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The characterizations of some special Frenet curves in Minkowski 3-space

Başak Özülkü Engin[1] and Ahmet Yücesan[2]*

Abstract

We derive a general differential equation satisfied by the distance function for non-null Frenet curves in Minkowski 3-space. By using this differential equation, we easily express the well-known characterizations of non-null some special Frenet curves which are pseudo-spherical curves and rectifying curves. Then we get a new characterization of general helix. Lastly, we characterize non-null pseudo-spherical curves with respect to centrode and co-centrode. Similarly, we derive a general differential equation satisfied by the distance function for null Frenet curves. By means of this differential equation we see that there is not exist null Frenet curve lies on pseudo-sphere and we get the well-known characterization of null rectifying curves. Finally, we find a new characterization for null general helix and we obtain characterization null general helix with respect to centrode and co-centrode.

Keywords

Pseudo-spherical curve, rectifying curve, general helix, centrode, co-centrode.

AMS Subject Classification

53A35, 53B30, 53C50.

[1] *Graduate School of Natural and Applied Sciences, Süleyman Demirel University,* 32200 *Isparta, TURKEY.*

[2] *Department of Mathematics, Süleyman Demirel University,* 32200 *Isparta, TURKEY.*

***Corresponding author**: [1] matematikbasak@gmail.com; [2]ahmetyucesan@sdu.edu.tr

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Contents

1. Introduction

One of the study topics of the theory of curves in differential geometry is to characterize some special curves. Some special curves in Euclidean 3−space $\mathbb{R}^3$ and Minkowski 3−space $\mathbb{R}^3_1$ are characterized by curvature and torsion of curves ([1–6, 9, 11, 13, 15–18, 20, 21]). In 2018, Deshmukh et al.[7] prove that every unit speed Frenet curve in Euclidean 3−space $\mathbb{R}^3$ satisfies a general differential equation based on curvature, torsion and distance function. As a result of this differential equation they give well-known characterizations of spherical and rectifying curves. Then, they derive a new characterization for general helices. Finally, they obtain a simple new characterization of spherical curves relative to the centrode and co-centrode.

As it is known, since a curve in Minkowski 3−space $\mathbb{R}^3_1$ has different causal characters, it is interesting to characterize some special curves in Minkowski 3−space. Therefore, in this study, we characterize both non-null and null special curves according to curvature, torsion and distance function. First of all, let us briefly recall the structure of Minkowski 3−space $\mathbb{R}^3_1$.

Let $\mathbb{R}^3_1$ be Minkowski 3−space (or the 3−dimensional Lorentzian space) with symmetric, bilinear and non-degenerate metric called as Lorentzian metric defined by

$$\langle u,v\rangle = -u_1v_1 + u_2v_2 + u_3v_3$$

for vectors $u=(u_1,u_2,u_3)$ and $v=(v_1,v_2,v_3)$ in Euclidean 3−space $\mathbb{R}^3$. The Lorentzian vector product of u and v is given by

$$u\times v = (u_3v_2 - u_2v_3, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1).$$

Thus, according to the structure of Lorentzian metric, a vector is named in three different categories as spacelike, timelike and null (lightlike). If $<u,u> > 0$ (or $u=0$), $<u,u> < 0$, $<u,u> = 0$ (and $u\neq 0$) then a vector u is named to be spacelike, timelike or null, respectively. We know that a curve in Minkowski 3−space $\mathbb{R}^3_1$ is called according to its velocity vector. Therefore, a curve is said to spacelike, timelike or null if its velocity vector is spacelike, timelike or null vector, respectively. Lastly, a surface is named non-degenerate (or degenerate) if induced metric on its tangent plane is non degenerate (or degenerate). As known, the most familiar non-degenerate surfaces are pseudo-sphere

$$S^2_1 = q^{-1}(r^2) = \{p\in\mathbb{R}^3_1 : -p_1^2 + p_2^2 + p_3^2 = r^2\}$$

and pseudo-hyperbolic space

$$H^2_0 = q^{-1}(-r^2) = \{p\in\mathbb{R}^3_1 : -p_1^2 + p_2^2 + p_3^2 = -r^2\}$$

([12, 14]).

