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On $\mathscr{P}$-energy of join of graphs

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Abstract

Given a graph $G=(V,E)$ with a vertex partition $\mathscr{P}$ of cardinality k, we associate to it a real matrix $A_{\mathscr{P}}(G)$, whose diagonal entries are the cardinalities of elements in $\mathscr{P}$ and off-diagonal entries are from the set $\{2,1,0,-1\}$. The $\mathscr{P}$-energy $E_{\mathscr{P}}(G)$ is the sum of the absolute values of eigenvalues of $A_{\mathscr{P}}(G)$. In this paper, we discuss $\mathscr{P}$-energy of the join of graphs using the concept of M-coronal of graphs and determine $\mathscr{P}$-energy for the complements of the join of graphs.

Keywords

Graph energy, partition energy, $\mathscr{P}$-energy, coronal of a graph.

AMS Subject Classification

05C15, 05C50, 05C69.

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Contents

1. Introduction

For present study, we consider graphs $G=(V,E)$ of order n and size m, which are finite and simple. All the definitions, terminologies and notations related with graph theoretic and spectral theoretic concepts are taken from [3, 15].

Since the time I. Gutman [4] introduced the concept of graph energy as the sum of the absolute values of the eigenvalues of the adjacency matrix $A(G)$ of a graph G, several variations and extensions of it has been introduced into the literature by researchers [1, 5, 6, 13]. One such variation is the concept of k-partition energy introduced by E. Sampathkumar et al. [13] based on the vertex partitions of a given graph. They defined *k-partition energy* $E_{P_k}(G)$ as the sum of the absolute values of the eigenvalues of the L-matrix $P_k(G)$ where the entries a_{ij} of $P_k(G)$ are defined as

(i) For $v_i, v_j \in V_r$, if $v_iv_j \in E(G)$ then $a_{ij}=2$ and if $v_iv_j \notin E(G)$ then $a_{ij}=-1$.

(ii) For $v_i \in V_r$ and $v_j \in V_s$, $a_{ij}=1$ if $v_iv_j \in E(G)$ whereas $a_{ij}=0$ for $v_iv_j \notin E(G)$.

Given a vertex partition $\mathscr{P}=\{V_1,V_2,\ldots,V_k\}$ of a graph G, by assigning to diagonal entries a_{ii} the value $|V_s|$ where $V_s \in \mathscr{P}$ is the set containing v_i, P. B. Joshi and M. Joseph defined $\mathscr{P}$-matrix $A_{\mathscr{P}}(G)$ and $\mathscr{P}$-energy $E_{\mathscr{P}}(G)$ which is the sum of the absolute values of eigenvalues of $A_{\mathscr{P}}(G)$ [8]. They have obtained some bounds of $E_{\mathscr{P}}(G)$, have determined the value of $E_{\mathscr{P}}(G)$ for some classes of graphs, and have examined the special cases when the $\mathscr{P}$-energy takes extreme values. In the present study, we determine $\mathscr{P}$-energy for the join of two graphs using the concept of M-coronal of a graph, a method adopted in [2, 10]. We further discuss the $\mathscr{P}$-energy for the complements of the join of graphs.

For two graphs G_1 and G_2, the join $G=G_1 \triangledown G_2$ is the graph obtained by joining every vertex of graph G_1 with every vertex of graph G_2 [7].

If M is a matrix associated with a graph G of order n, the M-coronal $\Gamma_M(\lambda)$ of G is the sum of elements of the matrix $(\lambda I_n - M)^{-1}$. That is, $\Gamma_M(\lambda)=\mathbf{1}_n{}^T(\lambda I_n - M)^{-1}\mathbf{1}_n$, where $\mathbf{1}_n$ is a column vector of order $n\times 1$ and $\mathbf{1}_n^T$ is its transpose and I_n is an identity matrix of order $n\times n$ [2]. This definition is a generalization of the coronal of a graph introduced by McLeman and McNicholas for the adjacency matrix of a graph [10]. Considering the $\mathscr{P}$-matrix $A_{\mathscr{P}}(G)$ associated with the vertex partition $\mathscr{P}$ of a graph G we define the *$\mathscr{P}$-coronal of G* and denote it by $\Gamma_{A_{\mathscr{P}}}(\lambda)=\mathbf{1}_n^T(\lambda I - A_{\mathscr{P}}(G))^{-1}\mathbf{1}_n$. We have adopted the method of first finding the characteristic polynomial $\phi_{\mathscr{P}}(G)$ of G in terms of the coronals of the component graphs and then determining the value $E_{\mathscr{P}}(G)$.

We would be referring to the following result proved by

Liu and Zhang in 2019 relating $\Gamma_M(\lambda)$ with a real matrix A, the identity matrix I_n and $J_{n\times n}$, a matrix of order $n\times n$ for which each element $a_{ij}=1$.

