Malaya Journal of Matematik, Vol. 8, No. 4, 2092-2096, 2020
https://doi.org/10.26637/MJM0804/0130

New results on $\mathscr{P}$-energy of join of graphs

Prajakta Bharat Joshi[1]* and Mayamma Joseph[2]

Abstract
The $\mathscr{P}$-energy of a graph G with a vertex partition $\mathscr{P}$ is the sum of the absolute values of the eigenvalues of its $\mathscr{P}$-matrix . In this article, we discuss the $\mathscr{P}$-energy of the join of graphs in the special case when the component graphs are either regular or complete bipartite.

Keywords
Graph energy, partition energy, $\mathscr{P}$-energy, coronal of a graph, join of graphs.

AMS Subject Classification
05C15, 05C50, 05C69.

[1,2] *Department of Mathematics, CHRIST(Deemed to be University), Bengaluru-560029, India.*
***Corresponding author**: [1] prajaktabharat.joshi@res.christuniversity.in; [2]mayamma.joseph@christuniversity.in
Article History: Received **24** September **2020**; Accepted **29** November **2020**

Contents

1. Introduction

The study of the energy of an arbitrary graph was initiated by Ivan Gutman, a chemist and mathematician in 1978 [3]. For a given graph G, its energy is sum of the absolute values of eigenvalues of the adjacency matrix $A(G)$ [3]. In last few decades, extensive studies have been done on graph energy and its variations [4–6].

Sampathkumar et al. [9] introduced the concept of k-partition energy of a graph $E_{P_k}(G)$ in 2015 using the idea of L-matrix that takes into consideration the vertex partitions of G. Prajakta and Mayamma [7] extended the concept of k-partition energy and initiated the study of $\mathscr{P}$-energy, the sum of absolute values of eigenvalues of $\mathscr{P}$-matrix $A_{\mathscr{P}}(G)$. The entries of $A_{\mathscr{P}}(G) = (a_{ij})_{n\times n}$ are given by

$$a_{ij} = \begin{cases} |V_r| & \text{if } i=j \text{ and } v_i = v_j \in V_r, \text{ for } r = 1,2,\dots k \\ 2 & \text{if } v_iv_j \in E(G) \text{ with } v_i, v_j \in V_r, \\ 1 & \text{if } v_iv_j \in E(G) \text{ with } v_i \in V_r \text{ and } v_j \in V_s \text{ for } r \neq s, \\ -1 & \text{if } v_iv_j \notin E(G) \text{ with } v_i, v_j \in V_r, \\ 0 & \text{otherwise.} \end{cases}$$

In their study on $\mathscr{P}$-energy, the authors have used the concept of the $\mathscr{P}$-coronal of a graph G $\Gamma_{A_{\mathscr{P}}}(\lambda)$ which is the sum of entries in $(\lambda I_n - A_{\mathscr{P}}(G))^{-1}$ where I_n is an identity matrix of order n [8] to find a generalized formula that gives the characteristic polynomial of the join of graphs [8]. The $\Gamma_{A_{\mathscr{P}}}(\lambda)$ in fact is a variation of M-coronal defined in [1] and is associated with the matrix $A_{\mathscr{P}}(G)$ corresponding to a graph G with the vertex partition $\mathscr{P}$.

It has been observed that $\mathscr{P}$-matrix of regular graphs have constant row sum. In this article we determine $\mathscr{P}$-energy of join of regular graphs along with that of the join of non-regular complete bipartite graphs. The following results found in [8] are required for further discussion.

Theorem 1.1. *[8] Let G_1 be a graph of order n_1 and G_2 be a graph of order n_2. If $\mathscr{P}_1$ is a vertex partition of G_1 and $\mathscr{P}_2$ is a vertex partition of G_2, then*

$$\phi_{\mathscr{P}}(G_1 \nabla G_2, \lambda) = \phi_{\mathscr{P}_1}(G_1,\lambda)\phi_{\mathscr{P}_2}(G_2,\lambda)[1 - \Gamma_{A_{\mathscr{P}_1}}(\lambda)\Gamma_{A_{\mathscr{P}_2}}(\lambda)] \tag{1.1}$$

where $\Gamma_{A_{\mathscr{P}_i}}(\lambda)$ is the $\mathscr{P}$-coronal of G_i corresponding to $A_{\mathscr{P}_i}(G_i)$, for $i = 1,2$.

