MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **10(01)**(2022), 1–19.
http://doi.org/10.26637/mjm1001/001

A new analytical method to solve Klein-Gordon equations by using homotopy perturbation Mohand transform method

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*R*eceived 12 August 2021; *A*ccepted 17 December 2021

Abstract. In this paper, we will study about Fractional-order partial differential equations in Mathematical Science and we will introduce and analyse fractional calculus with an integral operator that contains the Caputo- Fabrizio's fractional-order derivative. The advanced method is an appropriate union of the new integral transform named as 'Mohand transform' and the homotopy perturbation method. Some numerical examples are used to communicate the generality and clarity of the proposed method. We will also find the analytical solution of the linear and non-linear Klein-Gordan equation which originate in quantum field theory. The homotopy perturbation Mohand transform method (HPMTM) is a merged form of Mohand transform, homotopy perturbation method, and He's polynomials. Some numerical examples are used to indicate the generality and clarity of the proposed method.

Keywords: Mohand Transform, Homotopy Perturbation Method (HPM), Fractional Calculus

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Contents

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1. Introduction, Background and Preliminaries

Fractional calculus is an eminent phrase in each science and technology. Differential and crucial equations represent and outline various phenomena of technological knowledge and mold the difficulty in a new appearance. Fractional calculus is the generalization of regular differentiation and integration from linear to non-linear order. It extend with derivatives of actual or complex order.
In 1695 L'Hospital enquires Leibnitz that $D^n f$ could be what, if n is fractional. Leibnitz answers that it can be expand in the form of infinite series, such as $d^{1/2}xy$ and distant between infinite series also geometric series, we use only positive and negative integers in the finite series Leibniz also responded that $xd^{\left\{\frac{1}{2}\right\}} = x\sqrt{\mathrm{dx} : x}$ this is a clear paradox.

S.F. Larcroin developed a formula from a case of integer order which starts with$y = x^m$;m is a positive integer

$$\frac{d^n y}{\mathrm{dx}^n} = \frac{m!x^{m-n}}{(m-n)!};m \succeq n$$

Like this many other mathematicians gave their definitions and formula.Fractional calculus attracted some mathematical minds like Fourier, Euler, Marquis de Laplace, and plenty of others due primarily to its incontestable applications in such different fields of science and engineering. The literature is full-fledged by generating, growing, working, modifications, and generalization of the facts, formulae, and definitions relating to fractional calculus. A whole historical development and progress of fragmental calculus operators seem in books of Kilbas, Srivastava, and Trujillo [1], Miller and Ross [2], Nishimoto [3], Oldham and Spanier [4], Podlubny [5] and, Ross [6], etc.
In mathematical analysis, there are several fields wherever fractional calculus operators are usefully employed in numerous branches like integral and differential equations, special functions, integral transforms, operational calculus (see [6],[7]), etc. because it start to be used fractional calculus in various areas as numerous varieties of operators came to light-weight and by the time they were got changed.
Mohand Transform is derived from the classical Fourier integral. Based on the mathematical simplicity of the Mohand transform and its fundamental properties. Mohand transform was introduced by Mohand Mahgoub to facilitate the process of solving ordinary and partial differential equations in the time domain. Typically, Fourier, Laplace, Elzaki, Aboodh, kamal and Sumudu transforms are the convenient mathematical tools for solving differential equations.
Mohand transform and some of its fundamental properties are also used to solve differential equations.

2. Mohand Transform

2.1. Definition

A new transform called the Mohand transform defined for function of exponential order we consider functions in the set A defined by: For a given function in the set A, the constant M must be finite number,k_1, k_2 may be finite or infinite.

$$A = \{f(t) : \exists M, k_1, k_2 > 0.|f(t)| < \mathrm{Me}^{\frac{|t|}{k_j}}, \mathrm{if} t \in (-1)^j \times [0, \infty)\} \tag{2.1}$$

The Mohand transform denoted by the operator M (.) defined by the integral equations

$$M[f(t)] = R(v) = v^2 \int_0^\infty f(t)e^{-\mathrm{vt}}\, dt, t \geq 0, k_1 \leq v \leq k_2 \tag{2.2}$$

The variable v in this transform is used to factor the variable t in the argument of the function f . this transform has deeper Connection with the Laplace ,Elzaki, and Aboodh transform.
The purpose of this study is to show the applicability of this interesting new transform and its efficiency in solving the linear differential equations.

2.2. Mohand Transform and Different Types of Results

Mohand Transform is derived from the classical Fourier integral based on the mathematical simplicity of the Mohand transform with its fundamental properties.Mohand transform was introduced by MohandMahgoub to facilitate the process of solving ordinary and partial differential equations in the time domain.Mohandtransform defined for the function of exponential order we consider functions in the set A defined by:

$$A = f(t) : M, K_1, K_2 > 0.|f(t)| < Me^{\frac{|t|}{K_t}} if\ t \in (-1)^j \times [0, \infty)$$

where M must be finite number and K_1, K_2 may be finite or infinite, for a given function in set A.
The integral equation defines the operator M(.) which represents Mohand transform i.e

$$M[f(X)] = R(v) = v^2 \int_0^\infty e^{vt} f(t)dt, t \geq 0, K_1 \leq v \leq K_2 \tag{2.3}$$

The variable v in this transform is used to factor the variable t in the argument of the function f.
If $R_1(t)$ and $R_2(t)$ represents Mohand transform for functions $F_1(t)$ and $F_2(t)$ respectively,then Mohand transform of their convolution $F_1(t) * F_2(t)$ is given by

$$M(F_1(t) * F_2(t)) = \tfrac{1}{v^2} MF_1(t)MF_2(t)$$

$$M(F_1(t) * F_2(t)) = \frac{1}{v^2} R_1(t)R_2(t) \tag{2.4}$$

where $F_1(t) * F_2(t)$ is defined by

$$F_1(t) * F_2(t) = \int_0^t F_1(t-x)F_2(t)dt = \int_0^t F_1(t)F_2(t-x)dt \tag{2.5}$$

