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On $(1,2)^{\star}$-$\breve{g}$-normal and $(1,2)^{\star}$-$\breve{g}$-regular spaces

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Abstract
In this paper, we introduce the notions of $(1,2)^{\star}$-$\breve{g}$-normal space and $(1,2)^{\star}$-$\breve{g}$-regular space in bitopological spaces. We obtain several characterizations of $(1,2)^{\star}$-$\breve{g}$- normal space, $(1,2)^{\star}$-$\breve{g}$-regular space and some preservation theorems are given.

Keywords
Normal space, regular space, $(1,2)^{\star}$-$\breve{g}$-normal space and $(1,2)^{\star}$-$\breve{g}$-regular space.

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1. Introduction

In 1963, J.C.Kelly [1] expressed the geometrical existence of bitopological space that is a non empty set X together with two arbitrary topologies defined on X and it plays an important role to study the shapes of objects. General topologist have introduced and investigated different forms of open sets in bitopological space. As a generalization of closed sets, in 1970, N. Levine [2] initiated the study of so called g-closed sets. As the strong forms of g-closed sets, the notion of $\hat{g}$-closed sets ($=\omega$-closed sets) were introduced and studied by Veerakumar [11] and Sheik John [12]. Using g-closed sets, Munchi [5] introduced g-regular and g-normal spaces in topological spaces. In a similar way, Sheik John [12] was introduced ω-regular and ω-normal spaces using ω-closed sets in topological spaces. Recently, several researchers was introduced and studied many types of normal and regular spaces in topological spaces and bitopological spaces as so on. In this paper, we introduce the notions of $(1,2)^{\star}$-$\breve{g}$-normal space and $(1,2)^{\star}$-$\breve{g}$-regular space in bitopological spaces. We obtain several characterizations of $(1,2)^{\star}$-$\breve{g}$- normal space, $(1,2)^{\star}$-$\breve{g}$-regular space and some preservation theorems are given.

2. Preliminaries

Throughout this paper (X,τ_1,τ_2) and (Y,σ_1,σ_2) (or simply X and Y) represents the non-empty bitopological spaces on which no separation axiom are assumed, unless otherwise mentioned. For a subset H of X, $\tau_{1,2}$-$cl(H)$ and $\tau_{1,2}$-$int(H)$ represents the closure of H and interior of H respectively.

We recollect the following basic definitions which are used in this paper.

Definition 2.1. *Let S be a subset of X. Then S is said to be $\tau_{1,2}$-open [3] if $S = A\cup B$ where $A\in\tau_1$ and $B\in\tau_2$.*

The complement of $\tau_{1,2}$-open set is called $\tau_{1,2}$-closed.

Notice that $\tau_{1,2}$-open sets need not necessarily form a topology.

Definition 2.2. *[3] Let S be a subset of a bitopological space X. Then*

1. *the $\tau_{1,2}$-closure of S, denoted by $\tau_{1,2}$-$cl(S)$, is defined as $\cap\{F : S\subseteq F$ and F is $\tau_{1,2}$-closed$\}$.*
2. *the $\tau_{1,2}$-interior of S, denoted by $\tau_{1,2}$-$int(S)$, is defined as $\cup\{F : F\subseteq S$ and F is $\tau_{1,2}$-open$\}$.*

Definition 2.3. *[3] A subset A of a bitopological space (X,τ_1,τ_2) or X is said to be a $(1,2)^{\star}$-semi open set if $A\subseteq\tau_{1,2}$-$cl(\tau_{1,2}$-$int(A))$.*

The complement of the above mentioned set is called a closed set.

