Finite-time stability of nonlinear fractional systems with damping behavior

G. Arthi[1]* and N. Brindha[2]

Abstract
This paper concentrates with the problem of stability in the finite range of time for nonlinear system with multi term fractional-order and damping behavior. Utilizing the Mittag Leffler functions and generalized Gronwall inequality (GI), a sufficient criteria that ensure the finite time stability (FTS) for both condition $0 < \alpha_1 - \alpha_2 < 1$ and $1 \leq \alpha_1 - \alpha_2 < 2$. Finally, two numerical examples are carried out to verify the obtained results.

Keywords
Finite-time stability; Damped system; Fractional system.

AMS Subject Classification
26A33, 30E25, 34A12, 34A34, 34A37, 37C25, 45J05.

[1,2]*Department of Mathematics, PSGR Krishnammal College for Women, Coimbatore-641004, Tamil Nadu, India*
Corresponding author: [1] arthimath@gmail.com; [2]vnbrindha@gmail.com

Contents

1. Introduction

Calculus of fractional order (FO) is an extension for a traditional calculus which deals with functionals containing integer-order differentiation and integration. This notion has been developed by Leibniz and L'Hopital in 1695 where fractional derivatives was described. In recent years, FO systems have considerable attraction due to their capability to model complex phenomena. By using fractional derivative formulations, physical systems can be modeled more accurately. Also, fractional derivative can be used to modeling the structures in mathematical biology, several chemical processes and problems related to engineering. In real situations, the models generated by FO are more suitable rather than integer order. Since it is possible to model a higher order system by low order system by using FO derivatives. Application of fractional calculus established in stochastic dynamical systems, controlled thermonuclear fusion and plasma physics, image processing, nonlinear control theory [1, 7, 12, 19]. In [2, 3, 5, 13, 17], one can refer the potential applications of FO systems in physical problems description and control, complex practical systems, etc.

The traditional stability concepts like asymptotic stability, Lyapunov stability have been widely studied and these are deals with the problem whose operations described over the infinite interval of time [4, 10, 11, 18]. The concept of asymptotic and exponential stability imply the convergence of system's state to an equilibrium position over the infinite period. Most of the aforementioned results in many fields consider the problems correlate to the performance of convergency described over an interval of infinite period. But in practical process, the predominant analysis is that the characteristic of system in an interval of finite period, since it is too many phyiscally usable than concerning infinite time. In such case, the traditional methods are not appropriate. For such kind, the FTS method is proposed in 1950s. There are two kinds of stability concept over the interval of finite time. One is FTS i.e., the system's state of an asymptotic system reach the equilibrium position in a finite period and another one is fixed-time stability, that means the convergence time intervals have an identical upper-bounds in domain. FTS method is more practical and less conservative than the traditional stability methods. Also, this method is more applicable for analyzing the path of a system's state remains within the prescribed bounds over a finite interval of time. In comparison with asymptotic and other type of stability, the FTS has been

utilized to control the path of a space vehicle from an initial stage to a terminal stage in a described interval of time and also the greater values of the system's states should be reached in all those applications, for example, in the existence of saturation. FTS approach frequently occurs in various practical problems.

In [9], the authors investigated the stability in finite range of time for the system of fractional order with delay equation by make use of the Mittag-Leffler delay type matrix. Hei and Wu [6] analyzed the stability in finite range of time for the fractional impulsive systems with delay by proposed few conditions. By utilizing generalized GI, FTS for the time delayed systems with FO have been proposed in [8], also the FTS analyzed for nonlinear system of FO in [14]. In [20], the authors studied the FTS result for nonlinear FO system involving discrete time delay. For FO there are several approaches to the generalization of integration and differentiation, for example, the Riemann-Liouville, Grunwald-Letnikov, and Caputo derivative approach. This generalization enables one to describe absolutely noninteger order integrals or derivatives. The advantage of using Caputo approach is we can define the initial condition as same as the initial condition defined for integer order models. For this advantage, in this work, we consider the FO Caputo derivatives. However, as far as we know, few results are reported on the FTS of FO systems. The central concept of this work is to study the FTS for nonlinear multi term fractional system by using Mittag Leffler function and GI for both orders $0 < \alpha_1 - \alpha_2 < 1$ and $1 \leq \alpha_1 - \alpha_2 < 2$.

