Malaya Journal of Matematik, Vol. 8, No. 4, 2149-2157, 2020
https://doi.org/10.26637/MJM0804/0140

M-Polynomial and topological indices of Hanoi graph and generalized wheel graph

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Abstract
There are plenty of topological indices used in chemistry to study the chemical behavior and physical properties of molecular graphs. In the literature, several results are computed for degree based topological indices like "first Zagreb index, second Zagreb index, modified second Zagreb index, generalized Randić index, inverse Randić index, symmetric sum division index, harmonic index, inverse sum index, augmented Zagreb index". In this paper, we have investigated the aforesaid degree based topological indices for Hanoi graph and generalized wheel graph with the help of M-polynomial.

Keywords
Topological indices, M-polynomial, Hanoi graph H_n, generalized wheel graph W_n^m.

AMS Subject Classification
05C07, 05C09, 05C31, 05C76, 05C92.

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Article History: Received **16** September **2020**; Accepted **26** November **2020**

Contents

1. Introduction

In chemical graph theory, atoms are represented by vertices and chemical bonds are represented by edges. We assume $\mathtt{G} = (\mathtt{V}, \mathtt{E})$ be a graph, where $\mathtt{V}$ is the set of objects called vertices and $\mathtt{E}$ is the set of unordered pair of elements of $\mathtt{V}$ called edges of the graph $\mathtt{G}$. $d(v)$ denotes the degree of vertex v is the number of edges incident on v in a graph $\mathtt{G}$. A topological index is a graph invariant which is mostly applicable in chemistry. There are many degree based graph invariants such as Zagreb index, Randić index, SSD index, inverse sum index, ABC index, harmonic index etc., have been studied in the literature (see [1], [2] and [3]).

In 2015, Deutsch & Klavžar [4] proposed M-polynomial, by which several topological indices based on degree, were determined in [5] and [6]. The M-polynomial is a recent polynomial containing vast information about degree based topological indices of a graph. The M-polynomial of a graph $\mathtt{G}$ is denoted as $M(\mathtt{G}:p,q)$ and defined as $M(\mathtt{G}:p,q) = \sum_{\delta \le i \le j \le \Delta} m_{i,j}(\mathtt{G}) p^i q^j$, where $\delta = min\{d_v : v \in \mathtt{V}\}$, $\Delta = max\{d_v : v \in \mathtt{V}\}$ and $m_{i,j}(\mathtt{G})$ is the number of edges uv such that $d(u) = i$ and $d(v) = j$.

In 1972, Gutman & Trinajstić [7] proposed the "first Zagreb index" $M_1(\mathtt{G})$ and they found "second Zagreb index" $M_2(\mathtt{G})$ in 1975 (see [8]). These were named Zagreb group indices which are defined by $M_1(\mathtt{G}) = \sum_{v_i \in \mathtt{V}} d^2(v_i) = \sum_{v_i v_j \in \mathtt{E}} \{d(v_i) + d(v_j)\}$ and $M_2(\mathtt{G}) = \sum_{v_i v_j \in \mathtt{E}} d(v_i) d(v_j)$. The theoretical description and properties of two Zagreb indices can be obtained from several papers and articles (see [9], [10] and [11]). ${}^m M_1(\mathtt{G})$ and ${}^m M_2(\mathtt{G})$ are the "modified first and second Zagreb index" of a graph $\mathtt{G}$ respectively, which are defined by

$$ {}^m M_1(\mathtt{G}) = \sum_{v_i \in \mathtt{V}} \frac{1}{d^2(v_i)} \quad \text{and} \quad {}^m M_2(\mathtt{G}) = \sum_{v_i v_j \in \mathtt{E}} \frac{1}{d(v_i) d(v_j)}. $$

In [12], E. Estrada devised a graph invariant "atom-bond connectivity index” which is defined by

$$ABC(\mathtt{G}) = \sum_{v_iv_j\in \mathtt{E}} \sqrt{\frac{d(v_i)+d(v_j)-2}{d(v_i)d(v_j)}}.$$

It is useful to find the thermodynamical properties of alkanes. It has been revised by Furtula et al. in [13] and they put forward a new index “augmented Zagreb index”. It is defined as $A(\mathtt{G}) = \sum_{v_iv_j\in \mathtt{E}} \left(\frac{d(v_i)d(v_j)}{d(v_i)+d(v_j)-2}\right)^3$.

In 1975, Randić [14] unveiled the degree based graph invariant “Randić index”. It is defined by

$$R(\mathtt{G}) = \sum_{v_iv_j\in \mathtt{E}} \frac{1}{\sqrt{d(v_i)d(v_j)}}.$$

In 1998, Bollobás & Erdös [15] and Amic et al. [16] introduced the generalized Randić index. It is defined as $R_\alpha(\mathtt{G}) = \sum_{v_iv_j\in \mathtt{E}}(d(v_i)d(v_j))^\alpha$, where α is a non zero real number. For more details about generalized randić index, we refer the book [17]. The inverse Randić index of a graph $\mathtt{G}$ is defined by $RR_\alpha(\mathtt{G}) = \sum_{v_iv_j\in \mathtt{E}} \frac{1}{(d(v_i)d(v_j))^\alpha}$.

