Malaya Journal of Matematik, Vol. 8, No. 4, 2186-2189, 2020
https://doi.org/10.26637/MJM0804/0146

Introduction of color class dominating sets in graphs

A. Vijayalekshmi[1]* and A. E. Prabha[2]

Abstract
Let $G=(V,E)$ be a graph. In this paper, we define a new graph parameter called color class domination number of G. A color class dominating set of G is a proper coloring $\mathscr{C}$ of G with the extra property that every color class in $\mathscr{C}$ is dominated by a vertex in G. A color class dominating set is said to be a minimal color class dominating set if no proper subset of $\mathscr{C}$ is a color class dominating set of G. The color class domination number of G is the minimum cardinality taken over all minimal color class dominating sets of G and is denoted by $\gamma_\chi(G)$. Here we also obtain $\gamma_\chi(G)$ for Path graph, Cycle graph, Helm graph, Flower graph, Sunflower graph, Gear graph and Sunlet graph.

Keywords
Chromatic number, Domination number, Color class Dominating set, Color class domination number.

AMS Subject Classification
05C15, 05C69.

[1] *Department of Mathematics, S.T.Hindu College, Nagercoil-629002, Tamil Nadu, India.*
[2] *Research Scholar, Reg.No.12201, Department of Mathematics, S.T.Hindu College, Nagercoil-629002, Tamil Nadu, India.*
[1,2] Affliated to Manonmaniam Sundaranar University, Tirunelveli-627012, Tamil Nadu, India.
***Corresponding author**: [1]vijimath.a@gmail.com
Article History: Received **12** September **2020**; Accepted **21** November **2020**

Contents

1 Introduction 2186
2 Main Results 2187
References 2189

1. Introduction

All graphs considered in this paper are finite, undirected graphs and we follow standard definitions of graph theory as found in [3].

Let $G=(V,E)$ be a graph of order p. The open neighborhood $N(v)$ of a vertex $v \in V(G)$ consists of the set of all vertices adjacent to v.The closed neighborhood of v is $N[v]=N(v)\cup\{v\}$. For a set $S \subseteq V$, the open neighborhood $N(S)$ is defined to be $\bigcup_{v\in S} N(v)$ and the closed neighborhood of S is $N[S]=N(S)\cup S$. For any set H of vertices of G, the induced sub graph $<H>$ is the maximal sub graph of G with vertex set H.

A subset S of V is called a dominating set if every vertex in $V-S$ is adjacent to some vertex in S. A dominating set is a minimal dominating set if no proper subset of S is a dominating set of G. The domination number $\gamma(G)$ is the minimum cardinality taken over all minimal dominating sets of G. A γ-set is any minimal dominating set with cardinality γ. A proper coloring of G is an assignment of colors to the vertices of G such that adjacent vertices have different colors. The smallest number of colors for which there exists a proper coloring of G is called chromatic number of G and is denoted by $\chi(G)$.

The join G_1+G_2 of Graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph union $G_1 \cup G_2$ together with each vertex in V_1 is adjacent to every vertices in V_2. A path on n vertices denoted by P_n, is a connected graph with all but two vertices have degree 2 and $V(P_n)=\{v_i/1\le i\le n\}$ with $v_iv_{i+1}\in E(P_n)$ for $i<n$. A cycle graph is a graph on $n\ge 3$ vertices containing a single cycle through all vertices and is denoted by C_n. The Complete graph K_p has every pair of p vertices adjacent. A wheel graph on $n+1$ vertices is denoted by $W_{1,n}=K_1+C_n$. The helm graph H_n is the graph obtained from a wheel graph $W_{1,n}$ by adjoining a pendant edge at each vertex of the cycle C_n. The flower graph Fl_n is the graph obtained from a helm graph by joining each pendant vertex to the central vertex of the helm. The Sun flower graph Sf_n is the resultant graph obtained from the flower graph of wheel $W_{1,n}$ by adding pendant edges to the

central vertex. The Gear graph is a graph obtained by inserting an extra vertex between each pair of adjacent vertices on the wheel graph $W_{1,n}$. The sunlet SC_n graph on $2n$ vertices is obtained by attaching n pendant edges to a cycle graph C_n

2. Main Results

We introduce a new concept, color class dominating sets on graphs.