Definition 2.4. *[9] A subset A of a bitopological space (X,τ_1,τ_2) or X is said to be a $(1,2)^{\star}$-semi closure of A, denoted by $(1,2)^{\star}$-$scl(A)$, is defined as $\cap\{F : S\subseteq F$ and F is $(1,2)^{\star}$-semi closed$\}$.*

Definition 2.5. *A subset H of a bitopological space (X,τ_1,τ_2) or X is said to be*

1. *a $(1,2)^{*}$-generalized closed set (briefly, $(1,2)^{*}$-g-closed) [10] if $\tau_{1,2}$-$cl(H)\subseteq U$ whenever $H\subseteq U$ and U is $\tau_{1,2}$-open.*
2. *a $(1,2)^{*}$-weakly closed set (briefly, $(1,2)^{*}$-W-closed) [12] if $\tau_{1,2}$-$cl(H)\subseteq U$ whenever $H\subseteq U$ and U is $\tau_{1,2}$-semi open.*
3. *a $(1,2)^{\star}$-$\hat{\hat{g}}$-closed set [7] if $\tau_{1,2}$-$cl(H)\subseteq G$ whenever $H\subseteq G$ and G is $(1,2)^{\star}$-sg-open.*
4. *a $(1,2)^{\star}$-$\mathscr{G}$-closed set [6] if $(1,2)^{\star}$-$scl(H)\subseteq G$ whenever $H\subseteq G$ and G is $(1,2)^{\star}$-$\hat{\hat{g}}_1$-open.*
5. *a $(1,2)^{\star}$-$\breve{g}$-closed set [6] if $\tau_{1,2}$-$cl(H)\subseteq G$ whenever $H\subseteq G$ and G is $(1,2)^{\star}$-$\mathscr{G}$-open.*

The complements of the above mentioned closed sets are called their respective open sets.

Definition 2.6. *A function $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is said to be*

1. *a $(1,2)^{\star}$-continuous [4] if the inverse image of every $\sigma_{1,2}$-closed set of (Y,σ_1,σ_2) is $\tau_{1,2}$-closed set in (X,τ_1,τ_2).*
2. *a $(1,2)^{\star}$-$\breve{g}$-irresolute function [8] if the inverse image of every $(1,2)^{\star}$-$\breve{g}$-closed set in (Y,σ_1,σ_2) is $(1,2)^{\star}$-$\breve{g}$-closed in (X,τ_1,τ_2).*
3. *a $(1,2)^{\star}$-$\breve{g}$-continuous [8] if the inverse image of every $\tau_{1,2}$-closed set in (Y,σ_1,σ_2) is $(1,2)^{\star}$-$\breve{g}$-closed set in (X,τ_1,τ_2).*
4. *a $(1,2)^{\star}$-$\mathscr{G}$-irresolute function [8] if the inverse image of every $(1,2)^{\star}$-$\mathscr{G}$-closed in (Y,σ_1,σ_2) is $(1,2)^{\star}$-$\mathscr{G}$-closed set in (X,τ_1,τ_2).*
5. *a $(1,2)^{\star}$-pre-$\mathscr{G}$-closed [8] if $f(U)$ is $(1,2)^{\star}$-$\mathscr{G}$-closed in (Y,σ_1,σ_2), for each $(1,2)^{\star}$-$\mathscr{G}$-closed set U in (X,τ_1,τ_2).*
6. *a $(1,2)^{\star}$-weakly continuous [12] if the inverse image of every $\sigma_{1,2}$-closed set of (Y,σ_1,σ_2) is $(1,2)^{\star}$-semi closed set in (X,τ_1,τ_2).*

3. $(1,2)^{\star}$-$\breve{g}$-Normal Space

Definition 3.1. *A bitopological space (X,τ_1,τ_2) is called $(1,2)^{\star}$-$\breve{g}$-normal if for any pair of disjoint $(1,2)^{\star}$-$\breve{g}$-closed sets H and K, there exist disjoint $\tau_{1,2}$-open sets A and B such that $H\subseteq A$ and $K\subseteq B$.*

Theorem 3.2. *Let (X,τ_1,τ_2) be a bitopological space. Then the following are equivalent:*