Lemma 1.1. *[9] Let A be an $n\times n$ real matrix. Then*

$$\det(\lambda I_n - A - \alpha J_n) = [1-\alpha\Gamma_M(\lambda)]\det(\lambda I_n - A). \quad (1.1)$$

where α is a real number and λ is an eigenvalue of A.

We close this section by stating the following lemma and the definitions of k-complement and $k(i)$-complement of graphs required for further discussion.

Lemma 1.2. *[16] If M,N,P,Q are matrices where M is invertible and $S=\begin{pmatrix} M & N \\ P & Q\end{pmatrix}$, then $\det S = \det M\cdot\det[Q-PM^{-1}N]$.*

Definition 1.3. *[11] Let G be a graph and $P_k=\{V_1,V_2,\ldots,V_k\}$ be its vertex partition. Then the k-complement of G, $\overline{(G)}_k$ is obtained by removing edges between the vertices of V_i and V_j, for $i\neq j$ and adding edges between the vertices of V_i and V_j which are not in G.*

Definition 1.4. *[12] Let G be a graph and $P_k=\{V_1,V_2,\ldots,V_k\}$ be its vertex partition. Then the $k(i)$-complement of G, $\overline{(G)}_{k(i)}$ is obtained by removing edges of G joining the vertices within V_r and adding the edges from complement of G between the vertices V_r.*

2. Main results

In this section, we present a generalized expression for the characteristic polynomial of the graph $G_1\nabla G_2$ in terms of the characteristic polynomials and coronals of the component graphs. Further we obtain $\mathscr{P}$- energy of join of graphs with respect to specific vertex partitions. Recall that $\Gamma_{A_{\mathscr{P}}}(\lambda)$ denote the coronal corresponding to the matrix $A_{\mathscr{P}}(G)$.

Theorem 2.1. *If $\mathscr{P}_1$ and $\mathscr{P}_2$ are the vertex partitions of two graphs G_1 and G_2 of order n_1 and n_2 respectively, then*

$$\phi_{\mathscr{P}}(G_1\nabla G_2,\lambda)=\phi_{\mathscr{P}_1}(G_1,\lambda)\phi_{\mathscr{P}_2}(G_2,\lambda)[1-\Gamma_{A_{\mathscr{P}_1}}(\lambda)\Gamma_{A_{\mathscr{P}_2}}(\lambda)]. \quad (2.1)$$

Proof. For a vertex partition $\mathscr{P}=\mathscr{P}_1\cup\mathscr{P}_2$ of $G_1\nabla G_2$, the $\mathscr{P}$-matrix

$$A_{\mathscr{P}}(G_1\nabla G_2)=\begin{pmatrix} A_{\mathscr{P}_1}(G_1)_{n_1\times n_1} & J_{n_1\times n_2} \\ J_{n_2\times n_1} & A_{\mathscr{P}_2}(G_2)_{n_2\times n_2}\end{pmatrix}.$$

Therefore, its characteristic polynomial

$$\phi_{\mathscr{P}}(G_1\nabla G_2,\lambda)=\begin{vmatrix} \lambda I_{n_1}-A_{\mathscr{P}_1}(G_1)_{n_1\times n_1} & -J_{n_1\times n_2} \\ -J_{n_2\times n_1} & \lambda I_{n_2}-A_{\mathscr{P}_2}(G_2)_{n_2\times n_2}\end{vmatrix}.$$

By Lemma 1.2,

$$\phi_{\mathscr{P}}(G_1\nabla G_2,\lambda)=\left|\lambda I_{n_1}-A_{\mathscr{P}_1}(G_1)\right|\left|[\lambda I_{n_2}-A_{\mathscr{P}_2}(G_2)]-B\right|. \quad (2.2)$$

where

$$B=J_{n_2\times n_1}[\lambda I_{n_1}-A_{\mathscr{P}_1}(G_1)]^{-1}J_{n_1\times n_2}$$

On computing the value of B, it can be written as the product of a scalar quantity s and $J_{n_2\times n_2}$, where s is the sum of all the entries from $[\lambda I_{n_1}-A_{\mathscr{P}_1}(G_1)]^{-1}$. Therefore by the definition of the $\mathscr{P}$-coronal of a graph,

$$B=\Gamma_{A_{\mathscr{P}_1}}(\lambda)J_{n_2\times n_2}. \quad (2.3)$$

Thus from Equations (2.2) and (2.3),

$$\phi_{\mathscr{P}}(G_1\nabla G_2,\lambda)=\phi_{\mathscr{P}_1}(G_1,\lambda)\left|\lambda I_{n_2}-A_{\mathscr{P}_2}(G_2)-\Gamma_{A_{\mathscr{P}_1}}(\lambda)J_{n_2\times n_2}\right|.$$