In 1987, Fajtlowicz [18] introduced the harmonic index. It is defined as $H(\mathtt{G}) = \sum_{v_iv_j\in \mathtt{E}} \frac{2}{d(v_i)+d(v_j)}$. The relationship between the graph’s harmonic index and the eigen values was studied by Favaron et al. in [19]. The maximum and minimum values of harmonic index on simple connected graphs were determined by Zhong in [20]. The inverse sum index of a graph $\mathtt{G}$ is defined as $I(\mathtt{G}) = \sum_{v_iv_j\in \mathtt{E}} \frac{d(v_i)d(v_j)}{d(v_i)+d(v_j)}$. The values of inverse sum index across connected graphs, chemical graphs, trees and chemical trees were determined in [21]. Vukicevic [22] defined the symmetric sum division index of a graph $\mathtt{G}$ as $SSD(\mathtt{G}) = \sum_{v_iv_j\in \mathtt{E}} \frac{d^2(v_i)+d^2(v_j)}{d(v_i)d(v_j)}$.

2. Preliminaries

The aforesaid topological indices are also determined using M-polynomial of a graph. Let $F(p,q)$ be a M-polynomial of the graph $\mathtt{G}$ then

(i) first Zagreb index

$$M_1(\mathtt{G}) = [(D_p+D_q)F(p,q)]_{p=q=1}, \tag{2.1}$$

(ii) second Zagreb index

$$M_2(\mathtt{G}) = [D_qD_pF(p,q)]_{p=q=1}, \tag{2.2}$$

(iii) modified second Zagreb index

$${}^mM_2(\mathtt{G}) = [S_pS_qF(p,q)]_{p=q=1}, \tag{2.3}$$

(iv) generalized Randić index

$$R_\alpha(\mathtt{G}) = [D_p^\alpha D_q^\alpha F(p,q)]_{p=q=1}, \tag{2.4}$$

(v) inverse Randić index

$$RR_\alpha(\mathtt{G}) = [S_p^\alpha S_q^\alpha F(p,q)]_{p=q=1}, \tag{2.5}$$

(vi) symmetric sum division index

$$SSD(\mathtt{G}) = [(S_qD_p+S_pD_q)F(p,q)]_{p=q=1}, \tag{2.6}$$

(vii) harmonic index

$$H(\mathtt{G}) = 2[S_pJF(p,q)]_{p=1}, \tag{2.7}$$

$(viii)$ inverse sum index

$$I(\mathtt{G}) = [S_pJD_pD_qF(p,q)]_{p=q=1}, \tag{2.8}$$

(ix) augmented Zagreb index

$$A(\mathtt{G}) = [S_p^3Q_{-2}JD_p^3D_q^3F(p,q)]_{p=1}. \tag{2.9}$$

Degree based topological indices are broadly important in theoretical chemistry in which chemical graphs play an integral part. In this paper, we consider some degree based indices of two graphs namely Hanoi graph and generalized wheel graph. The following notations are used in this paper :

$$D_pF(p,q) = p\frac{\partial(F(p,q))}{\partial p},$$

$$D_qF(p,q) = q\frac{\partial(F(p,q))}{\partial q},$$

$$S_pF(p,q) = \int_0^p \frac{F(t,q)}{t}dt,$$

$$S_qF(p,q) = \int_0^q \frac{F(p,t)}{t}dt,$$

$$JF(p,q) = F(p,p),$$

$$Q_\alpha F(p) = p^\alpha F(p).$$

3. Main Results

3.1 Topological indices of Hanoi graph

The Hanoi graph H_n can be created by taking the vertices as odd binomial coefficients of Pascal’s triangle calculated on the integers from 0 to 2^n-1 and drawing an edge when coefficients are together diagonally or horizontally. The Hanoi graphs H_2 and H_3 are shown in Figure 1. A Hanoi graphs H_n has 3^n number of vertices and $\frac{3}{2}(3^n-1)$ number of edges.