Caputo fractional time derivative

$$D_t^\beta(h(t)) = \frac{M(\beta)}{1-\beta} \int_\alpha^t h'(x)e^{\left[-\beta\left(\frac{t-x}{1-\beta}\right)\right]}dx \tag{2.6}$$

$M(\alpha)$ is function of normalization such as $M(0) = M(1) = 1$.

$$M\{D_t^\beta(h(t))\} = \frac{M(\beta)}{1-\beta} v^2 \int_0^\infty e^{-vt} \int_\alpha^t h'(x)e^{\left[-\beta\left(\frac{t-x}{1-\beta}\right)\right]}dxdt \tag{2.7}$$

We use the convolution property of Mohand transform is defined as

$$M\{D_t^\beta(h(t))\} = \frac{M(\beta)}{1-\beta}[M(h'(x))] * M[e^{-\frac{\beta t}{1-\beta}}]$$

$$M\{D_t^\beta(h(t))\} = \frac{M(\beta)}{1-\beta}[vR(v) - h(0)v^2]\frac{v^2}{v-(\frac{-\beta}{1-\beta})}$$

$$M\{D_t^\beta(h(t))\} = M(\beta)[vR(v) - h(0)v^2]\frac{v^2}{v+(\beta(1-v))}$$

The solution of Caputo-Fabrizio fractional derivative is:

$$M\{D_t^\beta(h(t))\} = \frac{M(\beta)\left\{v^3 M[h(t)] - v^4 h(0)\right\}}{v+\beta(1-v)} \tag{2.8}$$

3. HPMTM for the model

In this section, we will study Klein-Gordon equation and its application by using homotopy perturbation Mohand transform method.

3.1. Solution of Klein-Gordon equation:

Klein-Gordon equation is

$$u_{\text{tt}}(x,t) - u_{\text{xx}}(x,t) + \text{a}u(x,t) = g(x,t) \tag{3.1}$$

with initial condition

$$u(x,0) = h(x), u_t(x,0) = f(x) \tag{3.2}$$

Taking the Mohand transform on both sides ofequ. (2.1), we get

$$M[u_{\text{tt}}(x,t)] = M[u_{\text{xx}}(x,t) - \text{a}u(x,t)] + M[g(x,t)] \tag{3.3}$$

Using the convolution property of Mohand transform, we get

$$v^2R(x,v) - v^3u(0) - v^2u(0) = M[u_{\text{xx}}(x,t) - \text{a}u(x,t)] + M[g(x,t)] \tag{3.4}$$

On simplifying and initial conditions, we get

$$R(x,v) = f(x) + \text{v}h(x) + \frac{1}{v^2}M[u_{\text{xx}}(x,t) - \text{a}u(x,t)] + \frac{1}{v^2}M[g(x,t)] \tag{3.5}$$

Taking inverse Mohand transform on both sides ofequ. (2.5), we get

$$u(x,t) = G(x,t) + M^{-1}[\frac{1}{v^2}M[u_{\text{xx}}(x,t) - \text{a}u(x,t)]] \tag{3.6}$$

where G(x,t) represents the term arising from the function and the specified initial conditions.
Using the HPM method, we get

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \tag{3.7}$$

Putting the equation (2.6) in equ. (2.7), we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = G(x,t) + p\left(M^{-1}\left[\frac{1}{v^2}M\left[\frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty} p^n u_n(x,t) - a\sum_{n=0}^{\infty} p^n u_n(x,t)\right]\right]\right) \tag{3.8}$$

On collecting the coefficients of exponents of p

$p^0 : u_0(x,t) = G(x,t)$

$$p^1 : u_1(x,t) = \left(M^{-1}\left[\frac{1}{v^2}M\left[\frac{\partial^2}{\partial x^2}u_0(x,t) - \mathrm{au}_0(x,t)\right]\right]\right) \tag{3.9}$$

$p^2 : u_2(x,t) = \left(M^{-1}\left[\frac{1}{v^2}M\left[\frac{\partial^2}{\partial x^2}u_1(x,t) - \mathrm{au}_1(x,t)\right]\right]\right)$

$p^3 : u_3(x,t) = \left(M^{-1}\left[\frac{1}{v^2}M\left[\frac{\partial^2}{\partial x^2}u_2(x,t) - \mathrm{au}_2(x,t)\right]\right]\right)$

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and similarly,

$$p^n : u_n(x,t) = \left(M^{-1}\left[\frac{1}{v^2}M\left[\frac{\partial^2}{\partial x^2}u_{(n-1)}(x,t) - \mathrm{au}_{(n-1)}(x,t)\right]\right]\right) \tag{3.10}$$

Hence, the solution is:

$$u(x,t) = \lim_{N\to\infty}\sum_{n=0}^{N} u_n(x,t) \tag{3.11}$$

3.2. Study of Mohand Transform Homotopy Perturbation Method (MTHPM)

Let a general non-linear non-homogeneous partial differential equation

$$\mathrm{Du}(x,t) + \mathrm{Ru}(x,t) + \mathrm{Nu}(x,t) = g(x,t) \tag{3.12}$$

With the initial conditions

$$u(x,0) = h(x), u_t(x,0) = f(x) \tag{3.13}$$

where D is the linear differential operator of order 2, R is a linear differential operator of less than D; N is the general nonlinear differential operator and is the source term.