1. *(X,τ_1,τ_2) is $(1,2)^{\star}$-$\breve{g}$-normal,*
2. *For each $(1,2)^{\star}$-$\hat{\hat{g}}$-closed set F and for each $(1,2)^{\star}$-$\breve{g}$-open set A containing F, there exists $\tau_{1,2}$-open set B containing F such that $\tau_{1,2}$-$cl(B)\subseteq A$,*
3. *For each pair of disjoint $(1,2)^{\star}$-$\breve{g}$-closed sets H and K in (X,τ_1,τ_2), there exists $\tau_{1,2}$-open set A containing H such that $\tau_{1,2}$-$cl(A)\cap K=\phi$,*
4. *For each pair of disjoint $(1,2)^{\star}$-$\breve{g}$-closed sets H and K in (X,τ_1,τ_2), there exist $\tau_{1,2}$-open sets A containing H and B containing K such that $\tau_{1,2}$-$cl(A)\cap\tau_{1,2}$-$cl(B)=\phi$.*

Proof. (1) $\Rightarrow$ (2). Let S be a $(1,2)^{\star}$-$\breve{g}$-closed set and A be $(1,2)^{\star}$-$\breve{g}$-open set such that $S\subseteq A$. Then $S\cap A^c=\phi$. By assumption, there exist $\tau_{1,2}$-open sets B and L such that $S\subseteq B, A^c\subseteq L$ and $B\cap L=\phi$ which implies $\tau_{1,2}$-$cl(B)\cap L=\phi$. Now $\tau_{1,2}$-$cl(B)\cap A^c\subseteq\tau_{1,2}$-$cl(B)\cap L=\phi$ and so $\tau_{1,2}$-$cl(B)\subseteq A$.

(2) $\Rightarrow$ (3). Let H and K be disjoint $(1,2)^{\star}$-$\breve{g}$-closed sets of (X,τ_1,τ_2). Since $H\cap K=\phi, H\subseteq K^c$ and K^c is $(1,2)^{\star}$-$\breve{g}$-open. By assumption, there exists $\tau_{1,2}$-open set A containing H such that $\tau_{1,2}$-$cl(A)\subseteq K^c$ and so $\tau_{1,2}$-$cl(A)\cap K=\phi$.

(3) $\Rightarrow$ (4). Let H and K be any two disjoint $(1,2)^{\star}$-$\breve{g}$-closed sets of (X,τ_1,τ_2). Then by assumption, there exists $\tau_{1,2}$-open set A containing H such that $\tau_{1,2}$-$cl(A)\cap K=\phi$. Since $\tau_{1,2}$-$cl(A)$ is $\tau_{1,2}$-closed, it is $(1,2)^{\star}$-$\breve{g}$-closed and so K and $\tau_{1,2}$-$cl(A)$ are disjoint $(1,2)^{\star}$-$\breve{g}$-closed sets in (X,τ_1,τ_2). Therefore again by assumption, there exists $\tau_{1,2}$-open set B containing H such that $\tau_{1,2}$-$cl(B)\cap\tau_{1,2}$-$cl(A)=\phi$.

(4) $\Rightarrow$ (1). Let H and K be any two disjoint $(1,2)^{\star}$-$\breve{g}$-closed sets of (X,τ_1,τ_2). By assumption, there exist $\tau_{1,2}$-open sets A containing H and B containing K such that $\tau_{1,2}$-$cl(A)\cap\tau_{1,2}$-$cl(B)=\phi$, we have $A\cap B=\phi$ and thus (X,τ_1,τ_2) is $(1,2)^{\star}$-$\breve{g}$-normal.