Theorem 3.1. *Let H_n be a Hanoi graph then the M-polynomial of H_n is given by*

$$M(H_n:p,q) = 6p^2q^3 + \frac{3}{2}(3^n-5)p^3q^3.$$

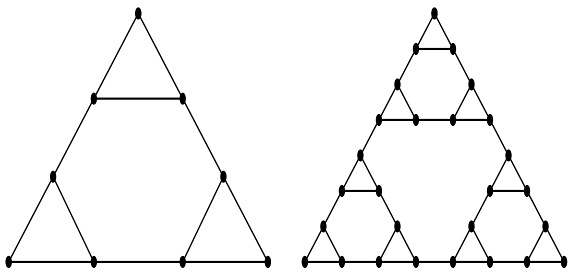


Figure 1. A representation of Hanoi graph H_2 and H_3

Proof. We have

$$| V(H_n) |= 3^n,$$

$$| E(H_n) |= \frac{3}{2}(3^n - 1).$$

The edge set of $H_n = (V,E)$ can be partitioned as :

$$E_{\{2,3\}} = \{uv \in E : d(u) = 2, d(v) = 3\},$$

$$E_{\{3,3\}} = \{uv \in E : d(u) = 3, d(v) = 3\}.$$

Now,

$$| E_{\{2,3\}} |= 6 = m_{2,3},$$

$$| E_{\{3,3\}} |= \frac{3}{2}(3^n - 5) = m_{3,3}.$$

Thus, the M-polynomial of H_n is

$$\begin{aligned} M(H_n : p,q) &= \sum_{0\le i\le j\le 3} m_{i,j}(H_n)p^i q^j \\ &= m_{2,3}(H_n)p^2q^3 + m_{3,3}(H_n)p^3q^3 \\ &= 6p^2q^3 + \frac{3}{2}(3^n - 5)p^3q^3. \end{aligned}$$

□

We denote M-polynomial of a Hanoi graph H_n by $F(p,q)$. i.e.,

$$M(H_n : p,q) = 6p^2q^3 + \frac{3}{2}(3^n - 5)p^3q^3 = F(p,q). \tag{3.1}$$

Now, we compute the following expressions :

$$\begin{aligned} D_pF(p,q) &= p\frac{\partial}{\partial p}\{6p^2q^3 + \frac{3}{2}(3^n - 5)p^3q^3\} \\ &= 12p^2q^3 + \frac{9}{2}(3^n - 5)p^3q^3, \end{aligned} \tag{3.2}$$

$$\begin{aligned} D_qF(p,q) &= q\frac{\partial}{\partial q}\{6p^2q^3 + \frac{3}{2}(3^n - 5)p^3q^3\} \\ &= 18p^2q^3 + \frac{9}{2}(3^n - 5)p^3q^3, \end{aligned} \tag{3.3}$$

$$\begin{aligned} S_pF(p,q) &= \int_0^p [\frac{1}{t}\{6t^2q^3 + \frac{3}{2}(3^n - 5)t^3q^3\}]dt \\ &= 3p^2q^3 + \frac{1}{2}(3^n - 5)p^3q^3, \end{aligned} \tag{3.4}$$

$$\begin{aligned} S_qF(p,q) &= \int_0^q [\frac{1}{t}\{6p^2t^3 + \frac{3}{2}(3^n - 5)p^3t^3\}]dt \\ &= 2p^2q^3 + \frac{1}{2}(3^n - 5)p^3q^3, \end{aligned} \tag{3.5}$$

and

$$JF(p,q) = F(p,p) \tag{3.6}$$

$$= 6p^5 + \frac{3}{2}(3^n - 5)p^6. \tag{3.7}$$

Theorem 3.2. *Let H_n be a Hanoi graph, then*
(i) first Zagreb index $M_1(H_n) = 3(3^{n+1} - 5)$,
(ii) second Zagreb index $M_2(H_n) = \frac{9}{2}(3^{n+1} - 1)$,
(iii) modified second Zagreb index ${}^mM_2(H_n) = \frac{1}{6}(3^n + 1)$.

Proof. Adding equation (3.2) and (3.3),

$$(D_p + D_q)F(p,q) = 30p^2q^3 + 9(3^n - 5)p^3q^3,$$

using equation (2.1), the first Zagreb index

$$\begin{aligned} M_1(H_n) &= [(D_p + D_q)F(p,q)]_{p=q=1} \\ &= 3(3^{n+1} - 5). \end{aligned}$$

From the equation (3.2) ,

$$\begin{aligned} D_qD_pF(p,q) &= q\frac{\partial}{\partial q}\{12p^2q^3 + \frac{9}{2}(3^n - 5)p^3q^3\} \\ &= 36p^2q^3 + \frac{27}{2}(3^n - 5)p^3q^3, \end{aligned}$$

using equation (2.2), the second Zagreb index

$$\begin{aligned} M_2(H_n) &= [D_qD_pF(p,q)]_{p=q=1} \\ &= \frac{9}{2}(3^{n+1} - 1). \end{aligned}$$

From the equation (3.5),

$$\begin{aligned} S_pS_qF(p,q) &= \int_0^p [\frac{1}{t}\{2t^2q^3 + \frac{1}{2}(3^n - 5)t^3q^3\}]dt \\ &= p^2q^3 + \frac{1}{6}(3^n - 5)p^3q^3, \end{aligned}$$

using equation (2.3), the modified second Zagreb index

$$
\begin{aligned}
{}^{m}M_2(H_n) &= [S_pS_qF(p,q)]_{p=q=1} \\
&= \frac{1}{6}(3^n+1).
\end{aligned}
$$

□

Theorem 3.3. *Let H_n be a Hanoi graph, then*
(i) generalized Randić index

$$R_\alpha(H_n) = \frac{3^{\alpha+1}}{2}(2^{\alpha+2}+3^{n+\alpha}-5.3^\alpha),$$

(ii) inverse Randić index

$$RR_\alpha(H_n) = \frac{1}{2^{\alpha-1}.3^{\alpha-1}} + \frac{3^n-5}{2.3^{2\alpha-1}},$$

where $\alpha > 0$.