Let $\gamma : I\subset\mathbb{R}\to\mathbb{R}^3_1$ be a unit speed that is $<\gamma'(s),\gamma'(s)> = \varepsilon_0$ (ε_0 being $+1$ or -1 according to γ is spacelike or timelike, respectively), where s is said to the arc length parameter. A unit speed curve $\gamma(s)$ in $\mathbb{R}^3_1$ is called a Frenet curve if it has a Frenet frame $\{T=\gamma'(s),N,B\}$, where T is the unit tangent vector of γ, N is the unit principal normal of γ and $\varepsilon_2 B = T\times N$ is the unit binormal vector of γ such that $\varepsilon_2 = <B,B> = sign(B) = \pm 1$. The Frenet frame provides the Frenet formulas given by

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon_1\kappa & 0 \\ -\varepsilon_0\kappa & 0 & \varepsilon_2\tau \\ 0 & -\varepsilon_1\tau & 0 \end{pmatrix}\begin{pmatrix} T \\ N \\ B \end{pmatrix}, \quad (1.1)$$

where $\varepsilon_0 = <T,T> = sign(T) = \pm 1$, $\varepsilon_1 = <N,N> = sign(N) = \pm 1$. Also $\kappa=\kappa(s)$ and $\tau=\tau(s)$ are the curvature and torsion of γ, respectively (see [12, 14]). By using Frenet equations (1.1), we have

$$<\gamma(s),T>' = \varepsilon_0 + \varepsilon_1\kappa <\gamma(s),N>, \quad (1.2)$$

$$<\gamma(s),N>' = -\varepsilon_0\kappa <\gamma(s),T> + \varepsilon_2\tau <\gamma(s),B> \quad (1.3)$$

and

$$<\gamma(s),B>' = -\varepsilon_1\tau <\gamma(s),N>. \quad (1.4)$$

Now, we assume that γ is a null Frenet curve parameterized by the pseudo-arc parameter s in $\mathbb{R}^3_1$. We choose the $T=\gamma'(s)$ and $N=\gamma''(s)$, then there exists only one null frame $\{T,N,B\}$ for which $\gamma(s)$ is a framed null curve with Frenet equations:

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\tau & 0 & -1 \\ 0 & \tau & 0 \end{pmatrix}\begin{pmatrix} T \\ N \\ B \end{pmatrix}, \quad (1.5)$$

such that

$$\begin{aligned} <T,T> = <B,B> = <T,N> = <B,N> = 0, \\ <T,B> = <N,N> = 1, \end{aligned} \quad (1.6)$$

where $B=\gamma'''(s) - \tau\gamma'(s)$ and $\tau = \frac{1}{2} <\gamma'''(s),\gamma'''(s)>$ is called the lightlike curvature of γ (see [8, 10, 19]).

2. The characterization of some special non-null Frenet curves in Minkowski 3-space

In this section, we derive a general differential equation which satisfied by distance function for a unit speed Frenet curve in Minkowski 3−space $\mathbb{R}^3_1$. This differential equation plays a major key to characterization of curves as pseudo-spherical curve, rectifying curve and general helix. In line with our purpose, we firstly give the following theorem.

Theorem 2.1. *Let $\gamma=\gamma(s)$ be a unit speed Frenet curve in Minkowski 3−space $\mathbb{R}^3_1$. Then $\gamma(s)$ satisfies the differential equation given by*

$$\begin{aligned} &\delta\varepsilon_1\varepsilon_2\rho\sigma h''' + \delta\varepsilon_1\varepsilon_2(\rho\sigma' + 2\rho'\sigma)h'' + \delta(\varepsilon_1\varepsilon_2(\sigma\rho')' \\ &+\varepsilon_0\varepsilon_2\frac{\sigma}{\rho} + \frac{\rho}{\sigma})h' + \delta\varepsilon_0\varepsilon_2(\frac{\sigma}{\rho})'h = \varepsilon_0\varepsilon_1\varepsilon_2(\sigma\rho')' + \varepsilon_0\frac{\rho}{\sigma}, \end{aligned} \quad (2.1)$$

where $\rho=\kappa^{-1}$, $\sigma=\tau^{-1}$, $h(s)=d(s)d'(s)$ and $\delta = sign(\gamma(s)) = \dfrac{<\gamma(s),\gamma(s)>}{|<\gamma(s),\gamma(s)>|} = \mp 1$.