The remaining part of the work consist of: The problem formulation, some necessary definitions and lemmas are provided in Section 2. Main result for FTS analysis are provided in Section 3. In Section 4, the efficiency of the proposed theorems are illustrated by numerical examples. Finally, Section 5 states the conclusion.

2. Preliminaries

This section provides system formulation and some useful properties to derive our required results. Consider the following nonlinear FO system with damping behavior

$$\begin{cases} {}^{C}_{0}D_t^{\alpha_1}y(t) - \mathscr{A} \, {}^{C}_{0}D_t^{\alpha_2}y(t) = f(t,y(t)), \ t \in L, \\ y(0) = y_0, \ y'(0) = y_1. \end{cases} \quad (2.1)$$

Here ${}^{C}_{0}D^{\alpha_1}$ indicates the caputo derivative of FO α_1 with lower limit zero and $L = [0,T]$, state vector $y(t) \in C(L,R^n)$, $\mathscr{A} \in \mathbb{R}^{n \times n}$ and $0 < \alpha_2 \leq 1$, $1 < \alpha_1 \leq 2$, $f : L \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function.

Definition 2.1. *Fractional integral for $h(t)$ interms of Riemann-Liouville with $\alpha_1 \in R^+$ is given by*

$$^{RL}D_{t_0,t}^{\alpha_1}h(t) = \frac{1}{\Gamma(\alpha_1)} \int_{t_0}^{t} (t-\theta)^{\alpha_1-1}h(\theta)d\theta, t > 0,$$

where $\Gamma(\alpha_1) = \int_0^\infty t^{\alpha_1-1}e^{-t}\mathrm{d}t$.

Definition 2.2. *Fractional derivative for $h(t)$ interms of Riemann-Liouville with $\alpha_1 \in R^+$ is given by*

$$^{RL}D_{t_0,t}^{\alpha_1}h(t) = \frac{d^n}{dt^n}\frac{1}{\Gamma(n-\alpha_1)}\int_{t_0}^{t}(t-\theta)^{n-\alpha_1-1} \\ h(\theta)d\theta,$$

with $n-1 < \alpha_1 < n \in Z^+$.

Definition 2.3. *Fractional derivative for $h(t)$ interms of Caputo with $\alpha_1 \in R^+$ is given by*

$$^{C}_{0}D_{t_0,t}^{\alpha_1}h(t) = \frac{1}{\Gamma(n-\alpha_1)}\int_{t_0}^{t}(t-\theta)^{n-\alpha_1-1} \\ h^{(n)}(\theta)d\theta,$$

with $n-1 < \alpha_1 < n \in Z^+$.

Definition 2.4. *[15, 16] The one parameter Mittag Leffler function is given by*

$$E_{\alpha_1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha_1+1)},$$

with $\alpha_1 > 0$, $Re(\alpha_1) > 0$ and $z \in C$. For parameters α_1 and α_2

$$E_{\alpha_1,\alpha_2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha_1+\alpha_2)},$$

with $\alpha_1, \alpha_2 \in C$, $Re(\alpha_1) > 0$, $Re(\alpha_2) > 0$, $z \in C$. By choosing $\alpha_2 = 1$, $E_{\alpha_1,1}(z) = E_{\alpha_1}(z)$.