By Lemma 1.1,

$$\phi_{\mathscr{P}}(G_1\nabla G_2,\lambda)=\phi_{\mathscr{P}_1}(G_1,\lambda)\left|\lambda I_{n_2}-A_{\mathscr{P}_2}(G_2)\right|\left[1-\Gamma_{A_{\mathscr{P}_1}}(\lambda)\Gamma_{A_{\mathscr{P}_2}}(\lambda)\right].$$

Hence, the characteristic polynomial of $A_{\mathscr{P}}(G_1\nabla G_2)$

$$\phi_{\mathscr{P}}(G_1\nabla G_2,\lambda)=\phi_{\mathscr{P}_1}(G_1,\lambda)\phi_{\mathscr{P}_2}(G_2,\lambda)[1-\Gamma_{A_{\mathscr{P}_1}}(\lambda)\Gamma_{A_{\mathscr{P}_2}}(\lambda)].$$

□

Remark 2.2. *For $\mathscr{P}=V(G_1\nabla G_2)$,*

$$\phi_{\mathscr{P}}(G_1\nabla G_2,\lambda)=\phi_{\mathscr{P}_1}(A,\lambda)\phi_{\mathscr{P}_2}(B,\lambda)[1-4\Gamma_A(\lambda)\Gamma_B(\lambda)]. \quad (2.4)$$

where $A=A_{\mathscr{P}_1}(G_1)+n_2I_{n_1}$ and $B=A_{\mathscr{P}_2}(G_2)+n_1I_{n_2}$.

It has been observed in [2, 3, 10] that the adjacency matrix of a regular graph has a constant row sum, but it is interesting to observe that in case of $\mathscr{P}$-matrix, not only regular graphs but non-regular graphs can also have $\mathscr{P}$-matrices with constant row sum. Note that, row sum is the sum of all the elements in a row of the given matrix.

We now attempt to determine the exact value of $E_{\mathscr{P}}(G_1\nabla G_2)$ in the special case when $A_{\mathscr{P}}(G_1)$ and $A_{\mathscr{P}}(G_2)$ has a constant row sums R_1 and R_2 respectively. First we prove the following lemma that gives the value of $\Gamma_{A_{\mathscr{P}}}(\lambda)$.

Lemma 2.3. *Let G be a graph of order n such that its $\mathscr{P}$-matrix $A_{\mathscr{P}}(G)$ corresponding to the partition $\mathscr{P}$ has a constant row sum R, then*

$$\Gamma_{A_{\mathscr{P}}}(\lambda)=\frac{n}{\lambda-R}. \quad (2.5)$$

Proof. Let $A_{\mathscr{P}}(G)$ be $\mathscr{P}$-matrix of G and R be its constant row sum. Thus,

$$A_{\mathscr{P}}(G)\mathbf{1}_n=R\mathbf{1}_n.$$

Therefore,

$$\Gamma_{A_{\mathscr{P}}}(\lambda)=\mathbf{1}_n^T[\lambda I_n-A_{\mathscr{P}}(G)]^{-1}\mathbf{1}_n=\frac{\mathbf{1}_n^T\mathbf{1}_n}{\lambda-R}=\frac{n}{\lambda-R}.$$

□

Using Theorem 2.1 and Lemma 2.3, we derive the characteristic polynomial of $\mathscr{P}$-matrix of join of two graphs G_1 and G_2 in the special case when $\mathscr{P}$-matrices of G_1 and G_2 have a constant row sum R_1 and R_2 respectively.

Lemma 2.4. *If G_1 and G_2 are graphs of oder n_1 and n_2 having vertex partitions $\mathscr{P}_1$ and $\mathscr{P}_2$ such that $A_{\mathscr{P}_1}(G_1)$ has a constant row sum R_1 and $A_{\mathscr{P}_2}(G_2)$ has a constant row sum R_2, then with respect to the vertex partition $\mathscr{P} = \mathscr{P}_1 \cup \mathscr{P}_2$ of $G_1 \triangledown G_2$,*

$$\phi_{\mathscr{P}}(G_1 \triangledown G_2, \lambda) = \phi_{\mathscr{P}_1}(G_1, \lambda)\phi_{\mathscr{P}_2}(G_2, \lambda)\left[\frac{(\lambda - R_1)(\lambda - R_2) - n_1 n_2}{(\lambda - R_1)(\lambda - R_2)}\right] \tag{2.6}$$

where $\mathscr{P}_1$ and $\mathscr{P}_2$ are the vertex partitions of G_1 and G_2.