Theorem 1.2. *[8] Let F be a graph of order t and let $\mathscr{P}_1$ be its vertex partition. If G is a graph of order n obtained by the join of k-copies of F such that $\mathscr{P}$ is its vertex partition, then for $\mathscr{P}$ of G such that it is the union of k-copies of $\mathscr{P}_1$,*

$$\phi_{\mathscr{P}}(G) = \left[\phi_{\mathscr{P}_1}(F,\lambda)\right]^k \left[1-(k-1)\Gamma_{A_{\mathscr{P}_1}}(\lambda)\right]\left[1+\Gamma_{A_{\mathscr{P}_1}}(\lambda)\right]^{(k-1)} \tag{1.2}$$

where $\Gamma_{A_{\mathscr{P}_1}}(\lambda)$ is the $\mathscr{P}$-coronal of F corresponding to $A_{\mathscr{P}_1}(F)$, for $i = 1,2$.

Theorem 1.3. *[8] Let G_i be a graph of order n_i and let $\mathscr{P}_i$ be its vertex partition, for $i=1,2$. If $A_{\mathscr{P}_i}(G_i)$ has a constant row sum R_i for $i=1,2$, then for a vertex partition $\mathscr{P}=\mathscr{P}_1\cup\mathscr{P}_2$ of $G_1\nabla G_2$,*

$$\phi_{\mathscr{P}}(G_1\nabla G_2,\lambda)=\phi_{\mathscr{P}_1}(G_1,\lambda)\phi_{\mathscr{P}_2}(G_2,\lambda)\left[\frac{(\lambda-R_1)(\lambda-R_2)-n_1n_2}{(\lambda-R_1)(\lambda-R_2)}\right] \quad (1.3)$$

where $\mathscr{P}_1$ and $\mathscr{P}_2$ are the vertex partitions of G_1 and G_2.

2. Regular graphs

In this section, we explore the $\mathscr{P}$-energy of join of regular graphs. We begin with a lemma that gives a property of $A_{\mathscr{P}}(G)$ when G is a regular graph.

Lemma 2.1. *If G is a regular graph with vertex partition $\mathscr{P}=\{V_1,V_2,\dots,V_k\}$ such that $|V_i|=l$, for $i=1,2,\dots,k$, then $A_{\mathscr{P}}(G)$ has a constant row sum $R=l+\frac{1}{n}(4m_1+2m_2-2m_3)$ where m_1 is the number of edges having end vertices are in same vertex partition, m_2, the number of edges with end vertices are in different partition and m_3 is the number of non-adjacent pairs of vertices.*

Proof. Consider a regular graph G a vertex partition $\mathscr{P}=\{V_1,V_2,\dots,V_k\}$ such that $|V_i|=l$ for each $V_i\in\mathscr{P}$. Since G is a regular graph and each diagonal entry of $A_{\mathscr{P}}(G)$ is l, sum of the elements of each row of $A_{\mathscr{P}}(G)$ is a constant, say R. Now we will determine the value of R in terms of the types of edges of G. It can be observed that, number of 1's, 2's and -1's are $2m_1$, $2m_2$ and $2m_3$, respectively. Therefore, the value of R is given by $R=l+\frac{1}{n}(4m_1+2m_2-2m_3)$. □

Note that, the quantities $m_1,m_2,m_3,$ and m'_1,m'_2,m'_3 used in Theorem 2.2 are taken in the same context as that of m_1,m_2 and m_3 mentioned in Lemma 2.1.

Theorem 2.2. *Let G_1 and G_2 be two regular graphs of order n_1 and n_2 with vertex partitions $\mathscr{P}_1$ and $\mathscr{P}_2$ respectively. If $\mathscr{P}_1=\{U_1,U_2,\dots,U_{k_1}\}$ and $\mathscr{P}_2=\{V_1,V_2,\dots,V_{k_2}\}$ such that $|U_i|=l_1$ and $|V_i|=l_2$, then for the vertex partition $\mathscr{P}=\mathscr{P}_1\cup\mathscr{P}_2$ of $G_1\nabla G_2$*

$$E_{\mathscr{P}}(G_1\nabla G_2)=\sum_{i=1,2}E_{\mathscr{P}_i}(G_i)-\sum_{i=1,2}\left|l_i+\frac{M_i}{n_i}\right|+\frac{1}{2}\left(|a_1|+|b_1|\right),$$

where $M_1=4m_1+2m_2-2m_3$ and $M_2=4m'_1+2m'_2-2m'_3$, $a_1=\left[l_1+l_2+\frac{M_1}{n_1}+\frac{M_2}{n_2}\right]+\left\{(l_1-l_2)^2+\left(\frac{M_1}{n_1}-\frac{M_2}{n_2}\right)+2l_1\left(\frac{M_1}{n_1}-\frac{M_2}{n_2}\right)+2l_2\left(\frac{M_2}{n_2}-\frac{M_1}{n_1}\right)+4n_1n_2\right\}^{\frac{1}{2}}$ and $b_1=\left[l_1+l_2+\frac{M_1}{n_1}+\frac{M_2}{n_2}\right]-\left\{(l_1-l_2)^2+\left(\frac{M_1}{n_1}-\frac{M_2}{n_2}\right)+2l_1\left(\frac{M_1}{n_1}-\frac{M_2}{n_2}\right)+2l_2\left(\frac{M_2}{n_2}-\frac{M_1}{n_1}\right)+4n_1n_2\right\}^{\frac{1}{2}}$.