Applying the Mohand transform on both sides ofequ. (2.12), we get

$$M[\mathrm{Du}(x,t)] = M[g(x,t)] - M[\mathrm{Ru}(x,t) + \mathrm{Nu}(x,t)] \tag{3.14}$$

Using the property of Mohand transform, we have

$$\left[v^2 R(x,v) - v^3 u(x,0) - v^2(x,0)\right] = M[g(x,t)] - M[\mathrm{Ru}(x,t) + \mathrm{Nu}(x,t)] \tag{3.15}$$

After the simplification and initial conditions, we get

$$R(x,v) = \mathrm{vh}(x) + f(x) + \frac{1}{v^2} M[g(x,t)] - \frac{1}{v^2} M[\mathrm{Ru}(x,t) + \mathrm{Nu}(x,t)] \tag{3.16}$$

Taking inverse Mohand transform on both sides of equ. (2.16), we get

$$u(x,t) = G(x,t) - M^{-1}\left[\frac{1}{v^2} M[\mathrm{Ru}(x,t) + \mathrm{Nu}(x,t)]\right] \tag{3.17}$$

where G(x,t) represents the term arising from the function and the specified initial conditions.

Now we use the HPM

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \tag{3.18}$$

and the non-linear term can be written as

$$\mathrm{Nu}(x,t) = \sum_{n=0}^{\infty} p^n H_n(x,t) \tag{3.19}$$

where is $H_n(x,t)$ He's polynomials and given by

$$H_n\left(u_0, u_1, u_2, ...u_n\right) = \frac{1}{n!} \frac{\partial^2}{\partial p^2} \left[N \sum_{n=0}^{\infty} p^i u_i\right]_{p=0} \tag{3.20}$$

Substituting the equ. (2.19) and equ. (2.18) in equ. (2.17), we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = G(x,t) - p\left(M^{-1}\left[\frac{1}{v^2} M\left[R\sum_{n=0}^{\infty} p^n u_n(x,t) + N\sum_{n=0}^{\infty} p^n H_n(u)\right]\right]\right) \tag{3.21}$$

On collecting the coefficient of exponents of p

$p^0 : u_0(x,t) = G(x,t)$

$$p^1 : u_1(x,t) = -M^{-1}\left[\frac{1}{v^2} M\left[\mathrm{R}u_0(x,t) + H_0(u)\right]\right] \tag{3.22}$$

$p^2 : u_2(x,t) = -M^{-1}\left[\frac{1}{v^2} M\left[\mathrm{R}u_1(x,t) + H_1(u)\right]\right]$

$p^3 : u_3(x,t) = -M^{-1}\left[\frac{1}{v^2} M\left[\mathrm{R}u_2(x,t) + H_2(u)\right]\right]$

$$\vdots$$

and similarly,

$$p^n : u_n(x,t) = -M^{-1}\left[\frac{1}{v^2}M\left[\mathrm{R}u_{n-1}(x,t) + H_{n-1}(u)\right]\right] \tag{3.23}$$

Hence, the solution is

$$u(x,t) = \lim_{N\to\infty}\sum_{n=0}^{N} u_n(x,t) \tag{3.24}$$

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + ...u_n(x,t) \tag{3.25}$$

4. Applications of the MTHPM

In this part, we applythe Mohand transform homotopy perturbation method (MTHPM) to solve the linear and nonlinear Klein-Gordon equation.

Example 1: Consider the linear Klein-Gordon equation

$$u_{\mathrm{tt}}(x,t) - u_{\mathrm{xx}}(x,t) + u(x,t) = 0 \tag{4.1}$$

With the initial conditions

$$u(x,0) = 0, u_t(x,0) = x \tag{4.2}$$

Taking Mohand transform on both sides of equ. (3.1), we get

$$M[u_{\mathrm{tt}}(x,t)] = M[u_{\mathrm{xx}}(x,t) - u(x,t)] \tag{4.3}$$

Using the convolution property of Mohand transform, we get

$$v^2R(x,v) - v^3u(0) - v^2u'(0) = M[u_{\mathrm{xx}}(x,t) - u(x,t)] \tag{4.4}$$

On simplifying and above initial conditions, we get

$$R(x,v) = x + \frac{1}{v^2}M[u_{\mathrm{xx}}(x,t) - u(x,t)] \tag{4.5}$$

Taking inverse Mohand transform on both sides of equ. (3.5), we get

$$u(x,t) = xt + M^{-1}[\frac{1}{v^2}M[u_{\mathrm{xx}}(x,t) - u(x,t)]] \tag{4.6}$$

Now we use the HPM and the non-linear term then we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = xt + M^{-1}[\frac{1}{v^2} M[\frac{\partial^2}{\partial x^2} \sum_{n=0}^{\infty} p^n u_n(x,t) - \sum_{n=0}^{\infty} p^n H_n(x,t)]] \tag{4.7}$$

Collecting the coefficients of exponents of p

$p^0 : u_0(x,t) = xt$

$$p^1 : u_1(x,t) = M^{-1}\left[\frac{1}{v^2} M\left[\frac{\partial^2}{\partial x^2} u_0(x,t) - u_0(x,t)\right]\right] = -x\frac{t^3}{3!} \tag{4.8}$$

$$p^2 : u_2(x,t) = M^{-1}\left[\frac{1}{v^2} M\left[\frac{\partial^2}{\partial x^2} u_1(x,t) - u_1(x,t)\right]\right] = -x\frac{t^5}{5!} \tag{4.9}$$

$$p^3 : u_3(x,t) = M^{-1}\left[\frac{1}{v^2} M\left[\frac{\partial^2}{\partial x^2} u_2(x,t) - u_2(x,t)\right]\right] = -x\frac{t^7}{7!} \tag{4.10}$$

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Similarly, we can obtain further values.