Proof. By Lemma 2.3,

$$\Gamma_{A_{\mathscr{P}_1}}(\lambda) = \frac{n_1}{\lambda - R_1} \tag{2.7}$$

and

$$\Gamma_{A_{\mathscr{P}_2}}(\lambda) = \frac{n_2}{\lambda - R_2}. \tag{2.8}$$

Therefore by substituting Equations (2.7) and (2.8) in (2.1),

$$\begin{aligned}\phi_{\mathscr{P}}(G_1 \triangledown G_2, \lambda) &= \phi_{\mathscr{P}_1}(G_1, \lambda)\phi_{\mathscr{P}_2}(G_2, \lambda)\left[1 - \frac{n}{\lambda - R_1}\frac{n}{\lambda - R_2}\right] \\ &= \phi_{\mathscr{P}_1}(G_1, \lambda)\phi_{\mathscr{P}_2}(G_2, \lambda)\left[\frac{(\lambda - R_1)(\lambda - R_2) - n_1 n_2}{(\lambda - R_1)(\lambda - R_2)}\right].\end{aligned}$$

□

Theorem 2.5. *Let G_1 and G_2 be two graph of order n_1 and n_1 respectively. Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be their respective vertex partitions. Then for a vertex partition $\mathscr{P} = \mathscr{P}_1 \cup \mathscr{P}_2$ of $G_1 \triangledown G_2$*

$$\begin{aligned}E_{\mathscr{P}}(G_1 \triangledown G_2) &= E_{\mathscr{P}_1}(G_1) + E_{\mathscr{P}_2}(G_2) - \sum_{i=1,2} |R_i| \\ &+ \frac{1}{2}\left\{\left|(R_1 + R_2) + \sqrt{(R_1 + R_2)^2 - 4(R_1 R_2 - n_1 n_2)}\right|\right. \\ &\left. + \left|(R_1 + R_2) - \sqrt{(R_1 + R_2)^2 - 4(R_1 R_2 - n_1 n_2)}\right|\right\}\end{aligned}$$

where R_i is a constant row sum of $A_{\mathscr{P}}(G_i)$, for $i = 1, 2$.

Proof. By Lemma 2.4, Equation (2.6) can be written as

$$\begin{aligned}&[\lambda - R_1][\lambda - R_2]\phi_{\mathscr{P}}(G_1 \triangledown G_2, \lambda) \\ &= \phi_{\mathscr{P}_1}(G_1, \lambda)\phi_{\mathscr{P}_2}(G_2, \lambda)[(\lambda - R_1)(\lambda - R_2) - n_1 n_2].\end{aligned} \tag{2.9}$$

Let the left hand side of the Equation (2.9) be $L(\lambda)$ and the right hand side be $R(\lambda)$. Thus, the roots of the equations $L(\lambda) = 0$ and $R(\lambda) = 0$ are same. Therefore, sum of the absolute values of the roots of these equations are also same. Thus,

$$\begin{aligned}\sum_{i=1,2} |R_i| + E_{\mathscr{P}}(G_1 \triangledown G_2) &= E_{\mathscr{P}_1}(G_1) + E_{\mathscr{P}_2}(G_2) \\ &+ \frac{1}{2}\left\{\left|(R_1 + R_2) + \sqrt{(R_1 + R_2)^2 - 4(R_1 R_2 - n_1 n_2)}\right|\right. \\ &\left. + \left|(R_1 + R_2) - \sqrt{(R_1 + R_2)^2 - 4(R_1 R_2 - n_1 n_2)}\right|\right\}.\end{aligned}$$

Hence,

$$\begin{aligned}E_{\mathscr{P}}(G_1 \triangledown G_2) &= E_{\mathscr{P}_1}(G_1) + E_{\mathscr{P}_2}(G_2) - \sum_{i=1,2} |R_i| \\ &+ \frac{1}{2}\left\{\left|(R_1 + R_2) + \sqrt{(R_1 + R_2)^2 - 4(R_1 R_2 - n_1 n_2)}\right|\right. \\ &\left. + \left|(R_1 + R_2) - \sqrt{(R_1 + R_2)^2 - 4(R_1 R_2 - n_1 n_2)}\right|\right\}.\end{aligned}$$

□

Remark 2.6. *As observed in [8], the $\mathscr{P}$-energy of a graph G corresponding to partition $\mathscr{P} = V(G)$ is the maximum and is referred to as the robust $\mathscr{P}$-energy $E_{\mathscr{P}_r}(G)$. Therefore, from Equations (2.4) and (2.5) it follows that robust $\mathscr{P}$-energy of $G_1 \triangledown G_2$*

$$\begin{aligned}E_{\mathscr{P}_r}(G_1 \triangledown G_2) &= \sum_{i=1}^{n_1} |\lambda_i + n_2| + \sum_{i=1}^{n_2} |\lambda_i' + n_1| - \sum_{i=1,2} |R_i'| \\ &+ \frac{1}{2}\left\{\left|(R_1' + R_2') + \sqrt{(R_1' + R_2')^2 - 4(R_1' R_2' - 4n_1 n_2)}\right|\right. \\ &\left. + \left|(R_1' + R_2') - \sqrt{(R_1' + R_2')^2 - 4(R_1' R_2' - 4n_1 n_2)}\right|\right\}\end{aligned}$$

where λ_i, λ_i' are the eigenvalues of $A_{\mathscr{P}_1}(G_1)$, $A_{\mathscr{P}_2}(G_2)$ respectively.