Proof. For the vertex partition $\mathscr{P}=\mathscr{P}_1\cup\mathscr{P}_2$ of $G_1\nabla G_2$, by Theorem 1.3 and Lemma 2.1, the characteristic polynomial $A_{\mathscr{P}}(G)$ is

$$\phi_{\mathscr{P}}(G_1\nabla G_2,\lambda)=\frac{\phi_{\mathscr{P}_1}(G_1,\lambda)\phi_{\mathscr{P}_2}(G_2,\lambda)\left\{[\lambda-(l_1+\frac{M_1}{n_1})][\lambda-(l_2+\frac{M_2}{n_2})]-n_1n_2\right\}}{[\lambda-(l_1+\frac{M_1}{n_1})][\lambda-(l_2+\frac{M_2}{n_2})]}$$

It can be written as

$$\begin{aligned}&[\lambda-(l_1+\frac{M_1}{n_1})][\lambda-(l_2+\frac{M_2}{n_2})]\phi_{\mathscr{P}}(G_1\nabla G_2,\lambda)\\&=\phi_{\mathscr{P}_1}(G_1,\lambda)\phi_{\mathscr{P}_2}(G_2,\lambda)\left\{[\lambda^2-\left[l_1+l_2+\frac{M_1}{n_1}+\frac{M_2}{n_2}\right]\lambda\right.\\&\qquad\left.+\left[l_1+l_2+\frac{M_1}{n_1}+\frac{M_2}{n_2}\right]-n_1n_2\right\}. \quad (2.1)\end{aligned}$$

Let left and right side of Equation (2.1) be $L(\lambda)$ and $R(\lambda)$ respectively. The roots of the equations $L(\lambda)=0$ and $R(\lambda)=0$ are same. Therefore, the sum of the absolute values of the roots of these equations are also same. Thus,

$$\begin{aligned}&|(l_1+\frac{M_1}{n_1})|+|(l_2+\frac{M_2}{n_2})|+E_{\mathscr{P}}(G_1\nabla G_2,\lambda)\\&=E_{\mathscr{P}_1}(G_1,\lambda)+E_{\mathscr{P}_2}(G_2,\lambda)+\frac{1}{2}\{|a_1|+|b_1|\}, \quad (2.2)\end{aligned}$$

where $M_1=4m_1+2m_2-2m_3$ and $M_2=4m'_1+2m'_2-2m'_3$, $a_1=\left[l_1+l_2+\frac{M_1}{n_1}+\frac{M_2}{n_2}\right]+\left\{(l_1-l_2)^2+\left(\frac{M_1}{n_1}-\frac{M_2}{n_2}\right)+2l_1\left(\frac{M_1}{n_1}-\frac{M_2}{n_2}\right)+2l_2\left(\frac{M_2}{n_2}-\frac{M_1}{n_1}\right)+4n_1n_2\right\}^{\frac{1}{2}}$ and $b_1=\left[l_1+l_2+\frac{M_1}{n_1}+\frac{M_2}{n_2}\right]-\left\{(l_1-l_2)^2+\left(\frac{M_1}{n_1}-\frac{M_2}{n_2}\right)+2l_1\left(\frac{M_1}{n_1}-\frac{M_2}{n_2}\right)+2l_2\left(\frac{M_2}{n_2}-\frac{M_1}{n_1}\right)+4n_1n_2\right\}^{\frac{1}{2}}$.

Therefore, on simplifying Equation (2.2) we get the required result. □

Corollary 2.3. *Let G_1 and G_2 be two regular graphs of order n_1 and n_2, with degrees r_1 and r_2 respectively. If $\mathscr{P}_1$ and $\mathscr{P}_2$ are the vertex partitions of G_1 and G_2, then corresponding to the vertex partition $\mathscr{P}=\mathscr{P}_1\cup\mathscr{P}_2$ of $G_1\nabla G_2$, then,*

(i) $E_{\mathscr{P}}(G_1\nabla G_2)=\sum_{i=1,2}E_{\mathscr{P}_i}(G_i)-\sum_{i=1,2}(3r_i+1)+\frac{1}{2}\{|a|+|b|\}$, where $a=[3(r_1+r_2)+2]+\sqrt{3(r_1-r_2)^2+4n_1n_2}$ and $b=[3(r_1+r_2)+2]-\sqrt{3(r_1-r_2)^2+4n_1n_2}$, provided $\mathscr{P}_1=\{V(G_1)\}$ and $\mathscr{P}_2=\{V(G_2)\}$.