Proof. From equation (3.3),

$$D_qF(p,q) = 18p^2q^3 + \frac{9}{2}(3^n-5)p^3q^3,$$

$$D_q^\alpha F(p,q) = 6.3^\alpha p^2q^3 + \frac{3.3^\alpha}{2}(3^n-5)p^3q^3,$$

$$
\begin{aligned}
D_pD_q^\alpha F(p,q) &= p\frac{\partial}{\partial p}\{6.3^\alpha p^2q^3 + \frac{3.3^\alpha}{2}(3^n-5)p^3q^3\} \\
&= 6.2.3^\alpha p^2q^3 + \frac{3.3.3^\alpha}{2}(3^n-5)p^3q^3,
\end{aligned}
$$

$$
\begin{aligned}
D_p^\alpha D_q^\alpha F(p,q) &= 6.2^\alpha.3^\alpha p^2q^3 + \frac{3.3^\alpha.3^\alpha}{2}(3^n-5)p^3q^3 \\
&= 2^{\alpha+1}3^{\alpha+1}p^2q^3 + \frac{3^{2\alpha+1}}{2}(3^n-5)p^3q^3,
\end{aligned}
$$

using equation (2.4), the generalized Randić index

$$
\begin{aligned}
R_\alpha(H_n) &= [D_p^\alpha D_q^\alpha F(p,q)]_{p=q=1} \\
&= \frac{3^{\alpha+1}}{2}(2^{\alpha+2}+3^{n+\alpha}-5.3^\alpha).
\end{aligned}
$$

From equation (3.5),

$$S_qF(p,q) = \frac{6}{3}p^2q^3 + \frac{3}{2.3}(3^n-5)p^3q^3,$$

$$S_q^\alpha F(p,q) = \frac{6}{3^\alpha}p^2q^3 + \frac{3}{2.3^\alpha}(3^n-5)p^3q^3,$$

$$
\begin{aligned}
S_pS_q^\alpha F(p,q) &= \int_0^p [\frac{1}{t}\{\frac{6}{3^\alpha}t^2q^3 + \frac{3}{2.3^\alpha}(3^n-5)t^3q^3\}]dt \\
&= \frac{6}{2.3^\alpha}p^2q^3 + \frac{3}{2.3.3^\alpha}(3^n-5)p^3q^3,
\end{aligned}
$$

$$
\begin{aligned}
S_p^\alpha S_q^\alpha F(p,q) &= \frac{6}{2^\alpha.3^\alpha}p^2q^3 + \frac{3}{2.3^\alpha.3^\alpha}(3^n-5)p^3q^3 \\
&= \frac{1}{3^{\alpha-1}.2^{\alpha-1}}p^2q^3 + \frac{3^n-5}{2.3^{2\alpha-1}}p^3q^3,
\end{aligned}
$$

using equation (2.5), the inverse Randić index

$$
\begin{aligned}
RR_\alpha(H_n) &= [S_p^\alpha S_q^\alpha F(p,q)]_{p=q=1} \\
&= \frac{1}{2^{\alpha-1}.3^{\alpha-1}} + \frac{3^n-5}{2.3^{2\alpha-1}}.
\end{aligned}
$$

□

Theorem 3.4. *Let H_n be a Hanoi graph, then the symmetric sum division index is given by*

$$SSD(H_n) = 3^{n+1}-2.$$

Proof. From equation (3.2),

$$D_pF(p,q) = 12p^2q^3 + \frac{9}{2}(3^n-5)p^3q^3,$$

$$
\begin{aligned}
S_qD_pF(p,q) &= \int_0^q [\frac{1}{t}\{12p^2t^3 + \frac{9}{2}(3^n-5)p^3t^3\}]dt \\
&= 4p^2q^3 + \frac{3}{2}(3^n-5)p^3q^3,
\end{aligned}
$$

from equation (3.3),

$$D_qF(p,q) = 18p^2q^3 + \frac{9}{2}(3^n-5)p^3q^3,$$

$$
\begin{aligned}
S_pD_qF(p,q) &= \int_0^p [\frac{1}{t}\{18t^2q^3 + \frac{9}{2}(3^n-5)t^3q^3\}]dt \\
&= 9p^2q^3 + \frac{3}{2}(3^n-5)p^3q^3,
\end{aligned}
$$

using equation (2.6), the symmetric sum division index

$$
\begin{aligned}
SSD(H_n) &= [(S_qD_p + S_pD_q)F(p,q)]_{p=q=1} \\
&= [13p^2q^3 + 3(3^n-5)p^3q^3]_{p=q=1} \\
&= 3^{n+1}-2.
\end{aligned}
$$

□

Theorem 3.5. *Let H_n be a Hanoi graph then*
(i) harmonic index $H(H_n) = \frac{1}{10}(5.3^n-1)$,
(ii) inverse sum index $I(H_n) = \frac{9}{20}(5.3^n-9)$,
(iii) augmented Zagreb index $A(H_n) = \frac{1}{2^7}(3^{n+7}-8887)$.