Hence the u(x,t) is

$$\begin{aligned} u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + ... \\ &= x\left[\left(-\frac{t^7}{7!} + \frac{t^5}{5!} - \frac{t^3}{3!} + t...\right)....\right] \\ &= xsint \end{aligned} \tag{4.11}$$

Example 2: Consider the linear Klein-Gordon equation

$$u_{\text{tt}}(x,t) - u_{\text{xx}}(x,t) + u(x,t) = 2\text{sinx} \tag{4.12}$$

With the initial conditions

$$u(x,0) = \text{sinx}, u_t(x,0) = 1 \tag{4.13}$$

Applyingthe Mohand transform on both sides of equ. (3.14), we get

$$M[u_{\text{tt}}(x,t)] = M[u_{\text{xx}}(x,t) - u(x,t)] + M[2\text{sinx}] \tag{4.14}$$

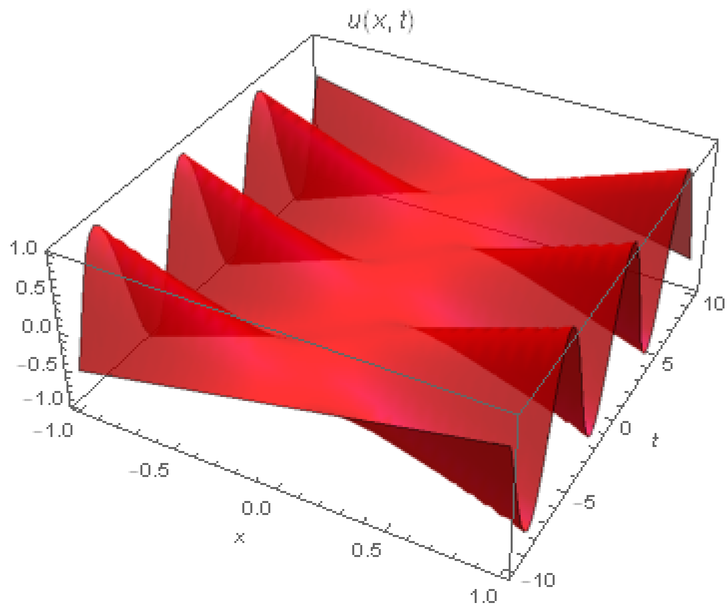

Figure 1: The Graph of $u(x,t) = xsint, t > 0. -\infty \leq x \leq \infty$

Using the convolution property of Mohand transform, we get

$$v^2 R(x,v) - v^3 u(x,0) - v^2 u'(x,0) = M[u_{xx}(x,t) - u(x,t)] + 2sinx(v) \tag{4.15}$$

After the simplification and above initial conditions, we get

$$R(x,v) = vsinx + 1 + 2\text{sinx}\frac{1}{v} + \frac{1}{v^2} M\left[u_{xx}(x,t) - u(x,t)\right] \tag{4.16}$$

Taking inverse Mohand transform on both sides of equ. (3.18), we get

$$u(x,t) = \text{sinx} + t^2\text{sinx} + t + M^{-1}\left[\frac{1}{v^2} M\left[u_{xx}(x,t) - u(x,t)\right]\right] \tag{4.17}$$

Now we use the HPM and the non-linear term then we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = \text{sinx} + t + t^2\text{sinx} + p\left(M^{-1}\left[\frac{1}{v^2} M[\left[\frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty} p^n u_n(x,t)] - \sum_{n=0}^{\infty} p^n u_n(u)\right]\right]\right) \tag{4.18}$$

Collecting the coefficients of exponents of p

$$p^0 : u_0(x,t) = sinx + t + t^2 sinx \tag{4.19}$$

$$p^1 : u_1(x,t) = M^{-1}\left[\frac{1}{v^2}M\left[\frac{\partial^2}{\partial x^2}u_0(x,t) - u_0(x,t)\right]\right]$$
$$= -\left[\frac{t^3}{3!} + \frac{t^4}{3!}\text{sinx} + t^2\text{sinx}\right] \tag{4.20}$$

$$p^2 : u_2(x,t) = M^{-1}\left[\frac{1}{v^2}M\left[\frac{\partial^2}{\partial x^2}u_1(x,t) - u_1(x,t)\right]\right]$$
$$= \left[\frac{t^5}{5!} + 8\frac{t^6}{6!}\text{sinx} + \frac{t^4}{3!}\text{sinx}\right] \tag{4.21}$$

$$p^3 : u_3(x,t) = M^{-1}\left[\frac{1}{v^2}M\left[\frac{\partial^2}{\partial x^2}u_2(x,t) - u_2(x,t)\right]\right]$$
$$= -\left[\frac{t^7}{7!} + 8\frac{t^6}{6!}\text{sinx} + 16\frac{t^8}{8!}\text{sinx}\right] \tag{4.22}$$

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Similarly, we can obtain further values.

Hence the u(x,t) is

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + ...$$
$$= \text{sinx} + \left[-\frac{t^7}{7!} + \frac{t^5}{5!} - \frac{t^3}{3!} + t\right]$$
$$= sinx + sint \tag{4.23}$$

Example 3: Consider the following nonlinear Klein-Gordon equation

$$u_{\text{tt}}(x,t) - u_{\text{xx}}(x,t) + u^2(x,t) = t^2x^2 \tag{4.24}$$

With the initial conditions

$$u(x,0) = 0, u_t(x,0) = x \tag{4.25}$$

Applyingthe Mohand transform on both sides of equ. (3.29), we get

$$M\left[u_{\text{tt}}(x,t)\right] = M\left[u_{\text{xx}}(x,t) - u^2(x,t)\right] + M\left[t^2x^2\right] \tag{4.26}$$

