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Near and closer relations in topology

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Abstract

Thamizharasi [13] studied the concepts of near relations and closer relations in topology in 2009. In this paper, some weak forms of near and closer relations are introduced and discussed. Some existing concepts in the literature of topology are characterized using these relations.

Keywords

Semiopen, Preopen, Regular open, b-open, p-set, q-set, Q-set.

AMS Subject Classification

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Contents

1. Introduction

In topological space the operators interior and closure are essential in applications of topology. The operators $Cl()$ and $Int()$ are level one operators of closure and interior. The operators $IntCl()$ and $ClInt()$ are the level two operators. Also $ClIntCl()$ and $IntClInt()$ are the level three operators. The two level and three level operators are used to define some nearly open sets in topological spaces. Thamizharasi defined and discussed near and closer relations using the level one operators in topology. Some weak forms of near and closer relations are introduced and discussed in this paper. Throughout this paper, (X,τ) is a topological space, A, B, D and E are the subsets of X and $\rho \in \{semi, \alpha, pre, \beta\}$.

2. Preliminaries

In this section, certain basic concepts and results in topology are given. The Interior and Closure operators on A are respectively denoted by $IntA$ and ClA.

Definition 2.1. *A is called regular open [10] if $A = IntClA$; semiopen [7] if there exists an open set U with $U \subseteq A \subseteq ClU$; preopen[8] if there exists an open set U with $A \subseteq U \subseteq ClA$; b-open[3] if $A \subseteq ClIntA \cup IntClA$; *b-open[5] if $A \subseteq ClIntA \cap IntClA$; $b^{\#}$-open[14] if $A = ClIntA \cup IntClA$; α-open[9] if $A \subseteq IntClIntA$; β-open[1] if $A \subseteq ClIntClA$; a p-set[11] if $ClIntA \subseteq IntClA$; a q-set[12] if $IntClA \subseteq ClIntA$ and a Q-set[6] if $ClIntA = IntClA$.*

The complements of the regular open, semiopen , preopen, α-open, β-open, b-open and $b^{\#}$-open sets are called the corresponding closed sets respectively. The interior and closure operators using the above sets can be defined in the usual way.

Lemma 2.2. *[2] $sIntA = A \cap ClIntA$; $pIntA = A \cap IntClA$; $\alpha IntA = A \cap IntClIntA$; $\beta IntA = A \cap ClIntClA$; $sClA = A \cup IntClA$; $pClA = A \cup ClIntA$; $\alpha ClA = A \cup ClIntClA$ and $\beta ClA = A \cup IntClIntA$.*

Lemma 2.3. *[13] A is near to B (briefly $A\ \mathcal{N}\ B$) if $IntA = IntB$ and A is closer to B (briefly $A\ \mathcal{C}\ B$) if $ClA = ClB$.*

3. Near Relations in Topology

Definition 3.1. **(i)** *$A\ s\mathcal{N}\ B$ if $sIntA = sIntB$*

(ii) *$A\ \alpha\mathcal{N}\ B$ if $\alpha IntA = \alpha IntB$*

(iii) *$A\ p\mathcal{N}\ B$ if $pIntA = pIntB$*

(iv) *$A\ \beta\mathcal{N}\ B$ if $\beta IntA = \beta IntB$*

Clearly $A\ \rho\mathcal{N}\ B$ if and only if $B\ \rho\mathcal{N}\ A$.