Definition 2.1. *Let G be a graph. A color class dominating set of G is a proper coloring $\mathscr{C}$ of G with the extra property that every color classes in $\mathscr{C}$ is dominated by a vertex in G. A color class dominating set is said to be a minimal color class dominating set if no proper subset of $\mathscr{C}$ is a color class dominating set of G. The color class domination number of G is the minimum cardinality taken over all minimal color class dominating sets of G and is denoted by $\gamma_\chi(G)$. This concept is illustrated by the following example.*

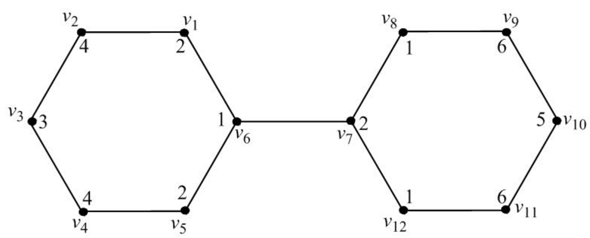

Figure 1

In figure 1, $\mathscr{C}_1 = \{v_6, v_8, v_{12}\}, \mathscr{C}_2 = \{v_1, v_5, v_7\}$, $\mathscr{C}_3 = \{v_3\}$, $\mathscr{C}_4 = \{v_2, v_4\}, \mathscr{C}_5 = \{v_{10}\}$ $\mathscr{C}_6 = \{v_9, v_{11}\}$. Then the color classes $\mathscr{C}_i, 1 \le i \le 6$ are dominated by vertices v_7, v_6, v_2 (or v_3 or v_4), v_3, v_9 (or v_{10} or v_{11}) and v_{10} respectively. So $\gamma_\chi(G) = 6$.

Theorem 2.2. *Let G be a graph of order p without isolated vertices. Then*

(i) $\chi(G) \le \gamma_\chi(G)$

(ii) $\max\{\chi(G), \gamma(G)\} \le \gamma_\chi(G) \le p$.

Proof. Since γ_χ-coloring of G is a proper coloring, $\chi(G) \le \gamma_\chi(G)$. Now, let be a γ_χ-coloring of G. Then for each color class $\mathscr{C}_i, 1 \le i \le \gamma_\chi(G)$, there exist a vertex $v_i \in V$ such that $\mathscr{C}_i$ is dominated by v_i. Let $S = \{v_1, v_2, \ldots, v_{\gamma_\chi}\}$, where $v_i \in \mathscr{C}_i, 1 \le i \le \gamma_\chi(G)$. Now, we have to show that S is a γ-set. Let $y \in V - S$. Then $y \in \mathscr{C}_i$ for some $i, 1 \le i \le \gamma_\chi(G)$. By the definition of γ_χ-coloring of G, y is adjacent to the vertex v_i of S. Then S is a γ-set. Therefore $\gamma(G) \le \gamma_\chi(G)$. since G is a graph of order p, G can be colored with at most p colors. Hence, $\max\{\chi(G), \gamma(G)\} \le \gamma_\chi(G) \le p$. □

Proposition 2.3. *For the Wheel graph $W_{1,n}$, $n \ge 3$,*

$$\gamma_\chi(W_{1,n}) = \begin{cases} 3 & \text{if } n \text{ is even} \\ 4 & \text{if } n \text{ is odd} \end{cases}$$

Theorem 2.4. *Let G be P_n or C_n. Then for $n > 3$,*

$$\gamma_\chi(P_n) = \gamma_\chi(C_n) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 (\bmod 4) \\ \frac{n}{2} + 1 & \text{if } n \equiv 2 (\bmod 4) \\ \left\lfloor \frac{n}{2} \right\rfloor + 1 & \text{if } n \equiv 1, 3 (\bmod 4) \end{cases}$$

Proof. Let $V(P_n) = \{v_i / 1 \le i \le n\}$ and $v_i v_{i+1} \in E(P_n)$ for $i < n$. Let $n > 4$. Let $\mathscr{C}$ be a γ_χ-coloring of P_n. We consider three cases.