Theorem 3.3. *If $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is bijective, $(1,2)^{\star}$-pre-$\mathscr{G}$-open, $(1,2)^{\star}$-$\breve{g}$-continuous and $\tau_{1,2}$-open and (X,τ_1,τ_2) is $(1,2)^{\star}$-$\breve{g}$-normal, then (Y,σ_1,σ_2) is $(1,2)^{\star}$-$\breve{g}$-normal.*

Proof. Let H and K be any disjoint $(1,2)^{\star}$-$\breve{g}$-closed sets of (Y,σ_1,σ_2). The function f is $(1,2)^{\star}$-$\breve{g}$-irresolute and so $f^{-1}(H)$ and $f^{-1}(K)$ are disjoint $(1,2)^{\star}$-$\breve{g}$-closed sets of (X,τ_1,τ_2). Since (X,τ_1,τ_2) is $(1,2)^{\star}$-$\breve{g}$-normal, there exist disjoint $\tau_{1,2}$-open sets A and B such that $f^{-1}(H)\subseteq A$ and $f^{-1}(K)\subseteq B$. Since f is $\tau_{1,2}$-open and bijective, we have $f(A)$ and $f(B)$ are $\tau_{1,2}$-open in (Y,σ_1,σ_2) such that $H\subseteq f(A), K\subseteq f(B)$ and $f(A)\cap f(B)=\phi$. Therefore, (Y,σ_1,σ_2) is $(1,2)^{\star}$-$\breve{g}$-normal.

Theorem 3.4. *If $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is $(1,2)^{\star}$-$\mathscr{G}$-irresolute, $(1,2)^{\star}$-$\breve{g}$-closed continuous injection and (Y,σ_1,σ_2) is $(1,2)^{\star}$-$\breve{g}$-normal, then (X,τ_1,τ_2) is $(1,2)^{\star}$-$\breve{g}$-normal.*

Proof. Let H and K be any disjoint $(1,2)^{\star}$-$\breve{g}$-closed subsets of (X,τ_1,τ_2). Since f is $(1,2)^{\star}$-$\mathscr{G}$-irresolute, $(1,2)^{\star}$-$\breve{g}$-closed, we have $f(H)$ and $f(K)$ are disjoint $(1,2)^{\star}$-$\breve{g}$-closed sets of (Y,σ_1,σ_2). Since (Y,σ_1,σ_2) is $(1,2)^{\star}$-$\breve{g}$-normal, there exist disjoint $\tau_{1,2}$-open sets A and B such that $f(H)\subseteq A$ and $f(K)\subseteq B$. i.e., $H\subseteq f^{-1}(A), K\subseteq f^{-1}(B)$ and $f^{-1}(A)\cap f^{-1}(B)=\phi$. Since f is $(1,2)^{\star}$-continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are $\tau_{1,2}$-open in (X,τ_1,τ_2), we have (X,τ_1,τ_2) is $(1,2)^{\star}$-$\breve{g}$-normal.

Theorem 3.5. *If $f:(X,\tau_1,\tau_2)\to(Y,\sigma_1,\sigma_2)$ is $(1,2)^{\star}$-weakly continuous, $(1,2)^{\star}$-$\breve{g}$-closed injection and (Y,σ_1,σ_2) is $(1,2)^{\star}$-$\breve{g}$-normal, then (X,τ_1,τ_2) is normal.*

Proof. Let H and K be any two disjoint $\tau_{1,2}$-closed sets of (X,τ_1,τ_2). Since f is injective and $(1,2)^{\star}$-$\breve{g}$-closed, $f(H)$ and $f(K)$ are disjoint $(1,2)^{\star}$-$\breve{g}$-closed sets of (Y,σ_1,σ_2). Since (Y,σ_1,σ_2) is $(1,2)^{\star}$-$\breve{g}$-normal, there exist $\tau_{1,2}$-open sets A and B such that $f(H)\subseteq A, f(K)\subseteq B$ and $\tau_{1,2}$-$cl(A)\cap\tau_{1,2}$-$cl(B)=\phi$. Since f is $(1,2)^{\star}$-weakly continuous. we have $H\subseteq f^{-1}(A)\subseteq\tau_{1,2}$-$int(f^{-1}(\tau_{1,2}\text{-}cl(A))), K\subseteq f^{-1}(B)\subseteq\tau_{1,2}$-$int(f^{-1}(\tau_{1,2}\text{-}cl(B)))$ and $\tau_{1,2}$-$int(f^{-1}(\tau_{1,2}\text{-}cl(A)))\cap\tau_{1,2}$-$int(f^{-1}(\tau_{1,2}\text{-}cl(B)))=\phi$. Therefore (X,τ_1,τ_2) is normal.