In the next theorem, we derive an expression for the characteristic polynomial for the join of k-copies of a graph F in terms of characteristic polynomial of F. Further, using this expression we obtain $E_{\mathscr{P}}(G)$ where G is the join of k-copies of F when $A_{\mathscr{P}}(F)$ has constant row sum.

Theorem 2.7. *Let F be a graph of order t and $\mathscr{P}_1$ be its vertex partition. If G is the join of k-copies of F, then for the vertex partition $\mathscr{P}$ of G having k elements, each of which is a $\mathscr{P}_1$,*

$$\phi_{\mathscr{P}}(G) = \left[\phi_{\mathscr{P}_1}(F, \lambda)\right]^k \left[1 - (k-1)\Gamma_{A_{\mathscr{P}_1}}(\lambda)\right]\left[1 + \Gamma_{A_{\mathscr{P}_1}}(\lambda)\right]^{(k-1)}. \tag{2.10}$$

Proof. By the choice of $\mathscr{P}$, we have the $\mathscr{P}$-matrix of G,

$$A_{\mathscr{P}}(G) = \begin{pmatrix} A_{\mathscr{P}_1}(F) & J & J & \dots & J \\ J & A_{\mathscr{P}_1}(F) & J & \dots & J \\ J & J & A_{\mathscr{P}_1}(F) & \dots & J \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ J & J & J & \dots & A_{\mathscr{P}_1}(F) \end{pmatrix}_{k \times k}.$$

Therefore, the characteristic polynomial of $A_{\mathscr{P}}(G)$

$$\phi_{\mathscr{P}}(G) = \left|\lambda I - A_{\mathscr{P}}(G)\right|.$$

Apply the following row and column operations on $\phi_{\mathscr{P}}(G)$,

(i) Add $\sum_{i=2}^{k} C_i$ to first column,

(ii) Subtract first row from i^{th} row, for $i = 2, 3, \ldots, k$.

Therefore, it becomes

$$\phi_{\mathscr{P}}(G,\lambda) = \left|\lambda I - A_{\mathscr{P}_1}(F) - (k-1)J\right| \left|\lambda I - A_{\mathscr{P}_1}(F) + J\right|^{(k-1)}.$$

By Lemma 1.1,

$$\begin{aligned}\phi_{\mathscr{P}}(G,\lambda) &= \left[1-(k-1)\Gamma_{A_{\mathscr{P}_1}}(\lambda)\right]\left|\lambda I - A_{\mathscr{P}_1}(F)\right| \\ &\qquad \left\{[1+\Gamma_{A_{\mathscr{P}_1}}(\lambda)]\left|\lambda I - A_{\mathscr{P}_1}(F)\right|\right\}^{(k-1)} \\ &= \left[\phi_{\mathscr{P}_1}(F,\lambda)\right]^k \left[1-(k-1)\Gamma_{A_{\mathscr{P}_1}}(\lambda)\right]\left[1+\Gamma_{A_{\mathscr{P}_1}}(\lambda)\right]^{(k-1)}.\end{aligned}$$

□

Remark 2.8. *For $\mathscr{P} = V(G)$,*

$$\phi_{\mathscr{P}}(G) = \left[\phi_{\mathscr{P}_1}(C,\lambda)\right]^k \left[1-2(k-1)\Gamma_C(\lambda)\right]\left[1+2\Gamma_C(\lambda)\right]^{(k-1)}$$

where $C = A_{\mathscr{P}_1}(F) + (n-t)I_t$.

Now, we consider the join of k-copies of a graph F wherein the $\mathscr{P}$-matrix of F has a constant row sum and obtain the corresponding characteristic polynomial in the the next result.

Lemma 2.9. *Let G be the join of k-copies of a graph F of order t. Let $\mathscr{P}$ and $\mathscr{P}_1$ be the vertex partitions of G and F respectively. If $A_{\mathscr{P}_1}(F)$ has constant row sum R, then for $\mathscr{P}$ such that it is the union of k-copies of $\mathscr{P}_1$,*

$$\phi_{\mathscr{P}}(G,\lambda) = \left[\frac{\phi_{\mathscr{P}_1}(F)}{\lambda - R}\right]^k \left[\lambda - R - (k-1)t\right]\left[\lambda - R + t\right]^{(k-1)}.$$

Proof. Let $A_{\mathscr{P}_1}(F)$ be a $\mathscr{P}$-matrix of F which has a constant row sum R, then by Lemma 2.3,

$$\Gamma_{A_{\mathscr{P}_1}}(\lambda) = \frac{t}{\lambda - R}. \tag{2.11}$$

Therefore, by Theorem 2.7 and Equation (2.11)

$$\begin{aligned}\phi_{\mathscr{P}}(G) &= \left[\phi_{\mathscr{P}_1}(F,\lambda)\right]^k \left[1-(k-1)\frac{t}{\lambda - R}\right]\left[1+\frac{t}{\lambda - R}\right]^{(k-1)} \\ &= \left[\frac{\phi_{\mathscr{P}_1}(F,\lambda)}{\lambda - R'}\right]^k \left[\lambda - R - (k-1)t\right]\left[\lambda - R + t\right]^{(k-1)}.\end{aligned}$$

Thus, the result holds. □

By using Lemma 2.9, we obtain the corresponding $\mathscr{P}$-energy.