Proof. From equation (3.7),

$$JF(p,q) = 6p^5 + \frac{3}{2}(3^n-5)p^6,$$

$$S_pJF(p,q) = \int_0^p [\frac{1}{t}\{6t^5 + \frac{3}{2}(3^n-5)t^6\}]dt = \frac{6}{5}p^5 + \frac{1}{4}(3^n-5)p^6,$$

using equation (2.7), the harmonic index

$$H(H_n) = 2[S_pJF(p,q)]_{p=1} = \frac{1}{10}[5.3^n - 1].$$

From equation (3.2),

$$D_pF(p,q) = 12p^2q^3 + \frac{9}{2}(3^n-5)p^3q^3,$$

$$D_qD_pF(p,q) = q\frac{\partial}{\partial q}\{12p^2q^3 + \frac{9}{2}(3^n-5)p^3q^3\} = 36p^2q^3 + \frac{27}{2}(3^n-5)p^3q^3,$$

$$JD_qD_pF(p,q) = 36p^5 + \frac{27}{2}(3^n-5)p^6,$$

$$S_pJD_qD_pF(p,q) = \int_0^p [\frac{1}{t}\{36t^5 + \frac{27}{2}(3^n-5)t^6\}]dt = \frac{36}{5}p^5 + \frac{27}{12}(3^n-5)p^6,$$

using equation (2.8), the inverse sum index

$$I(H_n) = [S_pJD_pD_qF(p,q)]_{p=q=1} = \frac{9}{20}(5.3^n - 9).$$

From equation (3.3),

$$D_qF(p,q) = 18p^2q^3 + \frac{9}{2}(3^n-5)p^3q^3,$$

$$D_q^3F(p,q) = 6.3^3p^2q^3 + \frac{3.3^3}{2}(3^n-5)p^3q^3,$$

$$D_pD_q^3F(p,q) = p\frac{\partial}{\partial p}\{6.3^3p^2q^3 + \frac{3.3^3}{2}(3^n-5)p^3q^3\} = 6.2.3^3p^2q^3 + \frac{3.3.3^3}{2}(3^n-5)p^3q^3,$$

$$D_p^3D_q^3F(p,q) = 6.2^3.3^3p^2q^3 + \frac{3.3^3.3^3}{2}(3^n-5)p^3q^3 = 2^43^4p^2q^3 + \frac{3^7}{2}(3^n-5)p^3q^3,$$

$$JD_p^3D_q^3F(p,q) = J\{2^43^4p^2q^3 + \frac{3^7}{2}(3^n-5)p^3q^3\} = 2^43^4p^5 + \frac{3^7}{2}(3^n-5)p^6,$$

$$Q_{-2}JD_p^3D_q^3F(p,q) = Q_{-2}\{2^43^4p^5 + \frac{3^7}{2}(3^n-5)p^6\} = 2^43^4p^3 + \frac{3^7}{2}(3^n-5)p^4,$$

$$S_pQ_{-2}JD_p^3D_q^3F(p,q) = \int_0^p [\frac{1}{t}\{2^43^4t^3 + \frac{3^7}{2}(3^n-5)t^4\}]dt = \frac{2^43^4}{3}p^3 + \frac{3^7}{2.4}(3^n-5)p^4,$$

$$S_p^3Q_{-2}JD_p^3D_q^3F(p,q) = \frac{2^43^4}{3^3}p^3 + \frac{3^7}{2.4^3}(3^n-5)p^4 = 2^4.3p^3 + \frac{3^7}{2^7}(3^n-5)p^4,$$

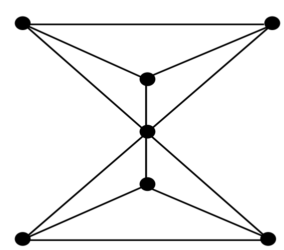
using equation (2.9), the augmented Zagreb index

$$A(H_n) = [S_p^3Q_{-2}JD_p^3D_q^3F(p,q)]_{p=1} = \frac{1}{2^7}(3^{n+7} - 8887).$$

□

3.2 Topological indices of generalized wheel graph

A generalized wheel graph $W_n^m = mC_n + K_1$ is defined as : take m copies of cycle $C_n; n \geq 3$ and then, join every vertex of m copies of cycle C_n with a vertex of a complete graph K_1. The generalized wheel graph W_3^2 is shown in Figure 2. For $m = 1$, the generalized wheel graph is turn out as wheel graph. A generalized wheel graph W_n^m has $mn+1$ number of vertices and $2mn$ number of edges.