4. $(1,2)^{\star}$-$\breve{g}$-Regular Space

Definition 4.1. *A bitopological space (X,τ_1,τ_2) is called a $(1,2)^{\star}$-$\breve{g}$-regular if for each $(1,2)^{\star}$-$\breve{g}$-closed set G and every point $p\notin G$, there exist disjoint $\tau_{1,2}$-open sets A and B such that $G\subseteq A$ and $p\subseteq B$.*

Theorem 4.2. *Let (X,τ_1,τ_2) be a bitopological space is a $(1,2)^{\star}$-$\breve{g}$-regular space if and only if for each $p\in X$ and $(1,2)^{\star}$-$\breve{g}$-neighbourhood N of p there exists $\tau_{1,2}$-open neighbourhood A of p such that $\tau_{1,2}$-$cl(A)\subseteq N$.*

Proof. Let N be any $(1,2)^{\star}$-$\breve{g}$-neighbourhood of p. Then there exists an $(1,2)^{\star}$-$\breve{g}$-open set F such that $p\in F\subseteq N$. Since F^c is $(1,2)^{\star}$-$\breve{g}$-closed and $p\notin F^c$, by hypothesis there exist $(1,2)^{\star}$-open sets A and B such that $F^c\subseteq A, p\in B$ and $A\cap B=\phi$ and so $B\subseteq A^c$. Now, $\tau_{1,2}$-$cl(B)\subseteq\tau_{1,2}$-$cl(A^c)=A^c$ and $F^c\subseteq A$ implies $A^c\subseteq F\subseteq N$. Therefore $\tau_{1,2}$-$cl(B)\subseteq N$.

Conversely, let G be any $(1,2)^{\star}$-$\breve{g}$-closed set and $p\notin G$. Then $p\in G^c$ and G^c is $(1,2)^{\star}$-$\breve{g}$-open and so G^c is a $(1,2)^{\star}$-$\breve{g}$-neighbourhood of p. By hypothesis, there exists $(1,2)^{\star}$-open neighbourhood B of p such that $p\in B$ and $\tau_{1,2}$-$cl(B)\subseteq G^c$ which implies $G\subseteq(\tau_{1,2}\text{-}cl(B))^c$. Then $(\tau_{1,2}\text{-}cl(B))^c$ is $\tau_{1,2}$-open set containing G and $B\cap(\tau_{1,2}\text{-}cl(B))^c=\phi$. Hence, X is $(1,2)^{\star}$-$\breve{g}$-regular.

Theorem 4.3. *For a bitopological space (X,τ_1,τ_2) is normal $\iff$ For every pair of disjoint $\tau_{1,2}$-closed sets H and K, there exist $(1,2)^{\star}$-$\breve{g}$-open sets F and G such that $H\subseteq F, K\subseteq G$ and $F\cap G=\phi$.*

Proof. Let H and K be disjoint $\tau_{1,2}$-closed subsets of (X,τ_1,τ_2). By hypothesis, there exist disjoint $\tau_{1,2}$-open sets (and hence $(1,2)^{\star}$-$\breve{g}$-open sets) F and G such that $H\subseteq F$ and $K\subseteq G$.