Using the convolution property of Mohand transform, we get

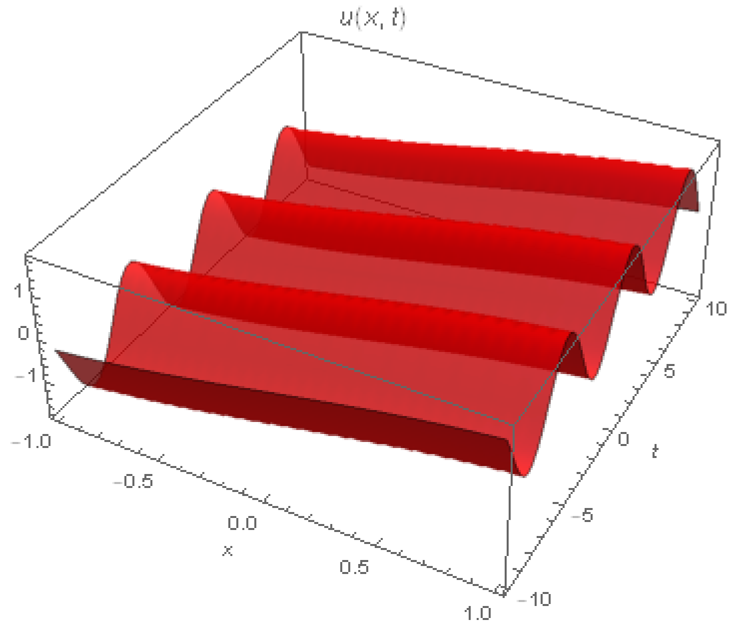

Figure 2: The Graph of $u(x,t) = sinx + sint, t > 0. -\infty \leq x \leq \infty$

$$v^2 R(x,v) - v^3 u(x,0) - v^2 u'(x,0) = 2x^2 \frac{1}{v} + M\left[u_{\text{xx}}(x,t) - u^2(x,t)\right] \tag{4.27}$$

On simplifying and above initial conditions, we get

$$R(x,v) = x + 2x^2 \frac{1}{v^3} + \frac{1}{v^2} M\left[u_{\text{xx}}(x,t) - u^2(x,t)\right] \tag{4.28}$$

Taking inverse Mohand transform on both sides of equ. (3.33), we get

$$u(x,t) = \text{xt} + \frac{x^2}{12} t^4 + \frac{1}{v^3} + M^{-1}\left[\frac{1}{v^2} M\left[u_{\text{xx}}(x,t) - u^2(x,t)\right]\right] \tag{4.29}$$

Now we use the HPM and the non-linear then we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = \text{xt} + \frac{x^2}{12} t^4 + \frac{1}{v^3} + p\left(M^{-1}\left[\frac{1}{v^2} M\left[\frac{\partial^2}{\partial x^2} \sum_{n=0}^{\infty} p^n u_n(x,t) - \sum_{n=0}^{\infty} p^n H_n(u)\right]\right]\right) \tag{4.30}$$

where $H_n(u)$ is represents the He's polynomial of nonlinear terms. The first few components of He's polynomials are given by

$$H_0(u) = (u_0)^2 \tag{4.31}$$

$$H_1(u) = 2u_0 u_1 \tag{4.32}$$

$$H_2(u) = 2u_0u_2 + (u_1)^2 \tag{4.33}$$

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Equating the multipliers of exponents of p

$$p^0 : u_0(x,t) = \text{xt} + \frac{x^2}{12}t^4 \tag{4.34}$$

$$\begin{aligned} p^1 : u_1(x,t) &= \left(M^{-1}\left[\frac{1}{v^2}M\left[\frac{\partial^2}{\partial x^2}u_0(x,t) - H_0(u)\right]\right]\right) \\ &= \left[\frac{t^{10}x^4}{12960} - \frac{t^7x^3}{252} + \frac{t^6}{180} - \frac{t^4x^2}{12}\right] \end{aligned} \tag{4.35}$$

$$\begin{aligned} p^2 : u_2(x,t) &= \left(M^{-1}\left[\frac{1}{v^2}M\left[\frac{\partial^2}{\partial x^2}u_1(x,t) - H_1(u)\right]\right]\right) \\ &= \left[\frac{t^{16}x^6}{18662400} + \frac{383t^{13}x^5}{15921360} - \frac{t^{12}x^2}{71280} + \frac{11t^{10}x^4}{45360} + \frac{t^7x^3}{252} - \frac{t^6}{180} - \frac{11\text{xt}^9}{22680}\right] \end{aligned} \tag{4.36}$$

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similarly, we can obtain further values.

Hence, the u(x,t) is

$$\begin{aligned} u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + ... \\ u(x,t) &= xt \end{aligned} \tag{4.37}$$

Example 4: Consider the following nonlinear Klein-Gordon equation

$$u_{\text{tt}}(x,t) - u_{\text{xx}}(x,t) + u^2(x,t) = 2x^2 - 2t^2 + t^4x^4 \tag{4.38}$$

With the initial conditions

$$u(x,0) = 0, u_t(x,0) = 0 \tag{4.39}$$

Applying the Mohand transform on both sides of equ. (3.45), we get

$$M\left[u_{\text{tt}}(x,t)\right] = M\left(2x^2 - 2t^2 + t^4x^4\right) + M\left[u_{\text{xx}}(x,t) - u^2(x,t)\right] \tag{4.40}$$

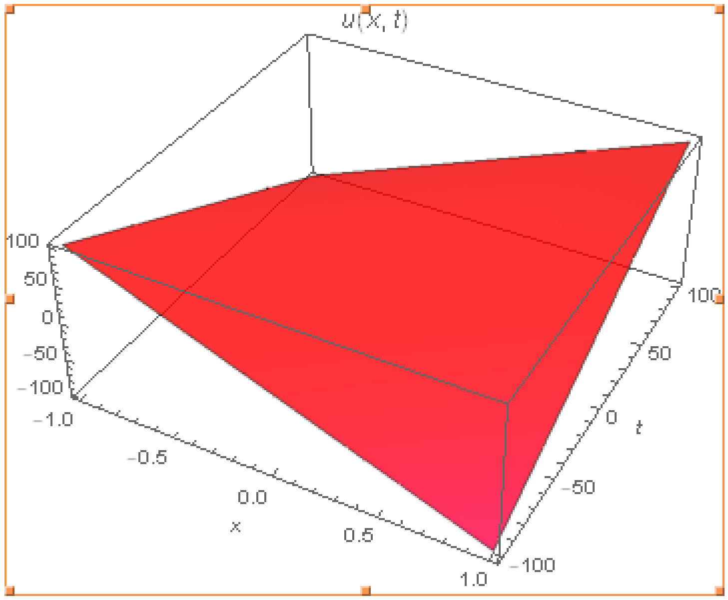

Figure 3: The Graph of $u(x,t) = xt, t > 0. -\infty \leq x \leq \infty$

Using the convolution property of Mohand transform, we get

$$v^2 R(x,v) - v^3 u(x,0) - v^2 u'(x,0) = 2x^2 v - \frac{4}{v} + \frac{24x^4}{v^3} + M\left[u_{\text{xx}}(x,t) - u^2(x,t)\right] \tag{4.41}$$