Definition 2.5. *[15] The Laplace transform for fractional derivative of $h(t)$ interms of Caputo is given by*

$$L\{{}^{C}_{0}D_t^{\alpha_1}h(t)\} = s^{\alpha_1}L(h(t)) - \sum_{r=0}^{m-1} s^{\alpha_1-r-1}h^{(r)}(0).$$

Furthermore, the Laplace transforms of Mittag-Leffler functions is given by

$$L\left[E_{\alpha_1,1}(\pm\lambda t^{\alpha_1})\right](s) = \frac{s^{\alpha_1-1}}{s^{\alpha_1}\mp\lambda},$$
$$Re(\alpha_1) > 0,$$
$$L\left[t^{\alpha_2-1}E_{\alpha_1,\alpha_2}(\pm\lambda t^{\alpha_1})\right](s) = \frac{s^{\alpha_1-\alpha_2}}{s^{\alpha_1}\mp\lambda},$$
$$Re(\alpha_1) > 0, \ Re(\alpha_2) > 0.$$

Definition 2.6. *[10] System (2.1) is finite time stable w.r.t $\{0,L,\delta,\varepsilon\}$, iff $\gamma < \delta$ implies $\|y(t)\| < \varepsilon$ for all $t \in L$, where $\gamma = max\{\|y(0)\|, \|y'(0)\|\}$ is the initial time of observation of system. Also, ε and δ are belongs to $\mathbb{R}^+$.*

Solution of (2.1) using Laplace and Inverse Laplace Transform is defined as

$$\begin{aligned} y(t) = \ & y_0 E_{\alpha_1-\alpha_2}(\mathscr{A}t^{\alpha_1-\alpha_2}) - \mathscr{A}y_0 t^{\alpha_1-\alpha_2} \\ & E_{\alpha_1-\alpha_2,\alpha_1-\alpha_2+1}(At^{\alpha_1-\alpha_2}) + y_1(t)E_{\alpha_1-\alpha_2,2} \\ & (\mathscr{A}t^{\alpha_1-\alpha_2}) + \int_0^t (t-\theta)^{\alpha_1-\alpha_2-1}E_{\alpha_1-\alpha_2,\alpha_1} \\ & (\mathscr{A}(t-\theta)^{\alpha_1-\alpha_2})f(\theta,y(\theta))d\theta \end{aligned} \quad (2.2)$$

Now, we impose the following assumption $(\mathbf{H_1})$: On $[0,T]$, function $f(t,y(t))$ satisfies the Lipschitz condition and $\exists\, M > 0$ such that

$$\|f(t,y(t))\| \quad \leq \quad M\|y(t)\|,\, for\; t \in L, y \in \mathbb{R}^n.$$

Lemma 2.7. *[8]*(**Generalized Gronwall Inequality**)
If $h(t) > 0$ & $v(t) > 0$ is locally integrable on $[0,T]$ and the continuous function $r(t) > 0$ is nondecreasing on $[0,T]$, $\alpha_1 > 0$, $r(t) \leq M$ with

$$h(t) \leq v(t) + r(t)\int_0^t (t-\theta)^{\alpha_1-1}h(\theta)\mathrm{d}\theta, 0 \leq t < T.$$

Then

$$h(t) \leq v(t) + \int_0^t \Big[\sum_{n=1}^{+\infty} \frac{[r(t)\Gamma(\alpha_1)]^n}{\Gamma(n\alpha_1)}(t-\theta)^{n\alpha_1-1}v(\theta)\Big]\mathrm{d}\theta,$$
$$0 \leq t < T.$$

Corollary 2.8. *From the assumption of above Lemma 2.7 and on $[0,T]$, $v(t)$ is a nondecreasing function. Then*

$$h(t) \leq v(t)E_{\alpha_1}\left(r(t)\Gamma(\alpha_1)\right)t^{\alpha_1}). \tag{2.3}$$