Proof. The square distance function of an unit speed Frenet curve $\gamma=\gamma(s)$ is

$$d^2(s) = \delta\langle\gamma(s),\gamma(s)\rangle,$$

where $\delta = sign(\gamma(s)) = \mp 1$. We begin with the derivative of the square distance function with respect to s. Then

$$d(s)d'(s) = \delta\langle\gamma(s),\gamma'(s)\rangle. \quad (2.2)$$

Substituting $h(s)=d(s)d'(s)$ and $\gamma'(s)=T$ in (2.2), we write

$$h(s) = \delta\langle\gamma(s),T\rangle. \quad (2.3)$$

Differentiation of (2.3) with respect to s and using (1.2), we obtain

$$h'(s) = \delta\varepsilon_0 + \delta\varepsilon_1\kappa\langle\gamma(s), N\rangle$$

or

$$\delta\varepsilon_1\rho(h'(s) - \delta\varepsilon_0) = \langle\gamma(s), N\rangle. \quad (2.4)$$

Similarly, we differentiate (2.4) with respect to s and using (1.3), we have

$$\begin{aligned}\delta\varepsilon_1\rho h''(s) + \delta\varepsilon_1\rho'(h'(s) - \delta\varepsilon_0) =& \varepsilon_0\kappa\langle\gamma(s), T\rangle \\ &+ \varepsilon_2\tau\langle\gamma(s), B\rangle\end{aligned} \quad (2.5)$$

If we write Eq. (2.3) in (2.5), then we see that

$$\begin{aligned}&\delta\varepsilon_1\varepsilon_2\rho\sigma h''(s) + \delta\varepsilon_1\varepsilon_2\sigma\rho' h'(s) - \varepsilon_0\varepsilon_1\varepsilon_2\sigma\rho' \\ &\quad + \delta\varepsilon_0\varepsilon_2\frac{\sigma}{\rho}h(s) = \langle\gamma(s), B\rangle.\end{aligned} \quad (2.6)$$

Finally, by differentiating (2.6) with respect to s, using (1.4) and (2.4) we find

$$\begin{aligned}&\delta\varepsilon_1\varepsilon_2\rho\sigma h'''(s) + \delta\varepsilon_1\varepsilon_2(\rho\sigma)'h''(s) + \delta\varepsilon_1\varepsilon_2(\sigma\rho')'h'(s) \\ &\quad + \delta\varepsilon_1\varepsilon_2\sigma\rho' h''(s) - \varepsilon_0\varepsilon_1\varepsilon_2(\sigma\rho')' + \delta\varepsilon_0\varepsilon_2(\frac{\sigma}{\rho})'h(s) \\ &\quad + \delta\varepsilon_0\varepsilon_2\frac{\sigma}{\rho}h'(s) = -\delta\tau\rho(h'(s) - \delta\varepsilon_0).\end{aligned} \quad (2.7)$$

If we arrange Eq. (2.7), then we obtain the required differential equation (2.1). □

2.1 Some conclusions regarding characterizations of pseudo-spherical curves and rectifying curves

Now, we can give the following consequences of Theorem 2.1 for characterization of pseudo-spherical curves and rectifying curves.

Corollary 2.2. *A unit speed Frenet curve γ in Minkowski $3-$space $\mathbb{R}^3_1$ is a pseudo-spherical curve if and only if γ satisfies the differential equation*

$$\varepsilon_1\varepsilon_2(\sigma\rho')' + \frac{\rho}{\sigma} = 0. \quad (2.8)$$

Proof. Let γ be a unit speed pseudo-spherical curve that is γ lies on the pseudo-sphere or the pseudo-hyperbolic space of radius r. Then

$$< \gamma(s), \gamma(s) >= \delta r^2$$

which implies the square distance function $d^2(s) = r^2$. So we see that $h = dd' = 0$. Substituting $h = 0$ and $h' = h'' = h''' = 0$ in the differential equation (2.1), we find Eq. (2.8).

Conversely, let γ be a unit speed Frenet curve which satisfies Eq. (2.8). Multiplying with $2\sigma\rho'$ both sides of Eq. (2.8) gives

$$2\varepsilon_1\varepsilon_2(\sigma\rho')(\sigma\rho')' + 2\rho\rho' = 0. \quad (2.9)$$

By integrating (2.9), we obtain

$$\varepsilon_1\varepsilon_2(\sigma\rho')^2 + \rho^2 = const.,$$

that is, γ is unit speed pseudo-spherical curve implies from [15–18]. □

Corollary 2.3. *A unit speed Frenet curve γ in Minkowski $3-$space $\mathbb{R}^3_1$ is a rectifying curve if and only if γ satisfies the differential equation*

$$(s+c)\left(\frac{\sigma}{\rho}\right)' + \frac{\sigma}{\rho} = 0, \quad (2.10)$$

for some constant c.