Conversely, let H and K be $\tau_{1,2}$-closed subsets of (X,τ_1,τ_2). Then by assumption, $H\subseteq A, K\subseteq B$ and $A\cap B=\phi$, where A and B are disjoint $(1,2)^{\star}$-$\breve{g}$-open sets. Since H and K are $(1,2)^{\star}$-$\mathscr{G}$-closed by [[8], Definition 2.6]. $H\subseteq\tau_{1,2}$-$int(A)$ and $K\subseteq\tau_{1,2}$-$int(B)$. Further, $\tau_{1,2}$-$int(A)\cap\tau_{1,2}$-$int(B)=\tau_{1,2}$-$int(A\cap B)=\phi$.

Theorem 4.4. *Let (X,τ_1,τ_2) be a bitopological space is $(1,2)^{\star}$-$\breve{g}$-regular $\iff$ for each $(1,2)^{\star}$-$\breve{g}$-closed set G of (X,τ_1,τ_2) and each $p\in G^c$ there exist $\tau_{1,2}$-open sets A and B of (X,τ_1,τ_2) such that $p\in A, G\subseteq B$ and $\tau_{1,2}$-$cl(A)\cap\tau_{1,2}$-$cl(B)=\phi$.*

Proof. Let G be a $(1,2)^{\star}$-$\breve{g}$-closed set of a bitopological space (X,τ_1,τ_2) and $p\notin G$. Then there exist $\tau_{1,2}$-open sets A_0 and B of (X,τ_1,τ_2) such that $p\in A_0, G\subseteq B$ and $A_0\cap B=\phi$, which implies $A_0\cap\tau_{1,2}$-$cl(B)=\phi$. Since $\tau_{1,2}$-$cl(B)$ is $\tau_{1,2}$-closed, it is $(1,2)^{\star}$-$\breve{g}$-closed and $p\notin\tau_{1,2}$-$cl(B)$. Since (X,τ_1,τ_2) is $(1,2)^{\star}$-$\breve{g}$-regular, there exist $\tau_{1,2}$-open sets F and T of (X,τ_1,τ_2) such that $p\in F, \tau_{1,2}$-$cl(B)\subseteq T$ and $F\cap T=\phi$, which implies $\tau_{1,2}$-$cl(F)\cap T=\phi$. Let $A=A_0\cap F$, then A and B are $\tau_{1,2}$-open sets of (X,τ_1,τ_2) such that $p\in A, G\subseteq B$ and $\tau_{1,2}$-$cl(A)\cap\tau_{1,2}$-$cl(B)=\phi$.

On the other hand side is trivial.

Theorem 4.5. *Let (X,τ_1,τ_2) be a bitopological space. Then the following are equivalent:*

1. *(X,τ_1,τ_2) is $(1,2)^{\star}$-$\breve{g}$-regular,*
2. *For each point $p\in X$ and for each $(1,2)^{\star}$-$\breve{g}$-neighbourhood N of p, there exists $\tau_{1,2}$-open neighbourhood B of p such that $\tau_{1,2}$-$cl(B)\subseteq N$,*
3. *For each point $p\in X$ and for each $(1,2)^{\star}$-$\breve{g}$-closed set G not containing p, there exists $\tau_{1,2}$-open neighbourhood B of p such that $\tau_{1,2}$-$cl(B)\cap G=\phi$.*

Proof. (1) ⇒ (2). It is obvious.

(2) ⇒ (3). Let $p\in X$ and G be a $(1,2)^{\star}$-$\breve{g}$-closed set such that $p\notin G$. Then G^c is a $(1,2)^{\star}$-$\breve{g}$-neighbourhood of p and by hypothesis, there exists $\tau_{1,2}$- open neighbourhood B of p such that $\tau_{1,2}$-$cl(B)\subseteq G^c$ and hence $\tau_{1,2}$-$cl(B)\cap G=\phi$.