On simplification and above initial conditions, we get

$$R(x,v) = \frac{2x^2}{v} - \frac{4}{v^3} + \frac{24x^4}{v^5} + \frac{1}{v^2} M\left[u_{\text{xx}}(x,t) - u(x,t)\right] \tag{4.42}$$

Taking inverse Mohand transform on both sides ofequ. (3.49), we get

$$u(x,t) = t^2 x^2 - \frac{t^4}{6} + \frac{x^4}{30} t^6 + M^{-1}\left[\frac{1}{v^2} M\left[u_{\text{xx}}(x,t) - u(x,t)\right]\right] \tag{4.43}$$

Now we use the HPM and the non-linear term we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = t^2 x^2 - \frac{t^4}{6} + \frac{x^4}{30} t^6 + p\left(M^{-1}\left[\frac{1}{v^2} M\left[\frac{\partial^2}{\partial x^2} \sum_{n=0}^{\infty} p^n u_n(x,t) - \sum_{n=0}^{\infty} p^n H_n(u)\right]\right]\right) \tag{4.44}$$

where $H_n(u)$ is represents the He's polynomial of nonlinear terms. The first few components of He's polynomials are given by

$$H_0(u) = (u_0)^2 \tag{4.45}$$

$$H_1(u) = 2u_0 u_1 \tag{4.46}$$

$$H_2(u) = 2u_0 u_2 + (u_1)^2 \tag{4.47}$$

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Equating the multipliers of exponents of p

$$p^0 : u_0(x,t) = t^2x^2 - \frac{t^4}{6} + \frac{x^4}{30}t^6 \tag{4.48}$$

$$\begin{aligned} p^1 : u_1(x,t) &= \left(\frac{1}{M}\left[\frac{1}{v^2}M\left[\frac{\partial^2}{\partial x^2}u_0(x,t) - H_0 u\right]\right]\right) \\ &= \left[-\frac{532224t^{14}x^8}{14!} + \frac{4032t^{11}x^4}{39916800} - \frac{2688t^{10}x^6}{3628800} + \frac{288t^8x^2}{40320} - \frac{20t^8}{40320} + \frac{24t^7x^2}{5040} - \frac{t^6x^4}{30} + \frac{t^6}{6}\right] \end{aligned} \tag{4.49}$$

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similarly, we can obtain further values.

Hence, the u(x,t) is

$$\begin{aligned} u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + ... \\ u(x,t) &= x^2t^2 \end{aligned} \tag{4.50}$$

Example 5: Consider the following nonlinear Klein-Gordon equation

$$u_{\text{tt}}(x,t) - u_{\text{xx}}(x,t) + u^2(x,t) = 6\text{xt}\left(x^2 - t^2\right) + t^6x^6 \tag{4.51}$$

With the initial conditions

$$u(x,0) = 0, u_t(x,0) = x \tag{4.52}$$

Applyingthe Mohand transform on both sides of equ. (3.60), we get

$$M\left[u_{\text{tt}}(x,t)\right] = M\left[u_{\text{xx}}(x,t) - u^2(x,t)\right] + M\left[6\text{xt}\left(x^2 - t^2\right) + t^6x^6\right] \tag{4.53}$$

Using the convolution property of Mohand transform, we get

$$v^2R(x,v) - v^3u(x,0) - v^2u'(x,0) = 6x^3 - \frac{36x}{v^2} + \frac{720x^6}{v^5} + M\left[u_{\text{xx}}(x,t) - u^2(x,t)\right] \tag{4.54}$$

On simplification and above initial conditions, we get

$$R(x,v) = \frac{6x^3}{v^2} - \frac{36x}{v^4} + \frac{720x^6}{v^7} + \frac{1}{v^2}M\left[u_{\text{xx}}(x,t) - u^2(x,t)\right] \tag{4.55}$$

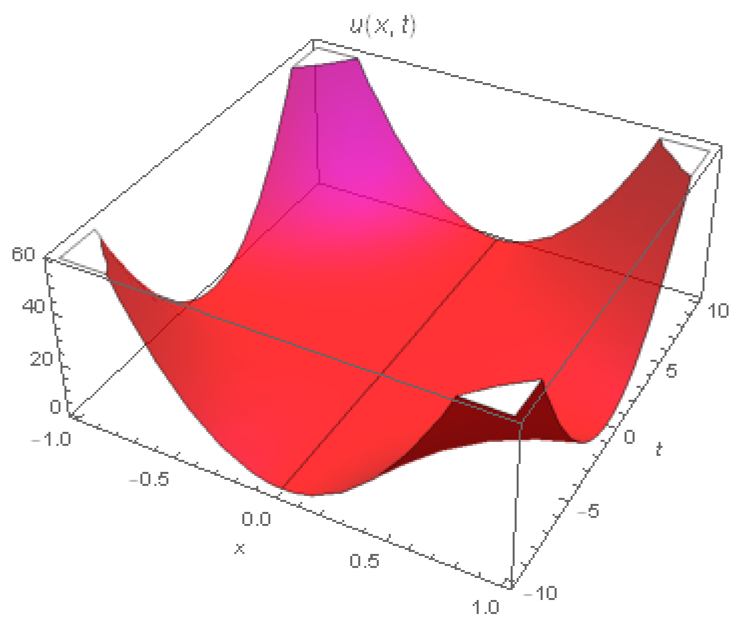

Figure 4: The Graph of $u(x,t) = x^2t^2, t > 0. -\infty \leq x \leq \infty$

Taking inverse Mohand transform on both sides ofequ. (3.64), we get

$$u(x,t) = \frac{6x^3}{v^2} - \frac{36x}{v^4} + \frac{720x^6}{v^7} + +M^{-1}\left[\frac{1}{v^2}M\left[u_{\text{xx}}(x,t) - u(x,t)\right]\right] \tag{4.56}$$