Theorem 2.10. *Let F be a graph of order t and G be the join of k-copies of F. Let $\mathscr{P}_1$ and $\mathscr{P}$ be the vertex partitions of G and F respectively. If $A_{\mathscr{P}_1}(F)$ has a constant row sum R, then for a vertex partition $\mathscr{P}$ such that it is the union of k-copies of $\mathscr{P}_1$,*

$$E_{\mathscr{P}}(G) = kE_{\mathscr{P}_1}(F) - kR + |R + (k-1)t| + (k-1)|R - t|. \tag{2.12}$$

Proof. By Lemma 2.9, the characteristic polynomial of $A_{\mathscr{P}}(G)$ can be written as,

$$(\lambda - R)^k \phi_{\mathscr{P}}(G,\lambda) = \left[\phi_{\mathscr{P}_1}(F)\right]^k \left[\lambda - R - (k-1)t\right]\left[\lambda - R + t\right]^{(k-1)}. \tag{2.13}$$

Now, consider the left hand side and the right hand side of the Equation (2.13) as $L_1(\lambda)$ and $R_1(\lambda)$ respectively. The roots of equation $L_1(\lambda) = 0$ and $R_1(\lambda) = 0$ are same. Therefore, the sum of the absolute values of their roots are also same. Thus,

$$kR + E_{\mathscr{P}}(G) = kE_{\mathscr{P}_1}(F) + |R + (k-1)t| + (k-1)|R - t|.$$

Therefore,

$$E_{\mathscr{P}}(G) = kE_{\mathscr{P}_1}(F) - kR + |R + (k-1)t| + (k-1)|R - t|.$$

Hence, the result holds. □

Remark 2.11. *The robust $\mathscr{P}$-energy of join G of k-copies of a graph F*

$$E_{\mathscr{P}_r}(G) = k\sum_{i=1}^{t} |\lambda_i + n - t| - kR' + |R' + 2(k-1)t| + (k-1)|R' - 2t|$$

where λ_i is an eigenvalue of $A_{\mathscr{P}_1}(F)$ for $i = 1, 2, \ldots, t$ and R' is the constant row sum of $A_{\mathscr{P}_1}(F) + (n-t)I_t$.

3. Complements of join of graphs

In this section, we determine the $\mathscr{P}$-energy of complement and generalized complements of join of graphs. It is to be noted that, the disjoint union of graphs is the complement of the join of those graphs [7].

We begin with the $\mathscr{P}$-energy of complement of join of k-copies of a graph.

Theorem 3.1. *Let G be a graph of order n obtained by the join of k-copies of a graph F of order t and $\mathscr{P}$ be its vertex partition such that it is the union of k-copies of vertex partition $\mathscr{P}_1$ of F. If $\overline{G}$ is the complement of G, then for the vertex partition $\mathscr{P}$ of $\overline{G}$,*

$$E_{\mathscr{P}}(\overline{G}) = \sum_{i=1}^{k} E_{\mathscr{P}_i}(\overline{F}).$$

Proof. For a vertex partition $\mathscr{P}$ such that it is the union of k-copies of $\mathscr{P}_1$, the $\mathscr{P}$-matrix of $\overline{G}$ is

$$A_{\mathscr{P}}(\overline{G})=\begin{pmatrix} A_{\mathscr{P}_1}(\overline{F}) & 0 & 0 & \dots & 0\\ 0 & A_{\mathscr{P}_2}(\overline{F}) & 0 & \dots & 0\\ 0 & 0 & A_{\mathscr{P}_3}(\overline{F}) & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \dots & A_{\mathscr{P}_k}(\overline{F})\end{pmatrix}_{k\times k} \tag{3.1}$$

where 0's are the block zero matrices of order $n_1\times n_1$. Therefore by Lemma 1.2 and Equation (3.1), we get

$$\phi_{\mathscr{P}}(\overline{G},\lambda)=\phi_{\mathscr{P}_1}(\overline{F},\lambda)\phi_{\mathscr{P}_2}(\overline{F},\lambda)\dots\phi_{\mathscr{P}_k}(\overline{F},\lambda).$$

Therefore,

$$E_{\mathscr{P}}(\overline{G})=E_{\mathscr{P}_1}(\overline{F})+E_{\mathscr{P}_2}(\overline{F})+\dots+E_{\mathscr{P}_k}(\overline{F}).$$

□

In the next theorem, we determine the characteristic polynomial of complement of the join of k-copies of F. It's proof is similar to that of Theorem 2.7.