Proof. We assume that γ is a unit speed rectifying curve that is the position vector of γ lies in the rectifying plane of γ. Then the square distance function is given by

$$d^2(s) = \mid \varepsilon_0 s^2 + c_1 s + c_2 \mid \quad (2.11)$$

for some constants c_1, c_2. By differentiating Eq. (2.11) and using $h(s) = d(s)d'(s)$, we find

$$h(s) = \delta\varepsilon_0(s+c),$$

where $c = \frac{\varepsilon_0 c_1}{2}$. Substituting $h(s) = \delta\varepsilon_0(s+c)$, $h'(s) = \delta\varepsilon_0$ and $h''(s) = h'''(s) = 0$ in Eq. (2.1), we have Eq. (2.10).

Conversely, assume that a unit speed Frenet curve γ satisfies Eq. (2.10). Then by integrating (2.10), we find

$$(s+c)(\frac{\sigma}{\rho}) = \overline{c}$$

for every constant $\overline{c}$. Hence we see that

$$\frac{\tau}{\kappa} = as + b$$

where $a = \frac{1}{\overline{c}}$ and $b = \frac{c}{\overline{c}}$. This implies that γ is a rectifying curve from [11]. □

Corollary 2.4. *A unit speed Frenet curve γ with $\delta = +1$ ($\delta = -1$, resp.) in Minkowski $3-$space $\mathbb{R}^3_1$ satisfies*

$$\varepsilon_1\langle\gamma(s), N\rangle^2 + \varepsilon_2\langle\gamma(s), B\rangle^2 = c^2 \quad (2.12)$$

for a constant c if and only if either γ is a pseudo-spherical curve or γ is a rectifying curve.

Proof. We suppose that γ is a unit speed Frenet curve with $\delta = +1$ ($\delta = -1$, resp.) which satisfies the condition (2.12). Taking into consideration the condition (2.12), we see that the square distance function of γ is

$$\delta d^2(s) = \varepsilon_0\langle\gamma(s), T\rangle^2 + c^2. \quad (2.13)$$

By differentiating (2.13) with respect to s and using $h(s) = d(s)d'(s)$, (2.2), (1.1) and (2.3), we find

$$h(s)(1 - \delta\varepsilon_0 h'(s)) = 0.$$

Hence either $h(s) = 0$ or $h^{'}(s) = \varepsilon_0\delta$ that is $h(s) = \delta\varepsilon_0(s+b)$ for constant b. As a conclusion either γ is a pseudo-spherical curve or γ is a rectifying curve.

Conversely, if a unit speed Frenet curve γ is a pseudo-spherical curve or γ is a rectifying curve, then γ satisfies (2.12). □

2.2 A new characterization of general helix

We gives a new characterization of general helix by using Theorem 2.1.

Theorem 2.5. *A unit speed Frenet curve γ in Minkowski $3-$space $\mathbb{R}^3_1$ is a general helix if and only if the function $h(s) = d(s)d^{'}(s)$ with respect to arc length parameter s satisfies the differential equation*

$$\varepsilon_1\delta(\rho h')' + \left(\varepsilon_2\frac{\rho}{\sigma^2} + \varepsilon_0\frac{1}{\rho}\right)\delta h = \varepsilon_0\varepsilon_1\rho' + \varepsilon_2\frac{\rho}{\sigma^2}(\varepsilon_0 s + b), \tag{2.14}$$

where $d^2(s) = \delta < \gamma(s), \gamma(s) >$, $\rho = \kappa^{-1}$, $\sigma = \tau^{-1}$ and $\delta = sign(\gamma(s)) = \pm 1$.