Lemma 2.9. *[4]*
(1) There exist M_1 and M_2 which are greater than or equal to one $\ni$ for any $\alpha_1 - \alpha_2(\in \mathbb{R}^+) < 1$,

$$\left\|E_{\alpha_1-\alpha_2,1}(\mathscr{A}t^{\alpha_1-\alpha_2})\right\| \leq M_1\left\|e^{\mathscr{A}t}\right\|,$$

$$\left\|E_{\alpha_1-\alpha_2,\alpha_1-\alpha_2}(\mathscr{A}t^{\alpha_1-\alpha_2})\right\| \leq M_2\left\|e^{\mathscr{A}t}\right\|, \tag{2.4}$$

here $\mathscr{A}$ indicates the matrix.
(2) Suppose $\alpha_1 - \alpha_2(\in \mathbb{R}^+) \geq 1$ then for $\gamma = 1,2,\alpha_1$

$$\left\|E_{\alpha_1-\alpha_2,\gamma}(\mathscr{A}t^{\alpha_1-\alpha_2})\right\| \leq \left\|e^{\mathscr{A}t^{\alpha_1-\alpha_2}}\right\| \tag{2.5}$$

In addition, if $\mathscr{A}$ is a stability matrix, then $\exists$ a constant $N \geq 1$ such that $t > 0$

$$\left\|E_{\alpha_1-\alpha_2,\gamma}(\mathscr{A}t^{\alpha_1-\alpha_2})\right\| \leq Ne^{-\eta t} for\; 0 < \alpha_1 - \alpha_2 < 1$$
$$\left\|E_{\alpha_1-\alpha_2,\gamma}(\mathscr{A}t^{\alpha_1-\alpha_2})\right\| \leq e^{-\eta t} for\; 1 \leq \alpha_1 - \alpha_2 < 2, \tag{2.6}$$

where η be the greatest eigenvalue of $\mathscr{A}$.

3. Main Results

Now, we derive the FTS for a nonlinear damped dynamical system for both fractional orders $0 < \alpha_1 - \alpha_2 < 1$ & $1 \leq \alpha_1 - \alpha_2 < 2$.

Theorem 3.1. *Choose $0 < \alpha_1 - \alpha_2 < 1$ with the assumption $(\mathbf{H_1})$, then FO system with damping behavior (2.1) is finite time stable provided that*

$$Ne^{-\eta t}\left[1 + \|\mathscr{A}\|t^{\alpha_1-\alpha_2} + t\right]$$
$$E_{\alpha_1-\alpha_2}\left(NM\,\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2}\right) < \frac{\varepsilon}{\delta}. \tag{3.1}$$

Proof. Taking norm on both sides of equation (2.2) we get the following,

$$\|y(t)\| \quad \leq \quad \|y_0\|\left\|E_{\alpha_1-\alpha_2}(\mathscr{A}t^{\alpha_1-\alpha_2})\right\| + \|\mathscr{A}\|\|y_0\|\,|t|^{\alpha_1-\alpha_2}$$
$$\left\|E_{\alpha_1-\alpha_2,\alpha_1-\alpha_2+1}(\mathscr{A}t^{\alpha_1-\alpha_2})\right\| + \|y_1\|\,|t|$$
$$\left\|E_{\alpha_1-\alpha_2,2}(\mathscr{A}t^{\alpha_1-\alpha_2})\right\| + \int_0^t |t-\theta|^{\alpha_1-\alpha_2-1}$$
$$\left\|E_{\alpha_1-\alpha_2,\alpha_1}(\mathscr{A}(t-\theta)^{\alpha_1-\alpha_2})\right\| \|f(\theta,y(\theta))\|\mathrm{d}\theta. \tag{3.2}$$

Using Lemma 2.9, equation (3.2) implies,

$$\|y(t)\| \quad \leq \quad \|y_0\|Ne^{-\eta t} + \|\mathscr{A}\|\|y_0\|\,|t|^{\alpha_1-\alpha_2}Ne^{-\eta t}$$
$$+ \|y_1\|\,|t|Ne^{-\eta t} + \int_0^t |t-\theta|^{\alpha_1-\alpha_2-1}Ne^{-\eta(t-\theta)}$$
$$\|f(\theta,y(\theta))\|\mathrm{d}\theta. \tag{3.3}$$