Proposition 3.2. *If $A\ \rho\mathcal{N}\ B$ then $A\ \mathcal{N}\ B$.*

Proof. Suppose $A\ s\mathcal{N}\ B$. Then $sIntA = sIntB$ that implies $A \cap ClIntA = sIntA = sIntB = B \cap ClIntB$.
Now $IntA \subseteq sIntA = sIntB = B \cap ClIntB \subseteq B \Rightarrow IntA \subseteq IntB$.
If A is pre-near to B then $pIntA = pIntB \Rightarrow A \cap IntClA = pIntA = pIntB = B \cap IntClB$ shows that $IntA \subseteq pIntB = B \cap IntClB \subseteq B \Rightarrow IntA \subseteq IntB$.
If A is α-near to B then $\alpha IntA = \alpha IntB \Rightarrow A \cap IntClIntA = \alpha IntA = \alpha IntB = B \cap IntClIntB$ that proves
$IntA \subseteq \alpha IntA = \alpha IntB = B \cap IntClIntB \subseteq B \Rightarrow IntA \subseteq IntB$.
If A is β-near to B then $\beta IntA = \beta IntB \Rightarrow A \cap IntClIntA = \beta IntA = \beta IntB = B \cap IntClIntB$. Now $IntA \subseteq \beta IntA = \beta IntB = B \cap IntClIntB \subseteq B \Rightarrow IntA \subseteq IntB$.
Thus we have proved that if A is ρ-near to B then $IntA \subseteq IntB$. Again if $A\ \rho\mathcal{N}\ B$ then $B\ \rho\mathcal{N}\ A \Rightarrow$ by our previous discussion $IntB \subseteq IntA$. This proves that if $A\ \rho\mathcal{N}\ B$ then $A\ \mathcal{N}\ B$. This completes the proof of the proposition. □

Lemma 3.3. *The relation $\mathcal{N}$ is an equivalence relation on the power set of X.*

Proof. Straight forward. □

The equivalence classes of the relation $\mathcal{N}$ are called the near classes of subsets of X. If A is a subset of X then $near[A] = \{B : A \text{ is near to } B\}$.

Proposition 3.4. *There is an one-to-one correspondence between the topology and the collection of near classes.*

Proof. For any open set O in (X,τ), a subset A of X is near to O iff $O = IntA$. Conversely every subset A of X is near to some open set in (X,τ). □

Proposition 3.5. (i) *If A is semiclosed then A is near to ClA.*

(ii) *If A is β-closed in (X,τ) then $A$is near to $ClIntA$.*

Proof. Suppose A is semiclosed in (X,τ). Then $IntA = IntClA \Rightarrow A$ is near to ClA. This proves (i). Suppose A is β-closed in (X,τ). Then $IntClIntA \subseteq A \Rightarrow IntA \subseteq IntClIntA \subseteq IntA$ so that $IntA = IntClIntA$ that establishes A is near to $ClIntA$. This proves (ii). □

Proposition 3.6. *Let $A\ \mathcal{N}\ B$ and $D\ \mathcal{N}\ E$. Then*

(i) $(A \cap D)\ \mathcal{N}\ (B \cap E)$

(ii) $(A \cap E)\ \mathcal{N}\ (B \cap D)$

(iii) *$A \cup D$ is not near to $B \cup E$*

Proof. Suppose $A\ \mathcal{N}\ B$ and $D\ \mathcal{N}\ E$. Then $IntA = IntB$ and $IntD = IntE$. Now $Int(A \cap D) = IntA \cap IntD = IntB \cap IntE = Int(B \cap E) \Rightarrow (A \cap D)\ \mathcal{N}\ (B \cap E)$. This proves (i) and the proof for (ii) is analog. Since $Cl(A \cup D) \neq ClA \cup ClD$, it follows that $A \cup D$ is not near to $B \cup E$. However examples can be constructed to establish this. This completes the proof of the proposition. □

Definition 3.7. *If $B \subseteq A$ and B is near to A then B is called a near subset of A in X.*

Proposition 3.8. *Let B be a near subset of A. Then*

(i) *If A is semiopen then B is semiopen.*

(ii) *If A is α-open then B is α-open.*

(iii) *If B is preclosed then A is preclosed.*

(iv) *If B is β-closed then A is β-closed.*

Proof. Suppose A is semiopen. Then $A \subseteq ClIntA$. Since A is near to B, $IntA = IntB \Rightarrow B \subseteq A \subseteq ClIntA = ClIntB$. This proves that B is semiopen that proves (i).
If A is α-open then $A \subseteq IntClIntA \Rightarrow B \subseteq A \subseteq IntClIntA = IntclIntB$, proving that B is α-open. This proves (ii).
Suppose B is preclosed. Then $ClIntB \subseteq B$. Since $A\ \mathcal{N}\ B$, $IntA = IntB \Rightarrow ClIntA = ClIntB \subseteq B \subseteq A$. This proves that A is pre closed that proves (iii).
If B is β-closed then $IntClIntA \subseteq B \Rightarrow IntClIntA = IntClIntB \subseteq B \subseteq A$ that shows that A is β-closed. This proves (iv). □

4. Closer Relations in Topology

Weak forms of closer relations are introduced and studied in this section. The existing closer relation is further investigated.