(ii) $E_{\mathscr{P}}(G_1\nabla G_2)=\sum_{i=1,2}E_{\mathscr{P}_i}(G_i)-\sum_{i=1,2}(2r_i-n_i+2)+\frac{1}{2}\{|a_2|+|b_2|\}$, where $a_2=[2(r_1+r_2)-(n_1+n_2)+4]+\{(2r_1+2r_2)^2+(n_1+n_2)^2+4r_1(n_2-n_1)+4r_2(n_1-n_2)\}^{\frac{1}{2}}$ $b_2=[2(r_1+r_2)-(n_1+n_2)+4]-\{(2r_1+2r_2)^2+(n_1+n_2)^2+4r_1(n_2-n_1)+4r_2(n_1-n_2)\}^{\frac{1}{2}}$ when $\mathscr{P}_1=\{\{u_1\},\{u_2\},\dots,\{u_{n_1}\}\}$ and $\mathscr{P}_2=\{\{v_1\},\{v_2\},\dots,\{v_{n_2}\}\}$.

By Theorem 1.2 and Lemma 2.1, we obtain the following result. We state it without proof as the proof is similar to that of Theorem 2.2.

Theorem 2.4. *Let H be a regular graph of order n_1 and $\mathscr{P} = \{V_1, V_2, \dots, V_t\}$ be its vertex partition such that $|V_i| = l_1$ for $i = 1,2,\dots,t$. Let G be the join of k-copies of H and $\mathscr{P}$ be its vertex partition such that it is the union of k-copies of $\mathscr{P}$. Then for $\mathscr{P}$*

$$E_{\mathscr{P}}(G) = kE_{\mathscr{P}_1}(H) + \frac{1}{n_1}|(k-1)n_1^2 + l_1 + M_1| + \frac{(k-1)}{n_1}|M_1 + l_1 n_1 - n_1^2| - \frac{k}{n_1}|l_1 n_1 + M_1|.$$

Corollary 2.5. *Let H be an r_1-regular graph of order n_1 and $\mathscr{P}$ be its vertex partition. Let G be the join of k-copies of H. If $\mathscr{P}$ is the vertex partition of G such that it is the union of k-copies of $\mathscr{P}$, then*

(i) $E_{\mathscr{P}}(G) = kE_{\mathscr{P}_1}(H) + |3r_1 + 1 - (k-1)n_1| + (k-1)|3r_1 + 1 - n_1| - k|3r_1 + 1|$ *when $\mathscr{P}_1 = V(H)$ and*

(ii) $E_{\mathscr{P}}(G) = kE_{\mathscr{P}_1}(H) + |2(r_1+1) - kn_1| + 2(k-1)|r + 1 - n_1| - k|2r_1 - n_1 + 2|$ *if $\mathscr{P}_1 = \{\{u_1\},\{u_2\},\dots,\{u_{n_1}\}\}$.*

Let G be the join of k regular graphs of order $n_1, n_2, \dots, n_k$ and degree $r_1, r_2, \dots, r_k$ respectively. Then the join G is a regular graph if $n_i - r_i = n_{i+1} - r_{i+1}$ [2]. In the next theorem, we use this condition and obtain the $\mathscr{P}$-energy of a regular graph which is the join of k graphs, each of which is regular.

Theorem 2.6. *Let $G_1, G_2, \dots, G_k$ be the regular graphs of order $n_1, n_2, \dots, n_k$ and degree $r_1, r_2, \dots, r_k$ respectively. Let $\mathscr{P}_1, \mathscr{P}_2, \dots, \mathscr{P}_k$ be their vertex partitions. If $G = G_1 \nabla G_2 \nabla \dots \nabla G_k$ is an r-regular graph of order n such that $r = n - s$, then for a vertex partition $\mathscr{P} = \bigcup_{i=1}^{k} \mathscr{P}_i$ of G*

$$E_{\mathscr{P}}(G) = \sum_{i=1}^{k} E_{\mathscr{P}_i}(G_i) - \sum_{i=1}^{k} R_i + \left(\sum_{i=1}^{k-1} n_i + R_k\right) + \sum_{i=1}^{k-1} |R_i - n_i| \quad (2.3)$$

where R_i is a constant row sum of $A_{\mathscr{P}_i}(G_i)$ and $s = n_i - r_i = n_{i+1} - r_{i+1}$, for $i = 1,2,\dots,k$.