(3) ⇒ (2). Let $p\in X$ and N be a $(1,2)^{\star}$-$\breve{g}$-neighbourhood of p. Then there exists a $(1,2)^{\star}$-$\breve{g}$-open set G such that $p\in F\subseteq N$. Since F^c is $(1,2)^{\star}$-$\breve{g}$- closed and $p\notin F^c$ by hypothesis there exists $\tau_{1,2}$-open neighbourhood N of p such that $\tau_{1,2}$-$cl(B)\cap F^c=\phi$. Therefore $\tau_{1,2}$-$cl(B)\subseteq F\subseteq N$.

Theorem 4.6. *For a subset H of bitopological space (X,τ_1,τ_2), then the following are equivalent.*

1. *(X,τ_1,τ_2) is $(1,2)^{\star}$-$\breve{g}$-regular.*
2. *$\tau_{1,2}$-$cl_{\theta}(H)=(1,2)^{\star}$-$\breve{g}$-$cl(H)$.*
3. *$\tau_{1,2}$-$cl_{\theta}(H)=H$ for each $(1,2)^{\star}$-$\breve{g}$-closed set of H.*

(1) $\Rightarrow$ (2). For any subset H of a bitopological space (X,τ_1,τ_2), we have always $H \subseteq (1,2)^{\star}$-$\breve{g}$-$cl(H) \subseteq \tau_{1,2}$-$cl_{\theta}(H)$. Let $p \in ((1,2)^{\star}$-$\breve{g}$-$cl(H))^c$. Then there exists a $(1,2)^{\star}$-$\breve{g}$-closed set G such that $p \in G^c$ and $H \subseteq G$. By assumption, there exist disjoint $\tau_{1,2}$-open sets A and B such that $p \in A$ and $G \subseteq B$. Now, $p \in A \subseteq \tau_{1,2}$-$cl(A) \subseteq B^c \subseteq G^c \subseteq H^c$ and therefore $\tau_{1,2}$-$cl(A) \cap H = \phi$. Thus, $p \in (\tau_{1,2}$-$cl_{\theta}(H))^c$ and hence $\tau_{1,2}$-$cl_{\theta}(H) = (1,2)^{\star}$-$\breve{g}$-$cl(H)$.

(2) $\Rightarrow$ (3). It is trivial.

(3) $\Rightarrow$ (1). Let G be any $(1,2)^{\star}$-$\breve{g}$-closed set and $p \in G^c$. Since G is $(1,2)^{\star}$-$\breve{g}$- closed, by assumption $p \in (\tau_{1,2}$-$cl_{\theta}(G))^c$ and so there exists $\tau_{1,2}$-open set A such that $p \in A$ and $\tau_{1,2}$-$cl(A) \cap G = \phi$. Then $G \subseteq (\tau_{1,2}$-$cl(A))^c$. Let $B = (\tau_{1,2}$-$cl(A))^c$. Then B is $\tau_{1,2}$-open such that $G \subseteq B$. Also, the sets A and B are disjoint and hence (X,τ_1,τ_2) is $(1,2)^{\star}$-$\breve{g}$-regular.

Theorem 4.7. *If (X,τ_1,τ_2) is a $(1,2)^{\star}$-$\breve{g}$-regular space and $f:(X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is bijective, $(1,2)^{\star}$-pre-$\mathscr{G}$-open, $(1,2)^{\star}$-$\breve{g}$-continuous and $\tau_{1,2}$-open, then (Y,σ_1,σ_2) is $(1,2)^{\star}$-$\breve{g}$-regular.*

Proof. Let G be any $(1,2)^{\star}$-$\breve{g}$-closed subset of (Y,σ_1,σ_2) and $b \notin G$. Since the function f is $(1,2)^{\star}$-$\breve{g}$-irresolute, we have $f^{-1}(G)$ is $(1,2)^{\star}$-$\breve{g}$-closed in (X,τ_1,τ_2). Since f is bijective, let $f(a) = b$, then $a \notin f^{-1}(G)$. By hypothesis, there exist disjoint $\tau_{1,2}$-open sets A and B such that $a \in A$ and $f^{-1}(G) \subseteq B$. Since f is $\tau_{1,2}$-open and bijective, we have $b \in f(A), G \subseteq f(B)$ and $f(A) \cap f(B) = \phi$. This shows that the space (Y,σ_1,σ_2) is also $(1,2)^{\star}$-$\breve{g}$-regular.

