, the sequence $\{4^n h(\frac{x}{2^n})\}$ converges for each $x \in A$.
Define $Q : A \to B$ as

$$
Q(x) = \lim_{n\to\infty} 4^n h\left(\frac{x}{2^n}\right), \forall x \in A
$$

Now from eq. (2.12)

$$
\lim_{n\to\infty} ||4^n g\left(\frac{x}{2^n}\right)-g(x),z|| \le \mu||x||^p\frac{2^{-p}}{1-2^{-(p-2)}}, \forall x \in A, z \in B.
$$

Therefore

$$
||h(x)-Q(x)-f(0),z|| \le \frac{\mu||x||^p}{2^p-4}, \quad \forall x \in A, z \in B.
$$

To prove Uniqueness of Q:
Let Q^* be another quadratic equation satisfying function eq. (2.7) and eq. (2.8)
Provided that Q and Q^* are quadratic,

$$
Q(\frac{x}{2^n}) = \frac{1}{4^n}Q(x),\, Q^*(\frac{x}{2^n}) = \frac{1}{4^n}Q^*(x), \forall x \in A
$$

Now, for $x \in A$

$$
\begin{aligned}
&||Q(x)-Q^*(x),z|| \\
&= 4^n||Q(\frac{x}{2^n})-Q^*(\frac{x}{2^n}),z|| \\
&\le 4^n\left[||Q(\frac{x}{2^n})-f(\frac{x}{2^n}),z||+||f(\frac{x}{2^n})-Q^*(\frac{x}{2^n}),z||\right] \\
&\le 2\mu 4^n\frac{||\frac{x}{2^n}||^p}{4-2^p} \\
&= 2.2^{np}\mu 4^n\frac{||x||^p}{4-2^p} \\
&= 2\,\mu\,2^{np-2n}\frac{||x||^p}{4-2^p} \\
&= 2\,\mu\,2^{n(p-2)}\frac{||x||^p}{4-2^p} \\
&\to 0, \quad as\, n \to \infty, \quad \forall z \in B.
\end{aligned}
$$

Therefore

$$
||Q(x)-Q^*(x),z|| = 0, \quad \forall z \in B.
$$
$$
Q(x) = Q^*(x), \quad \forall x \in A.
$$

■

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ISSN(P):2319 – 3786
Malaya Journal of Matematik
ISSN(O):2321 – 5666

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