Case(i): $n \equiv 0 (\bmod 4)$. For $= 1, 2, \ldots, \frac{n}{4}$, let

$$H_i = < v_{4i-3}, v_{4i-2}, v_{4i-1}, v_{4i} >$$

be the vertex induced sub graph of P_n. Then for each $i, 1 \le i \le \frac{n}{4}$ assign two distinct colors, say, $2i-1, 2i$ to the vertices $\{v_{4i-3}, v_{4i-1}\}$ and $\{v_{4i-2}, v_{4i}\}$ respectively, we get a γ_χ-coloring of P_n. So $\gamma_\chi(P_n) = \frac{2n}{4} = \frac{n}{2}$.

Case (ii): $n \equiv 2 (\bmod 4)$ since $n-2 \equiv 0 (\bmod 4), P_n$ is obtained from P_{n-2} followed by P_2. So $\gamma_\chi(P_n) = \gamma_\chi(P_{n-2}) + \gamma_\chi(P_2) = \frac{n}{2} + 1$.

Case (iii): $n \equiv 1, 3 (\bmod 4)$. When $n \equiv 1 (\bmod 4)$, since $n - 1 \equiv 0 (\bmod 4), P_n$ is obtained from P_{n-1} followed by P_1. So $\gamma_\chi(P_n) = \gamma_\chi(P_{n-1}) + \gamma_\chi(P_1) = \left\lfloor \frac{n}{2} \right\rfloor + 1$. When $n \equiv 3 (\bmod 4)$, as above $\gamma_\chi(P_n) = \gamma_\chi(P_{n-3}) + \gamma_\chi(P_3) = \left\lfloor \frac{n}{2} \right\rfloor + 1$. This γ_χ-coloring is true for C_n also. □

Theorem 2.5. *For the Helm graph $G = H_n, n \ge 3, \gamma_\chi(H_n) = n$.*

Proof. Let H_n be a helm graph with

$$V(H_n) = \{v\} \cup \{v_i / 1 \le i \le n\} \cup \{u_i / 1 \le i \le n\}.$$

Assign colors 1, 2 and n to the vertices $\{v_1, u_n\}, \{u_1, v_2, v_n\}$ and $\{v_{n-1}, v\}$ respectively. Assign color $i (3 \le i \le n-1)$ to the vertices $\{v_i, u_{i-1}\}$. The color classes $\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_n$ are dominated by v_n, v_1, v_{n-1} respectively. Also the color class $\mathscr{C}_i (3 \le i \le n-1)$ dominated by the vertex v_{i-1}. Hence $\gamma_\chi(H_n) = n$. □

Example 2.6.

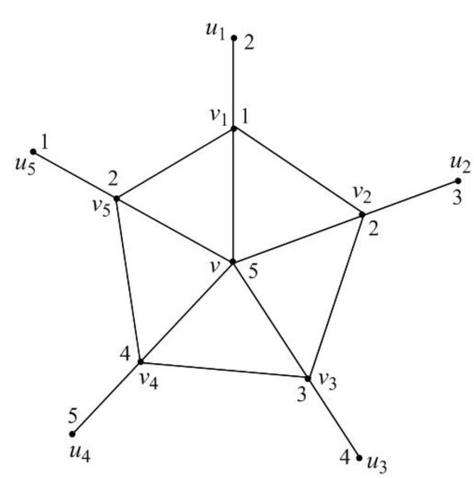

Figure 2

Theorem 2.7. *(i) If G is a flower graph Fl_n, $n \geq 3$,*

$$\gamma_\chi(Fl_n) = \begin{cases} 3 \text{ if } n \text{ is even} \\ 4 \text{ if } n \text{ is odd} \end{cases}$$

(ii) If G is a sunflower graph $Sf_n, n \geq 3$,

$$\gamma_\chi(Sf_n) = \begin{cases} 3 \text{ if } n \text{ is even} \\ 4 \text{ if } n \text{ is odd} \end{cases}$$