Proof. Let γ be a general helix with a vector u lying on the axis of γ. Assume, without loss of generality, that $< u, u >= \varepsilon$, where $\varepsilon = -1, 0, 1$. Then

$$\langle T, u\rangle = c$$

for constant c and

$$\langle N, u\rangle = 0.$$

On the other hands, we can write

$$u = \varepsilon_0\langle u, T\rangle T + \varepsilon_2\langle u, B\rangle B$$

and

$$\varepsilon = \varepsilon_0 c^2 + \varepsilon_2\langle u, B\rangle^2.$$

Hence we get

$$\langle u, B\rangle = \sqrt{\varepsilon\varepsilon_2 - \varepsilon_0\varepsilon_2 c^2}.$$

and we deduce that

$$u = \varepsilon_0 cT + \varepsilon_2\sqrt{\varepsilon\varepsilon_2 - \varepsilon_0\varepsilon_2 c^2}B. \tag{2.15}$$

Differentiating (2.15) with respect to s and using the Frenet equations (1.1) we find

$$\varepsilon_0\kappa c = \varepsilon_2\tau\sqrt{\varepsilon\varepsilon_2 - \varepsilon_0\varepsilon_2 c^2}. \tag{2.16}$$

Since

$$\langle\gamma(s), u\rangle' = c,$$

we get

$$\langle\gamma(s), u\rangle = cs + c_1 \tag{2.17}$$

for constant c_1. Substituting (2.15) in (2.17) and using (2.3), we have

$$\delta h(s) = \varepsilon_0 s + b - \frac{\varepsilon_0\varepsilon_2\sqrt{\varepsilon\varepsilon_2 - \varepsilon_0\varepsilon_2 c^2}}{c}\langle\gamma(s), B\rangle, \tag{2.18}$$

where $b = \dfrac{\varepsilon_0 c_1}{c}$. By differentiating (2.18) with respect to s and using (1.4) and (2.16), we obtain

$$\varepsilon_1\rho\left(\delta h'(s) - \varepsilon_0\right) = \langle\gamma(s), N\rangle.$$

Again by differentiating with respect to s of above the equation and using (1.3), (2.3) and (2.18), we get

$$\begin{aligned}\varepsilon_1\delta\rho h''(s) + \varepsilon_1\rho'(\delta h'(s) - \varepsilon_0) &= -\varepsilon_0\kappa\delta h(s) + \varepsilon_2\tau\langle\gamma(s), B\rangle \\ &= \frac{\varepsilon_0 c(\varepsilon_0 s + b)}{\sigma\sqrt{\varepsilon_2\varepsilon - \varepsilon_0\varepsilon_2 c^2}} \\ &\quad -\varepsilon_0\delta h\left(\frac{1}{\rho} + \frac{c}{\sigma\sqrt{\varepsilon\varepsilon_2 - \varepsilon_0\varepsilon_2 c^2}}\right).\end{aligned} \tag{2.19}$$

On the other hand, from Eq. (2.16) we see that

$$\frac{\rho}{\sigma} = \frac{\varepsilon_0\varepsilon_2 c}{\sqrt{\varepsilon\varepsilon_2 - \varepsilon_0\varepsilon_2 c^2}}. \tag{2.20}$$

Substituting (2.20) in (2.19), we obtain (2.14).

Conversely, assume that a unit speed Frenet curve γ satisfies Eq. (2.14). We multiply both sides of Eq. (2.14) by σ and differentiate with respect to s. Then we get

$$\begin{aligned}\varepsilon_1\varepsilon_2\delta\sigma\rho h''' + \varepsilon_1\varepsilon_2\delta(2\sigma\rho' + \sigma'\rho)h'' + \varepsilon_2\delta\Big[\varepsilon_1(\sigma\rho')' \\ + \left(\varepsilon_2\frac{\rho}{\sigma} + \varepsilon_0\frac{\sigma}{\rho}\right)\Big]h' + \varepsilon_2\delta\left(\varepsilon_2\frac{\rho}{\sigma} + \varepsilon_0\frac{\sigma}{\rho}\right)'h \\ -\varepsilon_0\varepsilon_1\varepsilon_2(\sigma\rho')' = \varepsilon_0\frac{\rho}{\sigma} + \left(\frac{\rho}{\sigma}\right)'(\varepsilon_0 s + b).\end{aligned} \tag{2.21}$$

By comparing (2.21) with (2.1), we see that

$$\left(\frac{\rho}{\sigma}\right)'(\delta h - \varepsilon_0 s - b) = 0.$$

If $h(s) = \delta(\varepsilon_0 s + b)$, then (2.14) reduces to

$$\varepsilon_0\left(\frac{1}{\rho}\right)(\varepsilon_0 s + b) = 0$$

which implies $\frac{1}{\rho} = 0$ that is $\kappa = 0$. Because γ is a unit speed Frenet curve, it is impossible. Thus $\left(\frac{\rho}{\sigma}\right)' = 0$, that is $\frac{\tau}{\kappa} = const$. Consequently, a unit speed Frenet curve γ is a general helix. □