Proof. For a vertex partition $\mathscr{P} = \mathscr{P}_1 \cup \mathscr{P}_2$ of G, by Theorem 1.3,

$$\phi_{\mathscr{P}}(G_1 \nabla G_2, \lambda) = \frac{\phi_{\mathscr{P}_1}(G_1,\lambda)\phi_{\mathscr{P}_2}(G_2,\lambda)[\lambda - (n_1 + R_2)][\lambda + n_1 - R_1]}{[\lambda - R_1][\lambda - R_2]} \quad (2.4)$$

Now, let $G = (G_1 \nabla G_2) \nabla G_3$ and $\mathscr{P} = \mathscr{P}_1 \cup \mathscr{P}_2 \cup \mathscr{P}_3$. Then Equation (2.4) becomes,

$$\phi_{\mathscr{P}}(G,\lambda) = \prod_{i=i}^{3} \frac{\phi_{\mathscr{P}_i}(G_i,\lambda)}{[\lambda - R_i]}[\lambda - (n_1 + n_2 + R_3)][\lambda + n_1 - R_1][\lambda + n_2 - R_2]$$

$$= \prod_{i=i}^{3} \frac{\phi_{\mathscr{P}_i}(G_i,\lambda)}{[\lambda - R_i]}[\lambda - (\sum_{i=1,2} n_i + R_3)] \prod_{i=1,2}[\lambda + n_i - R_i]$$

continuing in this way for $G_1, G_2, \dots, G_k$ and $\mathscr{P} = \bigcup_{i=1}^{k} \mathscr{P}_i$, we obtain

$$\phi_{\mathscr{P}}(G,\lambda) = \prod_{i=i}^{k} \frac{\phi_{\mathscr{P}_i}(G_i,\lambda)}{[\lambda - R_i]} \prod_{i=1}^{k-1}[\lambda - (R_i - n_i)]\left\{\lambda - \left[\sum_{i=1}^{k-1} n_i + R_k\right]\right\}. \quad (2.5)$$

Now, Equation (2.5) can be written as

$$\prod_{i=i}^{k}[\lambda - R_i]\phi_{\mathscr{P}}(G,\lambda) = \prod_{i=i}^{k}\phi_{\mathscr{P}_i}(G_i,\lambda)\prod_{i=1}^{k-1}[\lambda - (R_i - n_i)]\left\{\lambda - \left[\sum_{i=1}^{k-1} n_i + R_k\right]\right\}. \quad (2.6)$$

Consider the left hand side and the right hand side of the Equation (2.6) as $S_1(\lambda)$ and $S_2(\lambda)$ respectively. The roots of equation $S_1(\lambda) = 0$ and $S_2(\lambda) = 0$ are same. Therefore, the sum of the absolute values of their roots are also same. Thus,

$$\sum_{i=i}^{k}|R_i| + E_{\mathscr{P}}(G,\lambda) = \sum_{i=i}^{k} E_{\mathscr{P}_i}(G_i,\lambda) + \sum_{i=1}^{k-1}|R_i - n_i| + \left|\sum_{i=1}^{k-1} n_i + R_i\right|.$$

Therefore,

$$E_{\mathscr{P}}(G,\lambda) = \sum_{i=i}^{k} E_{\mathscr{P}_i}(G_i,\lambda) + \sum_{i=1}^{k-1}|R_i - n_i| + \left|\sum_{i=1}^{k-1} n_i + R_k\right| - \sum_{i=i}^{k}|R_i|.$$

Hence, the result holds. □

Corollary 2.7. *Let G_i be a regular graph of order n_i, degree r_i such that $\mathscr{P}_i = V(G_i)$, for $i = 1,2,\dots,k$. If $G = \bigcup_{i=1}^{k} G_i$ is a regular graph of degree r such that its vertex partition $\mathscr{P} = \bigcup_{i=1}^{k} \mathscr{P}_i$, then*

$$E_{\mathscr{P}}(G) = \sum_{i=1}^{k} E_{\mathscr{P}_i}(G_i) - \sum_{i=1}^{k}(3_{r_i} + 1) + \left(\sum_{i=1}^{k-1} n_i + 3r_k + 1\right) + \sum_{i=1}^{k-1}|3r_i + 1 - n_i|. \quad (2.7)$$

3. Complete bipartite graphs

In this section, we determine the $\mathscr{P}$-energy of join of complete bipartite graphs using Theorem 1.1 and 1.2. First we obtain the $\mathscr{P}$-coronal of a complete bipartite graph.