Definition 4.1. (i) *$A\ s\mathcal{C}\ B$ if $sClA = sClB$.*

(ii) *$A\ \alpha\mathcal{C}\ B$ if $\alpha ClA = \alpha ClB$.*

(iii) *$A\ p\mathcal{C}\ B$ if $pClA = pClB$.*

(iv) *$A\ \beta\mathcal{C}\ B$ if $\beta ClA = \beta ClB$.*
Clearly if $A\ \rho\mathcal{C}\ B$ then $B\ \rho\mathcal{C}\ A$.

Proposition 4.2. *If $A\ \rho\mathcal{C}\ B$ then $A\ \mathcal{C}\ B$.*

Proof. Suppose $A\ s\mathcal{C}\ B$. Then $sClA = sClB \Rightarrow A \cup IntClA = sClA = sClB = B \cap ClIntB$. Now $ClA \supseteq sClA = sClB = B \cup IntClB \supseteq B \Rightarrow ClA \supseteq ClB$.
Suppose $A\ p\mathcal{C}\ B$. Then $pClA = pClB \Rightarrow A \cup ClIntA = pClA = pClB = B \cup ClIntB$. Now $ClA \supseteq pClA = pClB = B \cup ClIntB \supseteq B \Rightarrow ClA \supseteq ClB$.
Suppose $A\ \alpha\mathcal{C}\ B$. Then $\alpha ClA = \alpha ClB \Rightarrow A \cup ClIntClA = \alpha ClA = \alpha ClB = B \cup ClIntClB$. Now $ClA \supseteq \alpha ClA = \alpha ClB = B \cup ClIntClB \supseteq B \Rightarrow ClA \supseteq ClB$.
Suppose $A\ \beta\mathcal{C}\ B$. $\beta ClA = \beta ClB \Rightarrow A \cup IntClIntA = \beta ClA = \beta ClB = B \cup IntClIntB$. Now $ClA \supseteq \beta ClA = \beta ClB = B \cup IntClIntB \supseteq B \Rightarrow ClA \supseteq ClB$. Thus it has been proved that if $A\ \rho\mathcal{C}\ B$ then $ClA \supseteq ClB$. Now if $A\ \rho\mathcal{C}\ B$ then $B\ \rho\mathcal{C}\ A$ then $ClB \supseteq ClA$. Therefore if $A\ \rho\mathcal{C}\ B$ then $ClA = ClB \Rightarrow A\ \mathcal{C}\ B$. This proves the proposition. □

Proposition 4.3. (i) *If $A\ \mathcal{N}\ B$ then $\alpha IntA\ \alpha\mathcal{C}\ \alpha IntB$.*

(ii) *If $A\ \mathcal{C}\ B$ then $\alpha ClA\ \alpha\mathcal{N}\ \alpha ClB$.*

Proof. Suppose $A \ \mathcal{N} \ B$. Then $IntA = IntB \Rightarrow ClIntA = ClIntB$ which further implies $\alpha Cl\alpha IntA = \alpha Cl\alpha IntB$. This proves that $\alpha IntA$ is α-closer to $\alpha IntB$.
Suppose $A \ \mathcal{C} \ B$. Then $ClA = ClB \Rightarrow IntClA = IntClB$ which further implies
$\alpha Int\alpha ClA = \alpha Int\alpha ClB$. This proves that αClA is α-near to αClB. This completes the proof of the proposition. □

Proposition 4.4. **(i)** *The set $A \ \mathcal{N} \ B$ iff $X\backslash A \ \mathcal{C} \ X\backslash B$.*