Using hypothesis $(\mathbf{H_1})$ in (3.3), we get

$$\|y(t)\| \quad \leq \quad \|y_0\|Ne^{-\eta t} + \|\mathscr{A}\|\|y_0\|\,|t|^{\alpha_1-\alpha_2}Ne^{-\eta t}$$
$$+ \|y_1\|\,|t|Ne^{-\eta t} + \int_0^t |t-\theta|^{\alpha_1-\alpha_2-1}Ne^{-\eta(t-\theta)}$$
$$M\|y(\theta)\|\mathrm{d}\theta. \tag{3.4}$$

Now multiply both sides of above equation by $e^{\eta t}$, this implies

$$e^{\eta t}\|y(t)\| \quad \leq \quad N\left[\|y_0\| + \|\mathscr{A}\|\|y_0\|\,|t|^{\alpha_1-\alpha_2} + \|y_1\|\,|t|\right]$$
$$+ \int_0^t |t-\theta|^{\alpha_1-\alpha_2-1}Ne^{\eta s}M\|y(\theta)\|\mathrm{d}\theta. \tag{3.5}$$

According to Lemma 2.7, let

$$h(t) \quad = \quad e^{\eta t}\|y(t)\|,$$
$$v(t) \quad = \quad N\left[\|y_0\| + \|A\|\|y_0\|\,|t|^{\alpha_1-\alpha_2} + \|y_1\|\,|t|\right],$$
$$r(t) \quad = \quad NM,$$

From the above it is easily understand that on $[0,T]$, $v(t)$ is nondecreasing function. Hence utilizing the Corollary 2.8 to (3.5), we get

$$h(t) \quad \leq \quad v(t)E_{\alpha_1-\alpha_2}(r(t)\Gamma(\alpha_1-\alpha_2)t^{(\alpha_1-\alpha_2)})$$
$$\leq \quad N\left[\|y_0\| + \|A\|\|y_0\|\,|t|^{\alpha_1-\alpha_2} + \|y_1\|\,|t|\right]$$
$$E_{\alpha_1-\alpha_2}(NM\,\Gamma(\alpha_1-\alpha_2)t^{(\alpha_1-\alpha_2)}). \tag{3.6}$$

Now replacing $h(t)$, which imply that

$$\|y(t)\| \quad \leq \quad Ne^{-\eta t}\left[\|y_0\| + \|\mathscr{A}\|\|y_0\|\,|t|^{\alpha_1-\alpha_2} + \|y_1\|\,|t|\right]$$
$$E_{\alpha_1-\alpha_2}(NM\,\Gamma(\alpha_1-\alpha_2)t^{(\alpha_1-\alpha_2)}). \tag{3.7}$$

When $\|y_0\| \leq \delta$, $\|y_1\| \leq \delta$, the above equation becomes,

$$\|y(t)\| \quad \leq \quad N\delta e^{-\eta t}\left[1 + \|\mathscr{A}\|\,|t|^{\alpha_1-\alpha_2} + |t|\right]$$
$$E_{\alpha_1-\alpha_2}(NM\,\Gamma(\alpha_1-\alpha_2)t^{(\alpha_1-\alpha_2)}),$$

From the statement of the theorem we can get the following,

$$\|y(t)\| \leq \varepsilon, for\; all\; t \in [0,T).$$

This implies (2.1) is finite time stable for the interval
$0 < \alpha_1 - \alpha_2 < 1$. □

Theorem 3.2. *If $1 \leq \alpha_1 - \alpha_2 < 2$ with the condition $(\mathbf{H_1})$ holds. Then system (2.1) is finite time stable provided that*

$$e^{-\eta t}\left[1 + \|\mathscr{A}\|t^{\alpha_1-\alpha_2} + t\right]E_{\alpha_1-\alpha_2}\left(M\,\Gamma(\alpha_1-\alpha_2)t^{\alpha_1-\alpha_2}\right) < \frac{\varepsilon}{\delta} \tag{3.8}$$

for any $t \in [0,T)$.