2.3 A new characterization of non-null pseudo-spherical curves with respect to centrode and co-centrode

Let γ be a unit speed Frenet curve in Minkowski 3−space $\mathbb{R}^3_1$. The centrode ∂ and the co-centrode ∂^* of γ are given by the Darboux vector

$$\partial = \tau T + \kappa B \tag{2.22}$$

and the co-Darboux vector

$$\partial^* = -\varepsilon_0 \kappa T + \varepsilon_2 \tau B, \tag{2.23}$$

respectively. We can clearly see that the co-Darboux vector is equal to the derivative of the principal normal vector N of γ that is $N^{'} = \partial^*$ and $\{N,\ \partial,\ \partial^*\}$ become an orthogonal frame along γ.

By differentiating (2.22) and using the Frenet equations (1.1) yields

$$\partial' = \tau' T + \kappa' B.$$

So $\partial^{'}$ is always orthogonal to the co-centrode ∂^* if and only if

$$\tau^{'}\kappa - \tau\kappa^{'} = 0$$

which implies that γ is a general helix. Therefore we can statement that the Frenet curve γ is a general helix if and only if $\partial^{'}$ is always orthogonal to the co-centrode ∂^*.

Now, we give the following a new characterization of non-null pseudo-spherical curves with respect to the centrode ∂ and the co-centrode ∂^* co-centrode.

Theorem 2.6. *Let γ be a unit speed Frenet curve in Minkowski 3−space $\mathbb{R}^3_1$. γ is a pseudo-spherical curve if and only if*

$$\frac{\tau}{\kappa} = \varepsilon_2 \frac{\langle \gamma(s), \partial^* \rangle}{\langle \gamma(s), \partial \rangle} \quad \textit{with} \quad \varepsilon_0 \tau^2 + \varepsilon_2 \kappa^2 \neq 0. \tag{2.24}$$

Proof. Let γ be a unit speed Frenet curve in Minkowski 3−space $\mathbb{R}^3_1$. Because $\{N,\ \partial,\ \partial^*\}$ is orthogonal along γ, by using (2.22) and (2.23) the position vector of $\gamma(s)$ can be written

$$\gamma(s) = \varepsilon_1 \langle \gamma(s), N \rangle N + \varepsilon_3 \frac{\langle \gamma(s), \partial \rangle}{\|\partial\|} \partial + \varepsilon_4 \frac{\langle \gamma(s), \partial^* \rangle}{\|\partial^*\|} \partial^*.$$

where $\varepsilon_3 = sign(\partial) = \pm 1$ and $\varepsilon_4 = sign(\partial^*) = \pm 1$. Then the square distance function of γ satisfies

$$\delta d^2 = \varepsilon_1 \langle \gamma(s), N \rangle^2 + \frac{\varepsilon_3}{\|\partial\|^2} \langle \gamma(s), \partial \rangle^2 + \frac{\varepsilon_4}{\|\partial^*\|^2} \langle \gamma(s), \partial^* \rangle^2. \tag{2.25}$$

If we take derivative of (2.25) with respect to s and use the Frenet equations (1.1), then we have

$$\begin{aligned}\delta d d' =& \varepsilon_1 \langle \gamma(s), N \rangle \langle \gamma(s), \partial^* \rangle - \frac{\langle \partial, \partial^{'} \rangle}{\|\partial\|^4} \langle \gamma(s), \partial \rangle^2 \\ &+ \frac{\varepsilon_3}{\|\partial\|^2} \langle \gamma(s), \partial \rangle (\varepsilon_0 \tau + \langle \gamma(s), \partial^{'} \rangle) \\ &- \frac{\langle \partial^*, \partial^{*'} \rangle}{\|\partial^*\|^4} \langle \gamma(s), \partial^* \rangle^2 \\ &+ \frac{\varepsilon_4}{\|\partial^*\|^2} \langle \gamma(s), \partial^* \rangle (-\kappa + \langle \gamma(s), \partial^{*'} \rangle).\end{aligned} \tag{2.26}$$