Now we use the HPM and the non-linear term we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = t^3x^3 - \frac{3\text{xt}^5}{10} + \frac{t^8x^6}{56} + p\left(M^{-1}\left[\frac{1}{v^2}M\left[\frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty} p^n u_n(x,t) - \sum_{n=0}^{\infty} p^n H_n(u)\right]\right]\right) \tag{4.57}$$

where $H_n(u)$ is represents the He's polynomial of nonlinear terms. The first few components of He's polynomials are given by

$$H_0(u) = (u_0)^2 \tag{4.58}$$

$$H_1(u) = 2u_0u_1 \tag{4.59}$$

$$H_2(u) = 2u_0u_2 + (u_1)^2 \tag{4.60}$$

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Equating the multipliers of exponents of p

$$p^0 : u_0(x,t) = t^3x^3 - \frac{3\text{xt}^5}{10} + \frac{t^8x^6}{56} \tag{4.61}$$

$$\begin{aligned} p^1 : u_1(x,t) &= \left(\frac{1}{M}\left[\frac{1}{v^2}M\left[\frac{\partial^2}{\partial x^2}u_0(x,t) - H_0u\right]\right]\right) \\ &= \left[\frac{t^{18}x^{12}}{653616} - \frac{3t^{15}x^7}{19600} + \frac{t^{13}x^9}{4368} + \frac{3t^{12}x^2}{4400} - \frac{53t^{10}x^4}{4200} + \frac{t^8x^6}{56} + \frac{3\text{xt}^5}{10}\right] \end{aligned} \tag{4.62}$$

...

similarly, we can obtain further values.

Hence, the u(x,t) is

$$\begin{aligned} u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + ... \\ u(x,t) &= x^3t^3 \end{aligned} \tag{4.63}$$

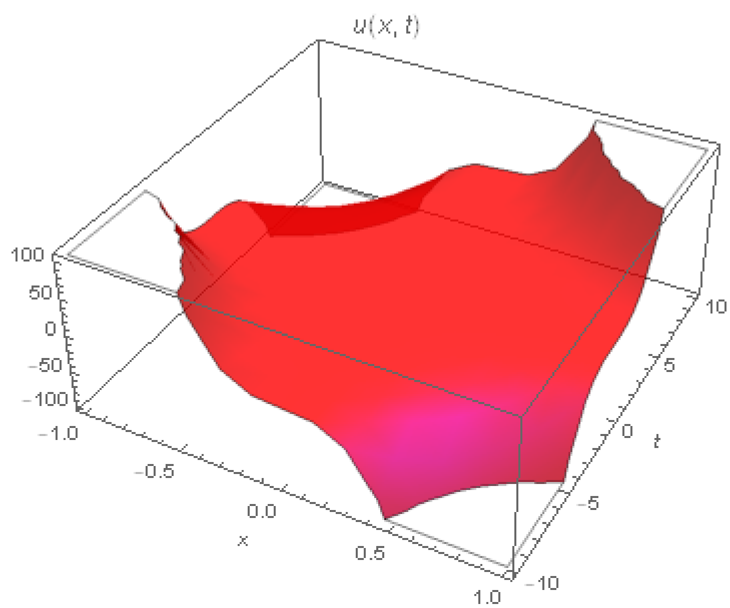

Figure 5: The Graph of $u(x,t) = x^3t^3, t > 0. -\infty \leq x \leq \infty$

5. Concluding Remarks and Observations

we have discussed the history, some definitions of fractional calculus, Riemann-Liouville differential and integral operator. We also knowing the Mittag-Leffler function and Caputo and Fabrizio fractional-order derivative. In this paper, we discussed some of the integral transforms (like Laplace Transform, Fourier Transform, and Mohand Transform). Homotopy perturbation Mohand transform method has been successfully operated to evaluating the linear and nonlinear Klein-Gordon equations with initial conditions. The method is good and simple to solve. In conclusion, the MTHPM may be considered as a nice simplification in numerical techniques and might find wide applications.

References

[1] A.A. Kilbas, H.M. Srivastava, and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam 2006.*

[2] K.S. Miller and B. Ross, *An Introduction to The Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons, New York, Chichester, Brisbane, Toronto, and Singapore, 1993.

[3] K. Nishimoto, *Fractional Calculus*, Vol. 1, Vol. 2 and Vol. 3, Descartes Press, Koriyama, Japan, (1984), (1987), (1989).

[4] K.B. Oldham, and J. Spanier, *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*, Academic Press, New York; and Dover Publications, New York, 1974.

[5] I. Podlubny, *Fractional Differential Equation*, Vol. 198, Academic Press, California, 1999.

[6] B. Ross, *Fractional Calculus and Its Applications*, (Proc. Internat. Conf., New Heaven, 1974), Lecture Notes in Math. Vol. 457, Springer Verlag, New York, 1978.

[7] S.G. Samko, A.A. Kilbas, and O.I. Marichev, *Fractional Integrals and Derivatives:Theory and Applications*, Gordon and Breach, Amsterdam, 1993.