Theorem 4.8. *If $f:(X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is $(1,2)^{\star}$-$\mathscr{G}$-irresolute $(1,2)^{\star}$-$\breve{g}$-closed continuous injection and (Y,σ_1,σ_2) is $(1,2)^{\star}$-$\breve{g}$-regular, then (X,τ_1,τ_2) is $(1,2)^{\star}$-$\breve{g}$-regular.*

Proof. Let G be any $(1,2)^{\star}$-$\breve{g}$-closed set of (X,τ_1,τ_2) and $a \notin G$. Since f is $(1,2)^{\star}$-$\mathscr{G}$-irresolute $(1,2)^{\star}$-$\breve{g}$-closed, $f(G)$ is $(1,2)^{\star}$-$\breve{g}$-closed in (Y,σ_1,σ_2) and $f(a) \notin f(G)$. Since (Y,σ_1,σ_2) is $(1,2)^{\star}$-$\breve{g}$-regular and so there exist disjoint $\tau_{1,2}$-open sets A and B in (Y,σ_1,σ_2) such that $f(a) \subseteq A$ and $f(G) \subseteq B$. i.e., $a \subseteq f^{-1}(A), G \subseteq f^{-1}(B)$ and $f^{-1}(A) \cap f^{-1}(B) = \phi$. Therefore (X,τ_1,τ_2) is $(1,2)^{\star}$-$\breve{g}$-regular.

Theorem 4.9. *If $f:(X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is $(1,2)^{\star}$-weakly continuous $(1,2)^{\star}$-$\breve{g}$-closed injection and (Y,σ_1,σ_2) is $(1,2)^{\star}$-$\breve{g}$-regular, then (X,τ_1,τ_2) is regular.*

Proof. Let G be any $\tau_{1,2}$-closed set of (X,τ_1,τ_2) and $a \notin G$. Since f is $(1,2)^{\star}$-$\breve{g}$-closed, $f(G)$ is $(1,2)^{\star}$-$\breve{g}$-closed in (Y,σ_1,σ_2) and $f(a) \notin f(G)$. Since (Y,σ_1,σ_2) is $(1,2)^{\star}$-$\breve{g}$-regular, there exist $\tau_{1,2}$-open sets A and B such that $f(a) \subseteq A, f(G) \subseteq B$ and $\tau_{1,2}$-$cl(A) \cap \tau_{1,2}$-$cl(B) = \phi$. Since f is $(1,2)^{\star}$-weakly continuous. $a \subseteq f^{-1}(A) \subseteq \tau_{1,2}$-$int(f^{-1}(\tau_{1,2}$-$cl(A))), G \subseteq f^{-1}(B) \subseteq \tau_{1,2}$-$int(f^{-1}(\tau_{1,2}$-$cl(B)))$ and $\tau_{1,2}$-$int(f^{-1}(\tau_{1,2}$-$cl(A))) \cap \tau_{1,2}$-$int(f^{-1}(\tau_{1,2}$-$cl(B))) = \phi$. Therefore, (X,τ_1,τ_2) is regular.

5. Conclusion

One must be in "love" with Mathematics is the intrinsic nature and beauty of Mathematics. As a result, the nature of inquisitiveness in a person gets always en-kindled and triggered by new theorems, axioms, even if it is mighty small in its nature or incredibly big.

Bitopology is applied to many fields such as Mathematics, Physics, Chemistry, Biology, Engineering and so on. This theory is definitely an eye opener for new research works. We can apply these findings into other research areas of general topology such as Fuzzy topology, Intuitionistic topology, Digital Topology, Nano Topology and so on.

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