On the other hand, from (2.22), (2.23) and (1.1) we get

$$\begin{aligned}\langle \gamma(s), \partial^{'} \rangle =& \tau^{'} \langle \gamma(s), T \rangle + \kappa^{'} \langle \gamma(s), B \rangle, \\ \langle \gamma(s), \partial^{*'} \rangle =& \varepsilon_0 \kappa^{'} \langle \gamma(s), T \rangle - \varepsilon_1 (\varepsilon_0 \kappa^2 + \varepsilon_2 \tau^2) \langle \gamma(s), N \rangle \\ &+ \varepsilon_2 \tau^{'} \langle \gamma(s), B \rangle.\end{aligned} \tag{2.27}$$

Substituting (2.22), (2.23) and (2.27) in (2.26), we arrive

$$\begin{aligned}\delta d d' =& -\frac{\varepsilon_0 \tau \tau' + \varepsilon_2 \kappa \kappa'}{\|\partial\|^4} (\tau^2 \langle \gamma(s), T \rangle^2 + \kappa^2 \langle \gamma(s), B \rangle^2 \\ &+ 2\kappa\tau \langle \gamma(s), T \rangle \langle \gamma(s), B \rangle) \\ &- \frac{\varepsilon_0 \kappa \kappa' + \varepsilon_2 \tau \tau'}{\|\partial^*\|^4} (\kappa^2 \langle \gamma(s), T \rangle^2 \\ &+ \tau^2 \langle \gamma(s), B \rangle^2 - 2\varepsilon_0 \varepsilon_2 \kappa \tau \langle \gamma(s), T \rangle \langle \gamma(s), B \rangle) \\ &+ \frac{1}{\varepsilon_0 \tau^2 + \varepsilon_2 \kappa^2} (\varepsilon_0 \tau^2 \langle \gamma(s), T \rangle + \tau \tau^{'} \langle \gamma(s), T \rangle^2 \\ &+ \varepsilon_0 \kappa \tau \langle \gamma(s), B \rangle + \kappa \kappa^{'} \langle \gamma(s), B \rangle^2) \\ &+ \frac{1}{\varepsilon_0 \kappa^2 + \varepsilon_2 \tau^2} (\varepsilon_0 \kappa^2 \langle \gamma(s), T \rangle + \kappa \kappa^{'} \langle \gamma(s), T \rangle^2 \\ &- \varepsilon_2 \kappa \tau \langle \gamma(s), B \rangle + \tau \tau' \langle \gamma(s), B \rangle^2).\end{aligned}$$

By a long direct computation, we find

$$\delta d d' = \frac{\varepsilon_0}{\varepsilon_0 \tau^2 + \varepsilon_2 \kappa^2} (\tau \langle \gamma(s), \partial \rangle - \varepsilon_2 \kappa \langle \gamma(s), \partial^* \rangle)$$

which implies that γ is a pseudo-spherical curve if and only if (2.24) holds. □

3. The characterization of some special null Frenet curves in Minkowski 3-space

Theorem 3.1. *Let $\gamma = \gamma(s)$ be a null Frenet curve in Minkowski 3−space $\mathbb{R}^3_1$. Then $\gamma(s)$ satisfies the differential equation given by*

$$h''' + \frac{2}{\sigma} h^{'} + \left(\frac{1}{\sigma}\right)^{'} h + \delta = 0, \tag{3.1}$$

where $\sigma = \tau^{-1}$, $h(s) = d(s)d'(s)$ and $\delta = sign(\gamma(s)) = \mp 1$.

Proof. We take derivative of the square distance function

$$d^2(s) = \delta < \gamma(s), \gamma(s) >$$

with respect to s. Then we arrive

$$h(s) = \delta < \gamma(s), T > . \tag{3.2}$$

By differentiating (3.2) with respect to s and using (1.5), (1.6) we have

$$h'(s) = \delta \langle \gamma(s), N \rangle. \tag{3.3}$$

Similarly, differentiating (3.3) with respect to s and using (1.5), (1.6) get

$$h''(s) = -\delta\tau\langle \gamma(s), T\rangle - \delta\langle \gamma(s), B\rangle. \tag{3.4}$$

Substitution (3.2) in (3.4), we obtain

$$h''(s) + \frac{1}{\sigma}h(s) = -\delta\langle \gamma(s), B\rangle. \tag{3.5}$$

Finally, if we take derivative of (3.5) with respect to s and using (3.3), (1.5) and (1.6), then we find the differential equation (3.1). □

We simply see the following consequences from Theorem 3.1.