[8] L. Debnath and R. P. Feynman, Recent applications of fractional calculus to science and engineering, *Int. J. Math. Math. Sci.*, **54**(2003), 3413–3442.

[9] A. Prakash, Analytical method for space-fractional telegraph equation by homotopy perturbation transform method, *Nonlinear Eng.*, **5(2)**(2016), 123–128.

[10] M. Safari, D. D. Ganji, and M. Moslemi, Application of he's variational iteration method and Adomian's decomposition method to the fractional KdV-Burgers Kuramoto equation, *Comput. Math. Appl.*, **58(11-12)**(2009), 2091–2097.

[11] J. Singh, D. Kumar, D. Baleanu, and S. Rathore, An efficient numerical algorithm for the fractional Drinfeld-Sokolov-Wilson equation, *Appl. Math. Comput.*, **335**(2018), 12–24.

[12] Y. Chen, I. Petras, and D. Xue, Fractional-order control-A tutorial, *Proc. Amer. Control Conf.*, 2009, 1397–1411.

[13] J. D. Singh Kumar and D. Baleanu, On the analysis of fractional diabetes model with exponential law, *Adv. Difference Equ.*, **231**(2018).

[14] J. Singh, D. Kumar, and A. Kılıçman, Numerical solutions of nonlinear fractional partial differential equations arising in spatial diffusion of biological populations, *Abstr. Appl. Anal.*, 2014, Art. no. 535793.

[15] K. Hosseini, Y. J. Xu, P. Mayeli, A. Bekir, P. Yao, Q. Zhou, and O. Guner, A study on the conformable time-fractional Klein-Gordon equations with quadratic and cubic nonlinearities, *Optoelectron. Adv. Mater.-Rapid Commun.*, **11(7-8)**(2017), 423–429.

[16] B. Batiha, M. S. M. Noorani, I. Hashim, and K. Batiha, Numerical simulations of systems of PDEs by variational iteration method, *Phys. Lett. A,* **372(6)**(2008), 822–829.

[17] A. M. Wazwaz, The variational iteration method for solving linear and non-linear systems of PDEs, *Comput. Math. Appl.*, **54(7-8)**(2007), 895–902.

[18] H. Khan, R. Shah, D. Baleanu, and M. Arif, An efficient analytical technique, for the solution of fractional-order telegraph equations, *Mathematics*, **7(5)**(2019), 426–436.

[19] R. Shah, H. Khan, P. Kumam, M. Arif, and D. Baleanu, Natural transform de-composition method for solving fractional-order partial differential equations with proportional delay, *Mathematics*, **7(6)**(2019), 532–540.

[20] A. M. Wazwaz, *Partial Differential Equations: Methods and Applications*, Leiden, The Netherlands: Balkema Publishers, 2002.

[21] M. A. Abdou, Approximate solutions of a system of PDEEs arising in physics, *Int. J. Nonlinear Sci.*, **12(3)**(2011), 305–312.

[22] O. Özkan and A. Kurt, On conformable double Laplace transform, *Opt.Quantum Electron.*, **50(2)**(2018), 103–110.

[23] Y. Çenesiz, D. Baleanu, A. Kurt, and O. Tasbozan, New exact solutions of Burgers' type equations with conformable derivative, *Waves Random Complex Media*, **27(1)**(2017), 103–116.

[24] G. C. Wu and D. Baleanu, Variational iteration method for fractional calculus-a universal approach by Laplace transform, *Adv. Difference Equ.*, **18**(2013).

[25] O. Özkan, Approximate analytical solutions of systems of fractional partial differential equations, *Karaelmas Sci. Eng. J.*, **7(1)**(2017), 63–67.

[26] R. Shah, H. Khan, P. Kumam, and M. Arif, An analytical technique to solve the system of nonlinear fractional partial differential equations, *Mathematics*, **7(6)**(2019), 505–515.

[27] P. S. Kumar, P. Gomathi, S. Gowri, and A. Viswanathan, Applications of Mohand transform to mechanics and electrical circuit problems, *Int. J.Res. Advent Technol.*, **6(10)**(2018), 2838–2840.

[28] S. Aggarwal and R. Chauhan, A comparative study ofMohand and Aboodh transform, *Int. J. Res. Advent Technol.*, **7(1)**(2019), 520–529.

[29] S. Aggarwal and R. Chaudhary, A comparative study of Mohand and Laplace transform, *J. Emerg. Technol. Innov. Res.*, **6(2)**(2019), 230–240.

[30] S. Aggarwal and S. D. Sharma, A comparative study of Mohand and Sumudu transform, *J. Emerg. Technol. Innov. Res.*, **6(3)**(2019), 145–153.

[31] S. Aggarwal, R. Chauhan, and N. Sharma, Mohand transform of Bessel's functions, *Int. J. Res. Advent Technol.*, **6(11)**(2018), 3034–3038.

[32] S. Aggarwal, N. Sharma, and R. Chauhan, Solution of linear Volterra integral equations of a second kind using Mohand transform, *Int. J. Res.Advent Technol.*, **6(11)**(2018), 3098–3102.

[33] S. Aggarwal, S. D. Sharma, and A. R. Gupta, A new application of Mohand transform for handling Abel's integral equation, *J. Emerg. Technol. Innov. Res.*, **6(3)**(2019), 600–608.

[34] M. Mohand and A. Mahgoub, The new integral transform 'Mohand' transform, *Adv. Theor. Appl. Math.*, **12(2)**(2017), 113–120.

[35] S. Aggarwal, R. Mishra, and A. Chaudhary, A comparative study ofMohand and Elzaki transform, *Global J. Eng. Sci. Researches*, **6(2)**(2019), 203–213.

[36] K. Kothari, U. Mehta, and J. Vanualailai, A novel approach of fractional order time-delay system modeling based on Haar wavelet, *ISA Trans.*, **80**(2018), 371–380.