Lemma 3.1. *Let $K_{r,s}$ be a complete bipartite graph of order $n = r + s$ such that $r \neq s$. Then for the vertex partition $\mathscr{P} = \{V_1, V_2\}$ where V_1 and V_2 are the two partite sets of $K_{r,s}$ and $\mathscr{P} = \{\{v_1\},\{v_2\},\dots,\{v_n\}\}$, the $\mathscr{P}$-coronal of $K_{r,s}$ is*

$$\Gamma_{A_{\mathscr{P}}}(\lambda) = \frac{n\lambda + 2rs - n}{\lambda^2 - 2\lambda - (rs - 1)}. \quad (3.1)$$

Proof. Let $X = diag((\lambda + s - 1)I_r, (\lambda + r - 1)I_s)$ be a diagonal matrix of order $n \times n$. Then

$$[\lambda - A_{\mathscr{P}}(K_{r,s})]X\mathbf{1}_n = [\lambda^2 - 2\lambda - (rs - 1)]\mathbf{1}_n.$$

Therefore,

$$\begin{aligned}\Gamma_{A_{\mathscr{P}}}(\lambda) &= \mathbf{1}_n^T[\lambda I_n - A_{\mathscr{P}}(K_{r,s})]^{-1}\mathbf{1}_n \\ &= \frac{\mathbf{1}_n^T X \mathbf{1}_n}{[\lambda^2-2\lambda-(rs-1)]} \\ &= \frac{n\lambda+2rs-n}{[\lambda^2-2\lambda-(rs-1)]}.\end{aligned}$$

□

Theorem 3.2. *Let K_{r_1,s_1} and K'_{r_2,s_2} be two complete bipartite graphs of order n_1 and n_2. Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be their vertex partitions respectively. Then for a vertex partition $\mathscr{P} = \mathscr{P}_1 \cup \mathscr{P}_2$ of $K_{r_1,s_1}\nabla K'_{r_2,s_2}$,*

$$\begin{aligned}E_{\mathscr{P}}(K_{r_1,s_1}\nabla K'_{r_2,s_2}) = E_{\mathscr{P}_1}(K_{r_1,s_1}) + E_{\mathscr{P}_2}(K'_{r_2,s_2}) - |(1+\sqrt{r_1s_1})| \\ -|(1-\sqrt{r_1s_1})| - |(1+\sqrt{r_2s_2})| - |(1-\sqrt{r_2s_2})| \\ +|c_1|+|c_2|+|c_3|+|c_4|\end{aligned}$$

where c_1, c_2, c_3 and c_4 are the roots of the quartic polynomial $\lambda^4-4\lambda^3-[r_1s_1+r_2s_2+n_1n_2-6]\lambda^2+2[r_1s_1(1-n_2)+r_2s_2(1-n_1)+n_1n_2-2]\lambda+[r_1s_1(2n_2-1)+r_2s_2(2n_1-1)-3r_1r_2s_1s_2-n_1n_2+1]$.

Proof. By Lemma 3.1 and Theorem 1.1,

$$\begin{aligned}\phi_{\mathscr{P}}(K_{r_1,s_1}\nabla K'_{r_2,s_2},\lambda) = \frac{\phi_{\mathscr{P}_1}(K_{r_1,s_1},\lambda)\phi_{\mathscr{P}_2}(K'_{r_2,s_2},\lambda)}{[\lambda^2-2\lambda-(r_1s_1-1)][\lambda^2-2\lambda-(r_2s_2-1)]} \\ \Big\{[\lambda^2-2\lambda-(r_1s_1-1)][\lambda^2-2\lambda-(r_2s_2-1)] \\ -[n_1\lambda+2r_1s_1-n_1][n_2\lambda+2r_2s_2-n_2]\Big\}.\end{aligned}$$

It can be written as,

$$\begin{aligned}&[\lambda^2-2\lambda-(r_1s_1-1)][\lambda^2-2\lambda-(r_2s_2-1)]\phi_{\mathscr{P}}(K_{r_1,s_1}\nabla K'_{r_2,s_2},\lambda) = \\ &\phi_{\mathscr{P}_1}(K_{r_1,s_1},\lambda)\phi_{\mathscr{P}_2}(K'_{r_2,s_2},\lambda)\Big\{[\lambda^2-2\lambda-(r_1s_1-1)][\lambda^2-2\lambda-(r_2s_2 \\ &-1)]-[n_1\lambda+2r_1s_1-n_1][n_2\lambda+2r_2s_2-n_2]\Big\}.\end{aligned}$$