$W_3^2 = 2C_3 + K_1$

Figure 2. A representation of a generalized wheel graph W_3^2

Theorem 3.6. *The M-polynomial of a generalized wheel graph W_n^m is given by*

$$M(W_n^m : p,q) = mn[p^3q^3 + p^3q^{mn}].$$

Proof. We have,

$$|V(W_n^m)| = mn + 1$$

and

$$|E(W_n^m)| = 2mn.$$

The edge set of a graph $W_n^m = (V,E)$ has the following two partitions :

$$E_{\{3,mn\}} = \{uv \in E : d(u) = 3, d(v) = mn\},$$

$$E_{\{3,3\}} = \{uv \in E : d(u) = 3, d(v) = 3\}.$$

Now,

$$\mid E_{\{3,mn\}} \mid = mn = m_{3,mn},$$

$$\mid E_{\{3,3\}} \mid = mn = m_{3,3}.$$

Thus, the M-polynomial of W_n^m

$$\begin{aligned} M(W_n^m : p,q) &= \sum_{0\le i\le j\le mn} m_{i,j}(W_n^m)p^iq^j \\ &= m_{3,mn}(W_n^m)p^3q^{mn} + m_{3,3}(W_n^m)p^3q^3 \\ &= mn\left(p^3q^{mn} + p^3q^3\right). \end{aligned}$$

□

Now, we consider M-polynomial of a generalized wheel graph W_n^m as $G(p,q)$. i.e.,

$$M(W_n^m : p,q) = mn(p^3q^{mn} + p^3q^3) = G(p,q). \tag{3.8}$$

Then, we compute the following expressions :

$$\begin{aligned} D_pG(p,q) &= p\frac{\partial}{\partial p}\{mn(p^3q^{mn} + p^3q^3)\} \\ &= 3mnp^3q^{mn} + 3mnp^3q^3, \end{aligned} \tag{3.9}$$

$$\begin{aligned} D_qG(p,q) &= q\frac{\partial}{\partial q}\{mn(p^3q^{mn} + p^3q^3)\} \\ &= m^2n^2p^3q^{mn} + 3mnp^3q^3, \end{aligned} \tag{3.10}$$

$$\begin{aligned} S_pG(p,q) &= \int_0^p [\frac{1}{t}\{mn(t^3q^{mn} + t^3q^3)\}]dt \\ &= \frac{mn}{3}p^3q^{mn} + \frac{mn}{3}p^3q^3, \end{aligned} \tag{3.11}$$

$$\begin{aligned} S_qG(p,q) &= \int_0^q [\frac{1}{t}\{mn(p^3t^{mn} + p^3t^3)\}]dt \\ &= p^3q^{mn} + \frac{mn}{3}p^3q^3, \end{aligned} \tag{3.12}$$

and

$$JG(p,q) = G(p,p) \tag{3.13}$$
$$= mn(p^{mn+3} + p^6). \tag{3.14}$$

Theorem 3.7. *Let W_n^m be a generalized wheel graph, then*
(i) first Zagreb index $M_1(W_n^m) = mn(mn+9)$,
(ii) second Zagreb index $M_2(W_n^m) = 3mn(mn+3)$,
(iii) modified second Zagreb index ${}^mM_2(W_n^m) = mn(\frac{1}{3mn} + \frac{1}{9})$.

Proof. Adding equations (3.9) and (3.10),

$$(D_p + D_q)G(p,q) = mn\{6p^3q^3 + (mn+3)p^3q^{mn}\},$$

using equation (2.1), the first Zagreb index

$$\begin{aligned} M_1(W_n^m) &= [(D_p + D_q)G(p,q)]_{p=q=1} \\ &= mn(mn+9). \end{aligned}$$

From equation (3.9)

$$D_pG(p,q) = 3mnp^3q^{mn} + 3mnp^3q^3,$$

$$\begin{aligned} D_qD_pG(p,q) &= q\frac{\partial}{\partial q}(3mnp^3q^{mn} + 3mnp^3q^3) \\ &= 3mn(mnp^3q^{mn} + 3p^3q^3), \end{aligned}$$

using equation (2.2), the second Zagreb index

$$M_2(W_n^m) = [D_qD_pG(p,q)]_{p=q=1} = 3mn(mn+3).$$

From equation (3.12)

$$S_qG(p,q) = p^3q^{mn} + \frac{mn}{3}p^3q^3$$

$$\begin{aligned} S_pS_qG(p,q) &= \int_0^p \{\frac{1}{t}(t^3q^{mn} + \frac{mn}{3}t^3q^3)\}dt \\ &= \frac{1}{3}p^3q^{mn} + \frac{mn}{9}p^3q^3, \end{aligned}$$

using equation (2.3), the modified second Zagreb index

$${}^mM_2(W_n^m) = [S_pS_qG(p,q)]_{p=q=1} = \frac{1}{3} + \frac{mn}{9}.$$

□

Theorem 3.8. *Let W_n^m be a generalized wheel graph, then*
(i) generalized Randić index $R_\alpha(W_n^m) = mn\{(3mn)^\alpha + 9^\alpha\}$,
(ii) inverse Randić index $RR_\alpha(W_n^m) = mn\{\frac{1}{(3mn)^\alpha} + \frac{1}{9^\alpha}\}$, where $\alpha > 0$.