Theorem 3.2. *Let F be a graph of order t and $\mathscr{P}_1$ be its vertex partition such that $\mathscr{P}_1=V(\overline{F})$. Let G be a graph of order n obtained by the join of k-copies of F and $\mathscr{P}$ be its vertex partition. Then for $\mathscr{P}=V(\overline{G})$,*

$$\phi_{\mathscr{P}}(\overline{G})=\left[\phi_{\mathscr{P}_1}(D,\lambda)\right]^k\left[1+(k-1)\Gamma_D(\lambda)\right]\left[1-\Gamma_D(\lambda)\right]^{(k-1)}.$$

where $D=A_{\mathscr{P}_1}(\overline{F})+(n-t)I_t$.

Now we consider a graph F whose $\mathscr{P}$-matrix has a constant row sum and find the corresponding $\mathscr{P}$-energy using Theorem 3.2.

Theorem 3.3. *Let G be a graph of order n obtained by the join of k-copies of a graph F of order t such that $\mathscr{P}$ and $\mathscr{P}_1$ are their vertex partitions respectively. Let $\overline{G}$ be the complement of G If $A_{\mathscr{P}_1}(\overline{F})$ has constant row sum R, then for $\mathscr{P}=V(\overline{G})$,*

$$E_{\mathscr{P}}(\overline{G})=k\sum_{i=1}^{t}|\lambda_i+n-t|-kR'+|R'-(k-1)t|+(k-1)|R'+t|. \tag{3.2}$$

Proof. By Lemma 2.3 and Theorem 3.2,

$$\phi_{\mathscr{P}}(\overline{G})=\left[\phi_{\mathscr{P}_1}(D,\lambda)\right]^k\left[1+(k-1)\frac{t}{\lambda-R'}\right]\left[1-\frac{t}{\lambda-R'}\right]^{(k-1)}$$

where $D=A_{\mathscr{P}_1}(\overline{F})+(n-t)I_t$ and $R'=R+n-t$.

$$\phi_{\mathscr{P}}(\overline{G})=\left[\phi_{\mathscr{P}_1}(D,\lambda)\right]^k\left[\frac{\lambda-R'+(k-1)t}{\lambda-R'}\right]\left[\frac{\lambda-R'-t}{\lambda-R'}\right]^{(k-1)}$$
$$=\left[\frac{\phi_{\mathscr{P}_1}(D)}{\lambda-R'}\right]^k\left[\lambda-R'+(k-1)t\right]\left[\lambda-R'-t\right]^{(k-1)}.$$

Thus, it can be written as

$$(\lambda-R')^k\phi_{\mathscr{P}}(G,\lambda)=\left[\phi_{\mathscr{P}_1}(D)\right]^k\left[\lambda-R'+(k-1)t\right]\left[\lambda-R'-t\right]^{(k-1)}.$$

Therefore,

$$kR'+E_{\mathscr{P}}(G,\lambda)=k\sum_{i=1}^{t}|\lambda_i+n-t|+|R'-(k-1)t|+(k-1)|R'+t|$$

$$E_{\mathscr{P}}(\overline{G})=k\sum_{i=1}^{t}|\lambda_i+n-t|-kR'+|R'-(k-1)t|+(k-1)|R'+t|.$$

where λ_i is an eigenvalue of $D=A_{\mathscr{P}_1}(\overline{F})+(n-t)I_t$.

□

Now, we obtain the $\mathscr{P}$-energy of complement of join of k graphs in the next result. We state it without proof as it's proof is similar to the proof of Theorem 3.1.

Theorem 3.4. *Let G be the join of k graphs $G_1,G_2,\dots,G_k$ such that $\mathscr{P}_1,\mathscr{P}_2,\dots,\mathscr{P}_k$ are the vertex partitions of $G_1,G_2,\dots,G_k$ respectively and $\mathscr{P}=\bigcup_{i=1}^{k}\mathscr{P}_i$ be the vertex partition of G. If $\overline{G}$ is the complement of G, then for the vertex partition $\mathscr{P}=\bigcup_{i=1}^{k}\mathscr{P}_i$ of $V(\overline{G})$,*

$$E_{\mathscr{P}}(\overline{G})=E_{\mathscr{P}_1}(G_1)+E_{\mathscr{P}_2}(G_2)+\dots+E_{\mathscr{P}_k}(G_k).$$

In the next theorem, we determine the $\mathscr{P}$-energy of k-complement of join of k graphs.