On taking factors of the first two terms on the left hand side, we get

$$\begin{aligned}&\Big\{(\lambda-(1+\sqrt{r_1s_1}))(\lambda-(1-\sqrt{r_1s_1}))(\lambda-(1+\sqrt{r_2s_2}))(\lambda-(1- \\ &\sqrt{r_2s_2}))\Big\}\phi_{\mathscr{P}}(K_{r_1,s_1}\nabla K'_{r_2,s_2},\lambda) = \phi_{\mathscr{P}_1}(K_{r_1,s_1},\lambda)\phi_{\mathscr{P}_2}(K'_{r_2,s_2},\lambda)\Big\{[\lambda^2- \\ &2\lambda-(r_1s_1-1)][\lambda^2-2\lambda-(r_2s_2-1)]-[n_1\lambda+2r_1s_1-n_1][n_2\lambda+ \\ &2r_2s_2-n_2]\Big\}.\end{aligned}$$

On simplifying the last two terms on the right hand side, we obtain

$$\begin{aligned}&(\lambda-(1+\sqrt{r_1s_1}))(\lambda-(1-\sqrt{r_1s_1}))(\lambda-(1+\sqrt{r_2s_2})) \\ &(\lambda-(1-\sqrt{r_2s_2}))\phi_{\mathscr{P}}(K_{r_1,s_1}\nabla K'_{r_2,s_2},\lambda) \\ &=\phi_{\mathscr{P}_1}(K_{r_1,s_1},\lambda)\phi_{\mathscr{P}_2}(K'_{r_2,s_2},\lambda)\Big\{\lambda^4-4\lambda^3-[r_1s_1+r_2s_2 \\ &+n_1n_2-6]\lambda^2+2[r_1s_1(1-n_2)+r_2s_2(1-n_1)+n_1n_2-2]\lambda \\ &+[r_1s_1(2n_2-1)+r_2s_2(2n_1-1)-3r_1r_2s_1s_2-n_1n_2+1]\Big\}.\end{aligned}$$

Thus,

$$\begin{aligned}&|(1+\sqrt{r_1s_1})|+|(1-\sqrt{r_1s_1})|+|(1+\sqrt{r_2s_2})|+|(1-\sqrt{r_2s_2})| \\ &+E_{\mathscr{P}}(K_{r_1,s_1}\nabla K'_{r_2,s_2},\lambda) = E_{\mathscr{P}_1}(K_{r_1,s_1},\lambda)+E_{\mathscr{P}_2}(K'_{r_2,s_2},\lambda) \\ &+|c_1|+|c_2|+|c_3|+|c_4| \quad (3.2)\end{aligned}$$

where c_1, c_2, c_3 and c_4 are the roots of the quartic polynomial $\lambda^4-4\lambda^3-[r_1s_1+r_2s_2+n_1n_2-6]\lambda^2+2[r_1s_1(1-n_2)+r_2s_2(1-n_1)+n_1n_2-2]\lambda+[r_1s_1(2n_2-1)+r_2s_2(2n_1-1)-3r_1r_2s_1s_2-n_1n_2+1]$.

Hence on simplifying Equation (3.2), we get the required result. □

Since the proof technique of the following theorem is same as that of Theorem 3.2, we state the next result without proof.

Theorem 3.3. *Let G_1 be a graph of order n_1 such that $A_{\mathscr{P}_1}(G_1)$ has a constant row sum R_1 and K_{r_1,s_1} be a complete bipartite graph of order n_2. Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be their vertex partitions respectively. Then for a vertex partition $\mathscr{P} = \mathscr{P}_1 \cup \mathscr{P}_2$ of $G_1\nabla K_{r_1,s_1}$,*

$$\begin{aligned}E_{\mathscr{P}}(G_1\nabla K_{r_1,s_1}) = E_{\mathscr{P}_1}(G_1)+E_{\mathscr{P}_2}(K_{r_1,s_1})-|R_1| \\ -|(1+\sqrt{r_1s_1})|+|d_1|+|d_2|+|d_3|\end{aligned}$$

where d_1, d_2 and d_3 are the roots of the cubic polynomial $\lambda^3-(R_1+2)\lambda^2-[2R_1-r_1s_1-n_1n_2+1]\lambda+[R_1(r_1s_1-1)-2r_1s_1s_2n_1+n_1n_2]$.

Now, we derive the expression for $\mathscr{P}$-energy of join of k-copies of $K_{r,s}$ in the next theorem.