Proof. From equation (3.10),

$$D_qG(p,q) = mn(mnp^3q^{mn} + 3p^3q^3),$$

$$D_q^\alpha G(p,q) = mn(m^\alpha n^\alpha p^3q^{mn} + 3^\alpha p^3q^3),$$

$$\begin{aligned} D_pD_q^\alpha G(p,q) &= p\frac{\partial}{\partial p}[mn\{m^\alpha n^\alpha p^3q^{mn} + 3^\alpha p^3q^3\}] \\ &= mn(m^\alpha n^\alpha p^3q^{mn} + 3^\alpha 3p^3q^3) \end{aligned}$$

$$D_p^\alpha D_q^\alpha G(p,q) = mn(3^\alpha m^\alpha n^\alpha p^3q^{mn} + 3^{2\alpha}p^3q^3),$$

using equation (2.4), the generalized Randić index

$$\begin{aligned}R_\alpha(W_n^m) &= [D_p^\alpha D_q^\alpha G(p,q)]_{p=q=1}\\ &= mn[3^\alpha m^\alpha n^\alpha p^3 q^{mn} + 3^{2\alpha} p^3 q^3]_{p=q=1}\\ &= mn\{(3mn)^\alpha + 9^\alpha\}.\end{aligned}$$

From equation (3.12),

$$S_q G(p,q) = mn(\frac{1}{mn} p^3 q^{mn} + \frac{1}{3} p^3 q^3),$$

$$S_q^\alpha G(p,q) = mn(\frac{1}{m^\alpha n^\alpha} p^3 q^{mn} + \frac{1}{3^\alpha} p^3 q^3),$$

$$\begin{aligned}S_p S_q^\alpha G(p,q) &= \int_0^p \{\frac{1}{t} mn(\frac{1}{m^\alpha n^\alpha} t^3 q^{mn} + \frac{1}{3^\alpha} t^3 q^3)\} dt\\ &= mn(\frac{1}{3.m^\alpha n^\alpha} p^3 q^{mn} + \frac{1}{3.3^\alpha} p^3 q^3),\end{aligned}$$

$$S_p^\alpha S_q^\alpha G(p,q) = mn\{\frac{1}{(3mn)^\alpha} p^3 q^{mn} + \frac{1}{3^{2\alpha}} p^3 q^3\},$$

using equation (2.5), the inverse Randić index

$$\begin{aligned}RR_\alpha(W_n^m) &= [S_p^\alpha S_q^\alpha G(p,q)]_{p=q=1}\\ &= mn[\frac{1}{(3mn)^\alpha} p^3 q^{mn} + \frac{1}{3^{2\alpha}} p^3 q^3]_{p=q=1}\\ &= mn\{\frac{1}{(3mn)^\alpha} + \frac{1}{9^\alpha}\}.\end{aligned}$$

□

Theorem 3.9. *Let W_n^m be a generalized wheel graph, then the symmetric sum division index*

$$SSD(W_n^m) = 3 + 2mn + \frac{(mn)^2}{3}.$$

Proof. From equation (3.9),

$$D_p G(p,q) = 3mnp^3 q^{mn} + 3mnp^3 q^3,$$

$$\begin{aligned}S_q D_p G(p,q) &= \int_0^q \{\frac{1}{t}(3mnp^3 t^{mn} + 3mnp^3 t^3)\} dt\\ &= 3p^3 q^{mn} + mnp^3 q^3,\end{aligned}$$

from equation (3.10),

$$D_q G(p,q) = (mn)^2 p^3 q^{mn} + 3mnp^3 q^3,$$

$$\begin{aligned}S_p D_q G(p,q) &= \int_0^p [\frac{1}{t}\{(mn)^2 t^3 q^{mn} + 3mnt^3 q^3\}] dt\\ &= \frac{(mn)^2}{3} p^3 q^{mn} + mnp^3 q^3,\end{aligned}$$

using equation (2.6), the symmetric sum division index

$$\begin{aligned}SSD(W_n^m) &= \{(S_q D_p + S_p D_q) G(p,q)\}_{p=q=1}\\ &= 3 + 2mn + \frac{(mn)^2}{3}.\end{aligned}$$

□

Theorem 3.10. *Let W_n^m be a generalized wheel graph, then*
(i) harmonic index $H(W_n^m) = 2mn(\frac{1}{mn+3} + \frac{1}{6})$,
(ii) inverse sum index is $I(W_n^m) = 3mn(\frac{mn}{mn+3} + \frac{1}{2})$,
(iii) augmented Zagreb index $A(W_n^m) = mn\{\frac{3^3 m^3 n^3}{(mn+1)^3} + \frac{3^6}{4^3}\}$.