Theorem 3.5. *For a graph $G=G_1\bigtriangledown G_2\bigtriangledown\dots\bigtriangledown G_k$ of order n such that $\mathscr{P}=\sum_{i=1}^{k}\mathscr{P}_i$,*

$$E_{\mathscr{P}}(\overline{(G)_k})=\sum_{i=1}^{k}E_{\mathscr{P}_i}(\overline{(G_i)_k}). \tag{3.3}$$

where $\overline{(G)_k}$ is the k-complement of the graph G and $\mathscr{P}_i$ is the vertex partition of G_i, for $i=1,2,\dots,k$.

Proof. The $\mathscr{P}$-matrix of $\overline{(G)_k}$ is the diagonal matrix whose diagonal entries are $A_{\mathscr{P}_1}\overline{(G_1)}_k,A_{\mathscr{P}_2}\overline{(G_2)}_k,\dots,A_{\mathscr{P}_k}\overline{(G_k)}_k$. Therefore, by Lemma 1.2

$$\phi_{\mathscr{P}}(\overline{(G)_k},\lambda)=\phi_{\mathscr{P}_1}(\overline{(G_1)_k},\lambda)\phi_{\mathscr{P}_2}(\overline{(G_2)_k},\lambda)\dots\phi_{\mathscr{P}_k}(\overline{(G_k)_k},\lambda).$$

Thus, the $\mathscr{P}$-energy of $\overline{(G)_k}$ is

$$E_{\mathscr{P}}(\overline{(G)_k})=E_{\mathscr{P}_1}(\overline{(G_1)_k})+E_{\mathscr{P}_2}(\overline{(G_2)_k})+\dots+E_{\mathscr{P}_k}(\overline{(G_k)_k}).$$

□

Remark 3.6. *Let G be the join of k graphs having the vertex partition $\mathscr{P}=V(G)$ and $\overline{(G)_1}$ be the 1-complement of G. Then the robust $\mathscr{P}$-energy of $\overline{(G)_1}$*

$$E_{\mathscr{P}_r}(\overline{(G)_1})=E_{\mathscr{P}_r}(G).$$

Now, we determine the characteristic polynomial of $k(i)$-complement of join of k-copies of a graph.

Theorem 3.7. *Let F be a graph of order t and let $\mathscr{P}_1$ be its vertex partition. If G is a graph of order n obtained by the join of k-copies of F such that $\mathscr{P}$ is its vertex partition, then for $\mathscr{P}$ of $\overline{(G)_{k(i)}}$ such that it is the union of k-copies of $\mathscr{P}_1$,*

$$\phi_{\mathscr{P}}(\overline{(G)_{k(i)}}) = \left[\phi_{\mathscr{P}_1}(\overline{(F)_{k(i)}},\lambda)\right]^k \left[1-(k-1)\Gamma_{A_{\mathscr{P}_1}}(\lambda)\right] \left[1+\Gamma_{A_{\mathscr{P}_1}}(\lambda)\right]^{(k-1)}. \quad (3.4)$$

We omit the proof of Theorem 3.7, since the process of obtaining Equation (3.4) is similar to that of Equation (2.10) in the proof of Theorem 2.7.

Now in Theorem 3.8 by considering the case when $A_{\mathscr{P}_1}(F)$ has a constant row sum R, we obtain the the corresponding $\mathscr{P}$-energy by the similar method as shown in the proof of Theorem 3.3. Therefore, we state the result directly without proof.

Theorem 3.8. *Let G be the join of k-copies of a graph F of order t having $\mathscr{P}$ and $\mathscr{P}_1$ as their vertex partitions respectively. If $A_{\mathscr{P}_1}(F)$ has constant row sum R, then for a vertex partition $\mathscr{P}$ of $\overline{(G)_{k(i)}}$ such that it is the union of k-copies of vertex partitions $\mathscr{P}_1$ of $\overline{(F)_{k(i)}}$,*

$$E_{\mathscr{P}}(\overline{(G)_{k(i)}}) = kE_{\mathscr{P}_1}(\overline{(F)_{k(i)}}) - kR + |R+(k-1)t| + (k-1)|R-t|.$$

The robust $\mathscr{P}$-energy of the $k(i)$-complement of join of k graphs is given by the following result.

Proposition 3.9. *Let G_i be graphs of order n_i for $i=1,2,\ldots,k$. If $G = G_1 \triangledown G_2 \triangledown \ldots \triangledown G_k$, then for the vertex partition $\mathscr{P} = V(\overline{(G)_{1(i)}})$*

$$E_{\mathscr{P}_r}(\overline{(G)_{1(i)}}) = \sum_{i=1}^{k}\sum_{j=1}^{n_i}\left|\lambda_{i_j} + (n-n_i)\right|$$

where $\overline{(G)_{1(i)}}$ is the $1(i)$-complement of the graph G of order n and λ_{i_j} is an eigenvalue of $A_{\mathscr{P}_i}(\overline{(G_i)_{1(i)}})$ for $j=1,2,\ldots,n_i$ where $i=1,2,\ldots,k$.

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