Proof. **(i)**
By the definition of flower graph, Fl_n is obtained from a helm graph by joining each pendant vertex to the central vertex. Let $V(Fl_n) = \{v_1, v_2, \ldots, v_{2n+1}\}$, where v_1 be the central vertex, $v_i (2 \leq i \leq n+1)$ be the vertices on the cycle C_n and $v_j (n+2 \leq i \leq 2n+1)$ be the vertices on the pendant edges of H_n such that $v_i (2 \leq i \leq n+1)$ is adjacent to v_{n+i} and v_1. We Consider two cases:
Case (i): n is even. Let $\mathscr{C} = \{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3\}$ be a proper coloring of Fl_n in which $\mathscr{C}_1 = \{v_1\}, \mathscr{C}_2 = \{v_2, v_4, \ldots, v_n\} \cup \{v_{n+3}, v_{n+5}, \ldots, v_{2n+1}\}$, $\mathscr{C}_3 = \{v_3, v_5, \ldots, v_{n+1}\} \cup \{v_{n+2}, v_{n+4}, \ldots, v_{2n}\}$.
Then the color class $\mathscr{C}_1$ dominated by the vertex v_2 and the color classes $\mathscr{C}_2$ and $\mathscr{C}_3$ are dominated by the vertex v_1.
Case (ii): n is odd. Let $\mathscr{C} = \{\mathscr{C}_1, \mathscr{C}_2, \mathscr{C}_3, \mathscr{C}_4\}$ be a proper coloring of Fl_n in which $\mathscr{C}_1 = \{v_1\}, \mathscr{C}_2 = \{v_2, v_4, \ldots, v_{n-1}\} \cup \{v_{n+3}, v_{n+5}, \ldots, v_{2n}\}, \mathscr{C}_3 = \{v_3, v_5, \ldots, v_n\} \cup \{v_{n+2}, v_{n+4}, \ldots, v_{2n+1}\}$ and $\mathscr{C}_4 = \{v_{n+1}\}$. Then the color class $\mathscr{C}_1$ dominated by the vertex v_2 and the color classes $\mathscr{C}_2, \mathscr{C}_3$ and $\mathscr{C}_4$ dominated by the vertex v_1. Therefore, the coloring $\mathscr{C}$ is a γ_χ-coloring of Fl_n and hence

$$\gamma_\chi(Fl_n) = \begin{cases} 3 \text{ if } n \text{ is even} \\ 4 \text{ if } n \text{ is odd} \end{cases}$$

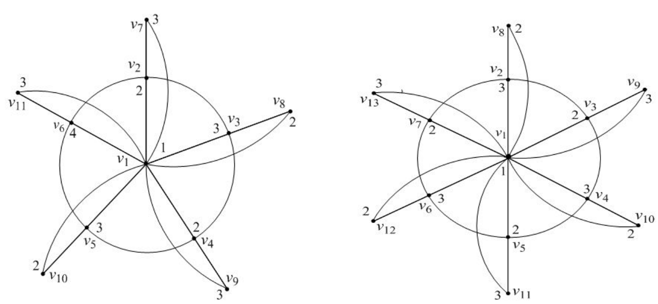

Figure 3. n odd and n even

(ii)
Let G be a sunflower graph Sf_n.Then G is a flower graph with pendant edges attached to the central vertex. As in Theorem $(2.7(i))$, we assign the same proper coloring of Fl_n with color 2 to the pendant vertices $\{v_{2n+2}, v_{2n+3}, \ldots, v_{3n+1}\}$ and we get the γ_χ-coloring of Sf_n. Hence

$$\gamma_\chi(Sf_n) = \begin{cases} 3 \text{ if } n \text{ is even} \\ 4 \text{ if } n \text{ is odd} \end{cases}$$

□

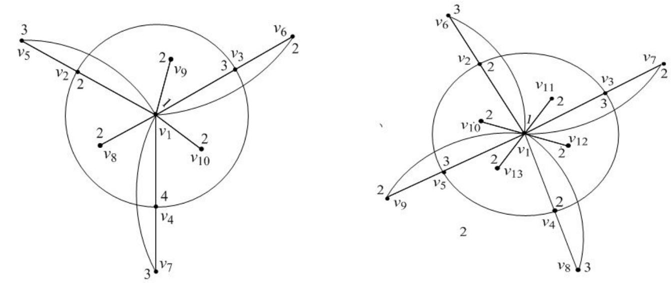

Figure 4. n odd and n even

Theorem 2.8. *The gear graph G_n has $\gamma_\chi(G_n) = \lceil \frac{n}{2} \rceil + 1$.*