Proof. From equation (3.14),

$$JG(p,q) = mn[p^{mn+3} + p^6],$$

$$\begin{aligned}S_p JG(p,q) &= \int_0^p [\frac{1}{t}\{mn(t^{mn+3} + t^6)\}] dt\\ &= mn(\frac{p^{mn+3}}{mn+3} + \frac{p^6}{6}),\end{aligned}$$

using equation (2.7), the harmonic index

$$\begin{aligned}H(W_n^m) &= 2[S_p JG(p,q)]_{p=1}\\ &= 2[mn(\frac{p^{mn+3}}{mn+3} + \frac{p^6}{6})]_{p=1}\\ &= 2mn(\frac{1}{mn+3} + \frac{1}{6}).\end{aligned}$$

From equation (3.9),

$$D_p G(p,q) = 3mnp^3 q^{mn} + 3mnp^3 q^3,$$

$$\begin{aligned}D_q D_p G(p,q) &= q\frac{\partial}{\partial q}(3mnp^3 q^{mn} + 3mnp^3 q^3)\\ &= 3mn\left(mnp^3 q^{mn} + 3p^3 q^3\right),\end{aligned}$$

$$\begin{aligned}JD_q D_p G(p,q) &= J\{3mn(mnp^3 q^{mn} + 3p^3 q^3)\}\\ &= 3mn\left(mnp^{mn+3} + 3p^6\right),\end{aligned}$$

$$\begin{aligned}S_p JD_q D_p G(p,q) &= \int_0^p [\frac{1}{t}\{3mn(mnt^{mn+3} + 3t^6)\}] dt\\ &= 3mn(\frac{mn}{mn+3} p^{mn+3} + \frac{1}{2} p^6),\end{aligned}$$

using equation (2.8), the inverse sum index

$$\begin{aligned}I(W_n^m) &= [S_p JD_p D_q G(p,q)]_{p=1}\\ &= 3mn(\frac{mn}{mn+3} + \frac{1}{2}).\end{aligned}$$

From equation (3.10),

$$D_q G(p,q) = m^2 n^2 p^3 q^{mn} + 3mnp^3 q^3,$$

$$D_q^3 G(p,q) = mn\left(m^3 n^3 p^3 q^{mn} + 3^3 p^3 q^3\right),$$

$$D_pD_q^3G(p,q) = p\frac{\partial}{\partial p}\{mn(m^3n^3p^3q^{mn}+3^3p^3q^3)\}$$
$$= mn\{3m^3n^3p^3q^{mn}+3^4p^3q^3\},$$

$$D_p^3D_q^3G(p,q) = mn\{3^3m^3n^3p^3q^{mn}+3^6p^3q^3\},$$

$$JD_p^3D_q^3G(p,q) = J\{mn(3^3m^3n^3p^3q^{mn}+3^6p^3q^3)\}$$
$$= mn(3^3m^3n^3p^{mn+3}+3^6p^6),$$

$$Q_{-2}JD_p^3D_q^3G(p,q) = Q_{-2}\{mn(3^3m^3n^3p^{mn+3}+3^6p^6)\}$$
$$= mn(3^3m^3n^3p^{mn+1}+3^6p^4),$$

$$S_pQ_{-2}JD_p^3D_q^3G(p,q)$$
$$= \int_0^p[\frac{1}{t}\{mn(3^3m^3n^3t^{mn+1}+3^6t^4)\}]dt$$
$$= mn\{\frac{3^3m^3n^3}{mn+1}p^{mn+1}+\frac{3^6}{4}p^4\},$$

$$S_p^3Q_{-2}JD_p^3D_q^3G(p,q) = mn\{\frac{3^3m^3n^3}{(mn+1)^3}p^{mn+1}+\frac{3^6}{4^3}p^4\},$$

using equation (2.9), the augmented Zagreb index

$$A(W_n^m) = [S_p^3Q_{-2}JD_p^3D_q^3G(p,q)]_{p=1}$$
$$= [mn\{\frac{3^3m^3n^3}{(mn+1)^3}p^{mn+1}+\frac{3^6}{4^3}p^4\}]_{p=1}$$
$$= mn\{\frac{3^3m^3n^3}{(mn+1)^3}+\frac{3^6}{4^3}\}.$$

□

4. Conclusion

In this paper, we have computed M-polynomial for Hanoi graph and generalized wheel graph. Using M-polynomial, we have derived nine degree based topological indices namely Zagreb group indices, modified second Zagreb index, generalized Randić index, inverse Randić index, symmetric sum division index, harmonic index, inverse sum index, augmented Zagreb index of graphs H_n and W_n^m. It is important to note that such topological indices may be determined directly via the formulas given in the introduction.

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