Corollary 3.2. *([14, 18]) There is no null Frenet curve lies pseudo-sphere in Minkowski* $3-$*space* $\mathbb{R}^3_1$.

Corollary 3.3. *([11]) A null Frenet curve* γ *in Minkowski* $3-$*space* $\mathbb{R}^3_1$ *is a rectifying curve if and only if* γ *satisfies the differential equation*

$$\frac{1}{\sigma} = as + b,$$

for some constants a and b.

3.1 A new characterization of null general helix

Note that a null Frenet curve parametrized by the pseudo-arc parameter is called a null helix if its lightlike curvature is constant. A null Frenet curve parametrized by the pseudo-arc parameter is a null general helix if and only if it is a null helix ([10]).

Now, we gives a new characterization of null general helix.

Theorem 3.4. *A null Frenet curve* γ *parametrized by the pseudo-arc parameter s in Minkowski* $3-$*space* $\mathbb{R}^3_1$ *is a null general helix if and only if the function* $h(s) = d(s)d'(s)$ *with respect to pseudo-arc parameter s satisfies the differential equation*

$$\delta h''(s) + \frac{2}{\sigma}\delta h(s) + s + \widetilde{c} = 0, \tag{3.6}$$

where $d^2(s) = \delta < \gamma(s), \gamma(s) >$, $\sigma = \tau^{-1}$ *and* $\delta = sign(\gamma(s)) = \pm 1$.

Proof. Let a null Frenet curve γ parametrized by the pseudo-arc parameter s be a null general helix. So there exists a vector $u \in \mathbb{R}^3_1$ such that $< T, u >$ is constant. Here we assume, without loss of generality, that $< u, u >= \varepsilon$, where $\varepsilon = -1, 0, 1$. Then

$$\langle T, u\rangle = c$$

for constant $c \neq 0$ and

$$\langle N, u\rangle = 0.$$

Therefore, we can write

$$u = \langle u, B\rangle T + cB$$

which implies that

$$\varepsilon = 2c\langle u, B\rangle.$$

Hence we find

$$\langle u, B\rangle = \frac{\varepsilon}{2c}$$

and we deduce that

$$u = \frac{\varepsilon}{2c}T + cB. \tag{3.7}$$

Differentiating (3.7) with respect to s and using the Frenet equations (1.5) we find

$$\tau = -\frac{\varepsilon}{2c^2}. \tag{3.8}$$

On the other hand, from

$$\langle \gamma(s), u\rangle' = c,$$

we get

$$\langle \gamma(s), u\rangle = cs + c_1 \tag{3.9}$$

for constant c_1. Substituting (3.7) in (3.9) and using (3.2), (3.5) and (3.8), we obtain (3.6).

Conversely, we suppose that a null Frenet curve γ parametrized by the pseudo-arc parameter s satisfies Eq. (3.6). By differentiating (3.6) with respect to s, we get

$$h'''(s) + (\frac{2}{\sigma})'h(s) + \frac{2}{\sigma}h'(s) + \delta = 0 \tag{3.10}$$

By comparing (3.10) with (3.1), we have

$$\left(\frac{1}{\sigma}\right)' h(s) = 0.$$

Since $h(s)$ can not be zero, $\left(\frac{1}{\sigma}\right)' = 0$, that is $\tau = const$. Consequently, a null Frenet curve γ is a null general helix. □

3.2 An other characterization of null general helix with respect to centrode and co-centrode

Let γ be a null Frenet curve in Minkowski 3−space $\mathbb{R}^3_1$. The centrode ∂ and the co-centrode ∂^* of γ are given by the Darboux vector

$$\partial = -\tau T + B \tag{3.11}$$

[19] and the co-Darboux vector

$$\partial^* = -\tau T - B,$$

respectively. We can clearly see that the co-Darboux vector is equal to the derivative of the principal normal vector N of γ that is $N' = \partial^*$.

By differentiating (3.11) and using the Frenet equations (1.5) yields

$$\partial' = -\tau' T.$$

So ∂' is always orthogonal to the co-centrode ∂^* if and only if

$$\tau' = 0$$

which implies that γ is a null general helix. Therefore we can give the following other characterization of null general helix with respect to the centrode ∂ and the co-centrode ∂^*.

Corollary 3.5. *γ is a null general helix in Minkowski 3−space $\mathbb{R}^3_1$ if and only if ∂' is always orthogonal to the co-centrode ∂^*.*

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