Proof. Let

$$V(G_n) = \{u\} \cup \{u_1, u_2, \ldots, u_n\} \cup \{v_1, v_2, \ldots, v_n\},$$

where v is the central vertex and $\deg(u_i) = 3$ and $\deg(v_i) = 2, 1 \leq i \leq n$. Assign distinct colors say, $i, 1 \leq i \leq \lceil \frac{n}{2} \rceil - 1$ to the vertices $\{v_{2i-1}, v_{2i}\}$ respectively. Also assign distinct colors say, $\lceil \frac{n}{2} \rceil$ and $\lceil \frac{n}{2} \rceil + 1$ to the vertices $\{u_1, u_2, \ldots, u_n\}$ and $\{v, v_n\}$ when n is odd and $\{u_1, u_2, \ldots, u_n\}$ and $\{v, v_{n-1}, v_n\}$ when n is even respectively, we get a γ_χ-coloring. Hence,

$$\gamma_\chi(G_n) = \left\lceil \frac{n}{2} \right\rceil + 1.$$

□

Example 2.9.

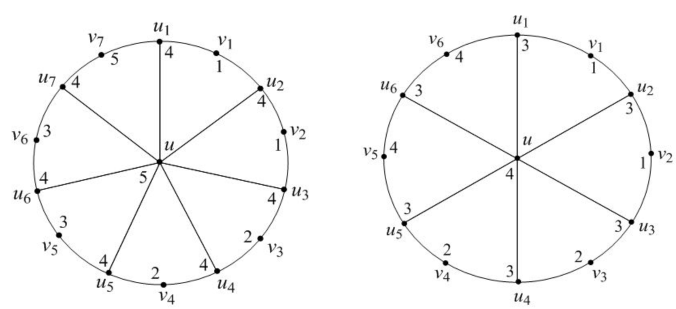

Figure 5. n odd and n even

Theorem 2.10. *The Sunlet graph SC_n has $\gamma_\chi(SC_n) = n$.*

Proof. Let

$$V(SC_n) = \{u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_n\}$$

with $\deg(u_i) = 3 (1 \leq i \leq n)$ and $\deg(v_i) = 1 (1 \leq i \leq n)$. We consider two cases:
Case (i). When n is even, assume color i, where $i = 1, 3, 5, \ldots, n-1$ to the vertices $\{u_i, v_{i+1}\}$ and color j, where $j = 2, 4, \ldots, n$ to the vertices $\{u_j, v_{j-1}\}$ respectively, we get the γ_χ-coloring of SC_n.
Case(ii). When n is odd, assign colors 1,2 and n to the vertices $\{u_1, v_n\}, \{u_2, u_n, v_1\}$ and $\{v_{n-1}\}$ respectively. Also assign color $i (3 \leq i \leq n-1)$ to the vertices $\{u_i, v_{i-1}\}$, we get a γ_χ-coloring. Thus $\gamma_\chi(SC_n) = n$. □

Example 2.11.

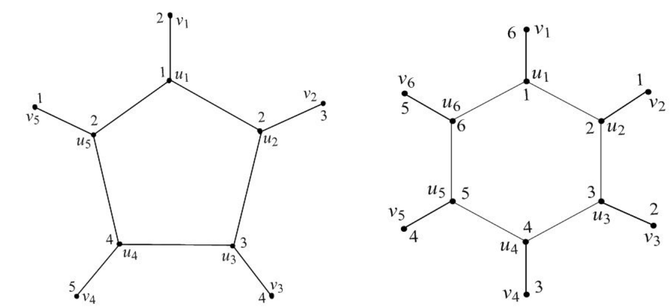

Figure 6. *n* odd and *n* even

References

[1] A.Vijayalekshmi, Total Dominator Colorings in Graphs, *International Journal of Advancements in Research & Technology,* 1(4)(2012), 1–10.

[2] R. M. Gera, *On Dominator Colorings in Graphs*, Graph theory Notes of New York, Network Academy of Sciences. LII , 25-30, 2007.

[3] F. Harrary, *Graph Theory*, Addition –Wesley Reading Mass, 1969.

[4] Terasa W. Haynes, Stephen T. Hedetniemi, Peter J Slater, *Domination in Graphs*, Marcel Dekker, New york, 1998.

⋆⋆⋆⋆⋆⋆⋆⋆⋆
ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
⋆⋆⋆⋆⋆⋆⋆⋆⋆