Theorem 3.4. *Let $K_{r,s}$ be a complete bipartite graph of order t, for $r \neq s$ and $\mathscr{P}_1$ be its vertex partition. Let G be the join of k-copies of $K_{r,s}$ and $\mathscr{P}$ be its vertex partition such that it is the union of k-copies of $\mathscr{P}_1$. Then*

$$\begin{aligned}E_{\mathscr{P}}(G) &= kE_{\mathscr{P}_1}(K_{r,s})-k(1+\sqrt{rs})-k(1-\sqrt{rs}) \\ &+\frac{1}{2}\Big\{\Big|t(k-1)+2+\sqrt{t^2(k-1)^2+4[2(k-1)+1]rs}\Big| \\ &+\Big|t(k-1)+2-\sqrt{t^2(k-1)^2+4[2(k-1)+1]rs}\Big| \\ &+\Big|t-2+\sqrt{t^2+4rs}\Big|+\Big|t-2-\sqrt{t^2+4rs}\Big|\Big\}.\end{aligned}$$

Proof. By Lemma 3.1, the $\mathscr{P}$-coronal of $K_{r,s}$ corresponding to $A_{\mathscr{P}_1}(K_{r,s})$ is given by

$$\Gamma_{A_{\mathscr{P}_1}}(\lambda) = \frac{t\lambda + 2rs - t}{\lambda^2 - 2\lambda - (rs-1)}. \tag{3.3}$$

By substituting Equation (3.3) in (1.2), we get

$$\phi_{\mathscr{P}}(G,\lambda) = \left[\frac{\phi_{\mathscr{P}_1}(K_{r,s})}{D}\right]^k \left[D-(k-1)N\right]\left[D+N\right]^{(k-1)}. \tag{3.4}$$

where $N = (t\lambda + 2rs - t)$ and $D = \lambda^2 - 2\lambda - (rs-1)$. Equation (3.4) can be written as

$$D^k\phi_{\mathscr{P}}(G,\lambda) = \left[\phi_{\mathscr{P}_1}(K_{r,s})\right]^k \left[D-(k-1)N\right]\left[D+N\right]^{(k-1)}. \tag{3.5}$$

Consider the left hand side and the right hand side of the Equation (3.5) as $S_1(\lambda)$ and $S_2(\lambda)$ respectively. The roots of equation $S_1(\lambda) = 0$ and $S_2(\lambda) = 0$ are same. Therefore, the sum of the absolute values of their roots are also same. To get this, we need to find out their roots.

1. The roots of D are $1 \pm \sqrt{rs}$, roots of $\left[D-(k-1)N\right]$.
2. The roots of $\left[D-(k-1)N\right]$ are

$$\frac{1}{2}\left\{t(k-1)+2 \pm \sqrt{t^2(k-1)^2 + 4[2(k-1)+1]rs}\right\}$$

3. Then roots of $\left[D+N\right]$ are

$$\frac{1}{2}[t-2+\sqrt{t^2+4rs}]$$

Thus, by Equation (3.5)

$$\begin{aligned} k(1+\sqrt{rs}) + k(1-\sqrt{rs})E_{\mathscr{P}}(G) &= kE_{\mathscr{P}_1}(K_{r,s}) \\ &+ \frac{1}{2}\left|t(k-1)+2+\sqrt{t^2(k-1)^2+4[2(k-1)+1]rs}\right| \\ &+ \frac{1}{2}\left|t(k-1)+2-\sqrt{t^2(k-1)^2+4[2(k-1)+1]rs}\right| \\ &+ \frac{1}{2}\left|t-2+\sqrt{t^2+4rs}\right| + \frac{1}{2}\left|t-2-\sqrt{t^2+4rs}\right|. \end{aligned} \tag{3.6}$$

On simplifying Equation (3.6), we get the required result. □

References

[1] S. Y. Cui and G. X. Tian, The spectrum and the signless Laplacian spectrum of coronae, *Linear Algebra Appl.* 437(7) (2012), 1692–1703.

[2] D. M. Cvetković , M. Doob and H. Sachs, *Spectra of Graphs*, Academic Press, New York, 1980.

[3] I. Gutman, The energy of a graph, *Ber. Math. Stat. Sekt. Forschungsz. Graz.* 103 (1978), 1–22.

[4] I. Gutman, The energy of a graph: old and new results in *Algebraic Combinatorics and Applications*, Springer, Berlin, Heidelberg, (2001), 196–211.

[5] I. Gutman and B. Furtula, The total π-electron energy saga, *Croat. Chem. Acta.* 90(3) (2017), 359–368, DOI: 10.5562/cca3189

[6] I. Gutman, X. Li, J. Zhang, *Graph Energy*, Springer, New York, 2012.

[7] P. B. Joshi and M. Joseph, $\mathscr{P}$-energy of graphs, *Acta Univ. Sapientiae, Info.* 12(1) (2020), 137–157.

[8] P. B. Joshi and M. Joseph, On $\mathscr{P}$-energy of join of graphs, Manuscript submitted for publication.

[9] E. Sampathkumar, S. V. Roopa, K. A. Vidya, M. A. Sriraj, Partition energy of a graph, *Proc. Jangjeon Math. Soc.* 18(4) (2015), 473–493.

⋆⋆⋆⋆⋆⋆⋆⋆⋆
ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
⋆⋆⋆⋆⋆⋆⋆⋆⋆