(ii) *The set $A \ \rho\mathcal{N} \ B$ iff $X\backslash A \ \rho\mathcal{C} \ X\backslash B$.*

Proof. Suppose $A \ \mathcal{N} \ B$. Then $IntA = IntB \Rightarrow Cl(X\backslash A) = Cl(X\backslash B)$. This shows that $X\backslash A \ \mathcal{C} \ X\backslash B$. Converse ease can be analogously proved. This proves (i) and the proof for (ii) is analog. □

Proposition 4.5. *Let $A \ \mathcal{C} \ B$ and $D \ \mathcal{C} \ E$.*

(i) $(A\cup D) \ \mathcal{C} \ (B\cup E)$.

(ii) $(A\cup E) \ \mathcal{C} \ (B\cup D)$.

(iii) *$(A\cap D)$ is not closer to $(B\cap E)$.*

Proof. Suppose $A \ \mathcal{C} \ B$ and $D \ \mathcal{C} \ E$. Then $ClA = ClB$ and $ClD = ClE$. Now $Cl(A\cup D) = ClA\cup ClD = ClB\cup ClE = Cl(B\cap E) \Rightarrow (A\cup D) \ \mathcal{C} \ (B\cup E)$. This proves (i) and the proof for (ii) analog. Since $Cl(A\cap D) \neq Cl(A)\cap Cl(D)$, it follows that $(A\cap D)$ is not closer to $(B\cap E)$. However examples can be constructed to establish that $(A\cap D)$ is not closer to $(B\cap E)$. This completes the proof of the proposition. □

Proposition 4.6. *The relation $\mathcal{C}$ is an equivalence relation on the power set of X.*

Proof. Obvious. □

The equivalence classes of the relation $\mathcal{C}$ are called the closer classes of subsets of X. If A is a subset of X then $closer[A] = \{B : A \textit{ is closer to } B\}$.

Proposition 4.7. *There is an one-to-one correspondence between the collection of closed sets and the collection of closer classes.*

Proof. For any closed set F in (X,τ), a subset A of X is closer to F iff $F = ClA$. Conversely every subset A of X is closer to some closed in (X,τ). □

Proposition 4.8. *For any subset A in (X,τ),*
$near[A] = closer[X\backslash A]$.

Proof. Follows from Proposition 4.4 □

Proposition 4.9. **(i)** *If A is semiopen then A is closer to $IntA$.*

(ii) *If A is β-open then A is closer to $IntClA$.*

Proof. Suppose A is semiopen in (X,τ). Then $ClA = ClIntA \Rightarrow A$ is closer to $IntA$. Suppose A is β-open in (X,τ). Then $A \subseteq ClIntClA \Rightarrow ClA = ClIntClA$ that proves A is closer to $IntClA$. □

Corollary 4.10. **(i)** *If A is closed or α-open then $A \ \mathcal{C} \ IntA$.*

(ii) *If A is preopen or b-open or $b^{\#}$-open then $A \ \mathcal{C} \ IntClA$.*

Proof. Since every regular closed is semiopen and since every α-open set is semiopen the assertion (i) follows from Proposition 4.9(i). Since preopen $\Rightarrow b$-open $\Rightarrow \beta$-open and since $b^{\#}$-open $\Rightarrow b$-open $\Rightarrow \beta$-open and assertion (ii) follows from Proposition 4.9(ii). This completes the proof of the corollary. □

Definition 4.11. *If $B \subseteq A$ and B is closer to A then B is called a closer subset of A in X.*