Proof. Proof is obtained by proceeding the same steps followed in Theorem 3.1 by using Lemma 2.9. □

Corollary 3.3. *The multi term fractional linear equation,*

$$\begin{cases} {}^{C}_{0}D^{\alpha_1}_t y(t) - \mathscr{A}\, {}^{C}_{0}D^{\alpha_2}_t y(t) = \mathscr{B}y(t),\ t \in L, \\ y(0) = y_0,\ y'(0) = y_1, \end{cases} \tag{3.9}$$

is finite time stable for $0 < \alpha_1 - \alpha_2 < 1$, if

$$Ne^{-\eta t}\left[1 + \|\mathscr{A}\| t^{\alpha_1 - \alpha_2} + t\right]$$

$$E_{\alpha_1 - \alpha_2}\left(\|\mathscr{B}\| N\, \Gamma(\alpha_1 - \alpha_2)t^{\alpha_1 - \alpha_2}\right) < \frac{\varepsilon}{\delta}. \tag{3.10}$$

Corollary 3.4. *System* (3.9) *is finite time stable for $1 \le \alpha_1 - \alpha_2 < 2$, if*

$$e^{-\eta t}\left[1 + \|\mathscr{A}\| t^{\alpha_1 - \alpha_2} + t\right] E_{\alpha_1 - \alpha_2}\left(\|\mathscr{B}\|\, \Gamma(\alpha_1 - \alpha_2)t^{\alpha_1 - \alpha_2}\right) < \frac{\varepsilon}{\delta}.$$

4. Example

Example 4.1. *Consider the nonlinear FO system with damping behavior*

$$\begin{cases} {}^{C}_{0}D^{\alpha_1}_t y_1(t) - {}^{C}_{0}D^{\alpha_2}_t y_2(t) = \sqrt{y_1^2(t) + 5}, \\ {}^{C}_{0}D^{\alpha_1}_t y_2(t) - {}^{C}_{0}D^{\alpha_2}_t y_1(t) = 0, \end{cases}$$

where $\alpha_1 = 1.25$ and $\alpha_2 = 0.75$. Now we consider the system (2.1) *with the following parameters*

$$\mathscr{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \text{ and } f(t, y(t)) = \begin{bmatrix} \sqrt{y_1^2(t) + 5} \\ 0 \end{bmatrix}.$$

Evidently, the hypothesis $(H1)$ is satisfied for $M = 1$. Now to validate the FTS condition (3.1) *w.r.t $\eta = 1$ and $\|\mathscr{A}\| = 1$ from Theorem* 3.1. *Let us choose $\delta = 0.05, N = 1.5, \varepsilon = 1$, then from inequality* (3.1), *we can obtain the estimated time of FTS is $T \approx 0.2301$.*

Example 4.2. *Consider the system* (3.9) *with the parameters $\alpha_1 = 1.25, \alpha_2 = 0.75$, $\mathscr{A} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}$ and $\mathscr{B} = \begin{bmatrix} 0 & -1 \\ 2 & 0 \end{bmatrix}$.*

Let us choose $N = 2$, $\varepsilon = 1$, $\delta = 0.05$. Now to validate the FTS condition (3.10) *w.r.t $\eta = 3$, $\|\mathscr{A}\| = 3.6503$ and $\|\mathscr{B}\| = 2$. Hence the inequality* (3.10) *implies,*

$$2e^{-3t}\left[1 + t + 3.6503t^{0.5}\right]E_{0.5}\left(7.09t^{0.5}\right) < 20.$$

From Corollary 3.3, *we can obtain the estimated time of FTS is $T \approx 0.502$.*

5. Conclusion

We analyzed the stability in the finite range of time for a nonlinear FO systems with damping behaviour. So far many authors investigated about the FTS result for linear and nonlinear fractional systems. In literature, the stability result in the finite range of time for this type of nonlinear system with damping behaviour not yet been studied. By using the Laplace and Inverse Laplace Transforms, Mittag Leffler function, Caputo derivative and GI, a few conditions are proposed to ensure the FTS result for both conditions $0 < \alpha_1 - \alpha_2 < 1$ and $1 \le \alpha_1 - \alpha_2 < 2$ involving damping behavior. Finally, the results are verified through examples.