Proposition 4.12. *Let B be a closer subset of A in (X,τ). Then*

(i) *if A is preopen then B is preopen.*

(ii) *if A is β-open then B is β-open.*

(iii) *if B is semiclosed then A is semiclosed.*

(iv) *if B is α-closed then A is α-closed.*

Proof. Suppose A is preopen. Then $A \subseteq IntClA$. Since A is closer to B, $ClA = ClB \Rightarrow B \subseteq A \subseteq IntClA = IntClB$. This proves that B is preopen that proves (i). Suppose A is β-open. Then $A \subseteq ClIntClA$. Since A is closer to B, $ClA = ClB \Rightarrow B \subseteq A \subseteq ClIntClA = ClIntClB$. This proves that B is β-open that proves (ii). Suppose B is semiclosed. Then $IntClB \subseteq B$. Since A is closer to B, $ClA = ClB \Rightarrow IntClA = IntClB \subseteq B \subseteq A$. This proves that A is semiclosed that proves (iii). Suppose B is α-closed. Then $ClIntClB \subseteq B$. Since A is closer to B, $ClA = ClB \Rightarrow ClIntClA = ClIntClB \subseteq B \subseteq A$. This proves that A is α-closed that proves (iv). This completes the proof of the proposition. □

Proposition 4.13. *Let $A \ \mathcal{N} \ B$ and $A \ \mathcal{C} \ B$ in (X,τ) and $B \subseteq A$. Then*

(i) *If A is b-open then B is b-open.*

(ii) *If A is $^*b$-open then B is $^*b$-open.*

(iii) *If B is b-closed then A is b-closed.*

(iv) *If B is $^*b$-closed then A is $^*b$-closed.*

Proof. If A is b-open, then $A \subseteq IntClA\cup ClIntA$. Since A is near to B and closer to B, $IntA = IntB$ and $ClA = ClB \Rightarrow ClIntA = ClIntB$ and $IntClA = IntClB$. Now $B \subseteq A \subseteq IntClA\cup ClIntA = IntClB\cup ClIntB \Rightarrow B$ is b-open. This proves (i).
Suppose A is $^*b$-open. Then $IntClA\cap ClIntA$. Since A is near to B and closer to B,$IntA = IntB$ and $ClA = ClB \Rightarrow ClIntA =$

$ClIntB$ and $IntClA = IntClB$. Now $B \subseteq A \subseteq IntClA \cap ClIntA = IntClB \cap ClIntB \Rightarrow B$ is $^*b$-open. This proves (ii).
Suppose B is b-closed. Then $IntClB \cap ClIntB \subseteq B$. Since A is near to B and closer to B, $IntA = IntB$ and $ClA = ClB$ $\Rightarrow ClIntA = ClIntB$ and $IntClA = IntClB$. Now $IntClA \cap ClIntA = IntClB \cap ClIntB \subseteq B \subseteq A \Rightarrow A$ is b-closed. This proves (iii).
Suppose B is $^*b$-closed. Then $IntClB \cap ClIntB \subseteq B$. Since A is near to B and closer to B,$IntA = IntB$ and $ClA = ClB$ $\Rightarrow ClIntA = ClIntB$ and $IntClA = IntClB$. Now $IntClA \cap ClIntA = IntClB \cap ClIntB \subseteq B \subseteq A \Rightarrow A$ is $^*b$-closed. This proves (iv). This completes the proof of the proposition. □

Proposition 4.14. *Let A be near to B and closer to B in* (X, τ). *Then*

(i) *A is a p-set if and only if B is a p-set.*

(ii) *A is a q-set if and only if B is a q-set.*

(iii) *A is a Q-set if and only if B is a Q-set.*

Proof. A is near to B and closer to B. Then $IntA = IntB$ and $ClA = ClB \Rightarrow ClIntA = ClIntB$ and $IntClA = IntClB$. Now it follows that A is a p-set iff $ClIntA \subseteq IntClA$ iff $ClIntB \subseteq IntClB$ iff B is a p-set, A is a q-set iff $ClIntA \supseteq IntClA$ iff $ClIntB \supseteq IntClB$ iff B is a q-set and A is a Q-set iff $ClIntA = IntClA$ iff $ClIntB = IntClB$ iff B is a Q-set. □

5. Conclusion

Nearly open sets namely regular open, semiopen, preopen, α-open, β-open sets and the corresponding closed sets are characterized using the near, ρ-near, closer and ρ-closer relations in topology. Further these relations also used to examine p-sets, q-sets and Q-sets. In particular it is proved that there is an one-to-one correspondence between the collection of near classes and the topology and that between the collection of closer classes and the collection closed sets.

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