References

[1] S. Abbas, M. Benchohra and G. M. N'Guérékata, *Topics in Fractional Differential Equations*, New York: Springer-Verlag, 2012.

[2] S. Das, *Functional Fractional Calculus for System Identification and Controls*, New York: Springer-Verlag, 2008.

[3] S. Das, *Functional Fractional Calculus*, New York: Springer-Verlag, 2011.

[4] M. De la Sen, About robust stability of Caputo linear fractional dynamic systems with time delays through fixed point theory, *Fixed Point Theory and Applications*, 2011 (2011) 867932.

[5] W. M. Haddad and A. L'Afflitto, Finite-time stabilization and optimal feedback control, *IEEE Transactions on Automatic Control*, 61 (2016) 1069-1074.

[6] X. Hei and R. Wu, Finite-time stability of impulsive fractional-order systems with time-delay, *Applied Mathematical Modelling*, 40 (2016) 4285-4290.

[7] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Application of Fractional Differential Equations*, Elsevier B.V, 2006.

[8] M. P. Lazarevic and A. M. Spasic, Finite-time stability analysis of fractional order time-delay systems: Gronwall's approach, *Mathematical and Computer Modelling*, 49 (2009) 475-481.

[9] M. Li and J. Wang, Finite time stability of fractional delay differential equations, *Applied Mathematics Letters*, 64 (2017) 170-176.

[10] C. Liang, W. Wei and J. Wang, Stability of delay differential equations via delayed matrix sine and cosine of polynomial degrees, *Advances in Difference Equations*, 1 (2017) 131.

[11] P. Mahajan, S. K. Srivastava and R. Dogra, Uniform practical stability of perturbed impulsive differential system in terms of two measures, *Malaya Journal of Matematik*, 7 (2019) 142-146.

[12] K. B. Oldham and J. Spanier, *The Fractional Calculus*, New York: Academic Press, 1974.

[13] I. Petras, *Fractional-Order Nonlinear Systems: Modeling, Analysis and Simulation*, Springer Science and Business Media, 2011.

[14] V. N. Phat and N. T. Thanh, New criteria for finite-time stability of nonlinear fractional-order delay systems: A Gronwall inequality approach, *Applied Mathematics Letters*, 83 (2018) 169-175.

[15] I. Podlubny, *Fractional Differential Equations*, Mathematics in Science and Engineering, Academic Press, New York, 1998.

[16] I. Podlubny, What Euler could further write, or the unnoticed "big bang" of the fractional calculus, *Fractional Calculus and Applied Analysis*, 16 (2013) 501-506.

[17] S. G. Smako, A. A. Kilbas and O. Marchev, *Fractional Integrals and Derivatives: Theory and Applications*, USA : Gordon and Breach Science Publishers, 1993.

[18] A. Sood and S. K. Srivastava, Lyapunov approach for stability of integro differential equations with non instantaneous impulse effect, *Malaya Journal of Matematik*, 4 (2016) 119-125.

[19] V. E. Tarasov, *Fractional Dynamics: Applications of Fractional*

Calculus to Dynamics of Particles, Fields and Media, New York: Springer-Verlag, 2011.

[20] F. Wang, D. Chen, X. Zhang and Y. Wu, Finite-time stability of a class of nonlinear fractional-order system with the discrete time delay, *International Journal of Systems Science*, 48 (2017) 984-993.