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The relation graphs of finite lattices

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Abstract

The relation graph of a finite lattice L is defined to be a simple graph with the elements of L as vertices and two distinct vertices are adjacent if and only if they are comparable in L. We investigate the properties of relation graphs and characterize those lattices whose relation graphs are complete. The association between the relation graphs of isomorphic lattices is studied.

Keywords

Lattice, simple graph, triangle free graph, isomorphism of lattices.

AMS Subject Classification

05C25.

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Contents

1. Introduction

Graph theory arrived in the world of Mathematics as an offshoot of Topology, via the works of the celebrated Mathematician, Leonhard Euler. However, even he might not have foreseen its tremendous growth. Today, Graph Theory is recognized as one of the most versatile branches of Mathematics, favoured as much for theoretical work, as it is for applications in fields as diverse as artificial intelligence, image processing, nanotechnology, economics and even genetics.

Lattice theory is another fascinating branch of Discrete Mathematics that find applications in several diverse fields. Lattices are basically structures that arise in the discipline of abstract algebra and have several applications, as illustrated in [5]. Perhaps because of their similarity in representation, studies have been carried out the relation between graphs and lattices, and attempts have been made to connect the two. One such attempt is the field of image processing, as explained in [6]. This paper is an exploration in combining these two areas of Discrete Mathematics– Graph Theory and Lattice Theory, inspired by [4]

Throughout this paper, a graph $G=(V,E)$ or simply G, is chosen such that it is finite, simple and undirected. Also, the lattice $L=(V,\leq)$ is finite and consequently bounded, with least element 0 and greatest element 1.

2. Preliminaries

In this section, we see a few basic definitions that are essential for the main body of work. All definitions pertaining to graph theory are taken from [2] and those pertaining to Lattice Theory are taken from [5].

Definition 2.1. *A graph $G=(V,E)$ consists of a finite non empty set V of vertices $v_i, i=1,2,3,\ldots,n$ together with a prescribed set E of unordered pairs of vertices, e_i, called edges. We write (v_i adj v_j) if v_i and $v_s s j$ are joined by an edge.*

Definition 2.2. *A graph $G=(V,E)$ is said to be complete if every pair of vertices is joined by an edge.*

Definition 2.3. *Two graphs G_1 and G_2 are said to be isomorphic graphs if there exists a one to one correspondence between their vertex sets that preserves adjacency. Symbolically, we write $G_1 \simeq G_2$.*

Definition 2.4. *A lattice $(L,\wedge,\vee)$ is a partially ordered set in which every pair of elements (a,b) has a greatest lower bound $a\vee b$ and a least upper bound $a\wedge b$. A lattice is also represented as $(L,\leq)$ where $\leq$ is the partial order or sometimes, merely as L with the operations understood implicitly.*

The dual of a lattice is the lattice obtained by replacing $\wedge$ with $\vee$, $\vee$ with $\wedge$ and $\leq$ with $\geq$ respectively.

Definition 2.5. *A chain is a lattice in which every pair of elements (a,b) is comparable ie; $a \leq b$ or $b \leq a$.*

Definition 2.6. *A bijective mapping $g : L_1 \to L_2$ from a lattice $(L_1, \wedge_1, \vee_1)$ to a lattice $(L_2, \wedge_2, \vee_2)$ is called a lattice isomorphism if $g(a \wedge_1 b) = g(a) \wedge_2 g(b)$ and $g(a \vee_1 b) = g(a) \vee_2 g(b)$ for all pairs (a,b) in L_1. Alternately, if we represent the lattices as $(L_1, \leq_1)$ and $(L_2, \leq_2)$, then the bijective mapping $g : L_1 \to L_2$ is called a lattice isomorphism if for all a,b in L_1, $a \leq_1 b$ if and only if $g(a) \leq_2 g(b)$.*
Symbolically, we write $L1 \simeq L_2$. Two lattices are said to be dually isomorphic if one is isomorphic to the dual of the other.

A lattice may be represented by means of a Hasse Diagram[5]. Since a lattice is a partially ordered set, direction in a Hasse diagram is implicitly understood. In [3], H. M. Mulder defined a diagraph as follows:

Definition 2.7. *The diagraph of L is the graph with V as vertex set and two vertices are adjacent if one covers the other as elements of the lattice. In other words, the diagraph of a lattice is its unoriented Hasse diagram.*

3. The Relation Graph

In this section, we define a new concept termed as a 'relation graph'. We proceed to prove certain structural properties of the relation graph.

Definition 3.1. *The relation graph $G(L)$ of a finite lattice L is defined to be a simple graph with the elements of L as vertices and two distinct vertices are adjacent if and only if they are comparable in L. The vertex corresponding to the element a of L is represented by $\wedge_a$.*

It is obvious that the diagraph of a lattice is a subgraph of its relation graph. Illustrated below is a lattice and its associated relation graph $G(L)$:

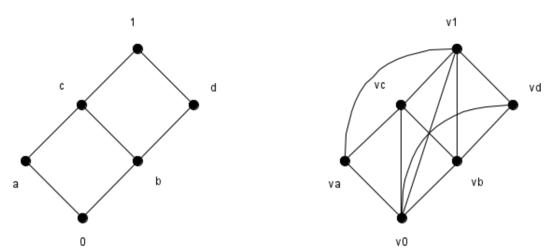

Fig.1:Lattice and its associated relation graph $G(L)$

Observation 3.2. $\{0,1\}$ *is the only lattice whose diagraph coincides with its relation graph.*

Observation 3.3. *The singleton lattice and the chain on two elements are the only lattices whose diagraphs and relation graphs coincide.*

Theorem 3.4. *The vertices of the relation graph corresponding to the least and greatest element are of degree $k-1, k$ being the number of elements in the lattice.*

Proof. Let the given lattice be $(L, \leq)$ with least element 0 and greatest element 1. Let $G(L)$ be the corresponding relation graph. Consider any element a in L with corresponding vertex v_a in $F(L)$. Then, $0 \leq a \leq 1$ for all a in L. Then by definition, we see that v_0 and v_1 are adjacent to all other vertices in $G(L)$. Hence, $degv_0 = degv_1 = k-1, k$ being the number of elements in L. □

Theorem 3.5. *The relation graph of a lattice is complete if and only if the lattice is a chain.*

Proof. Let $(L, \leq)$ be a lattice such that the relation graph $G(L)$ is complete. Then for all $a, b \in L$, either $a \leq b$ or $b \leq a$. Hence L is a chain. Conversely, let L be a chain. Then by definition, every vertex in $G(L)$ is adjacent to every other vertex in it. This implies $G(L)$ is complete. □

Theorem 3.6. *Isomorphic lattices have isomorphic relation graphs.*

Proof. Let $(L_1, \leq_1)$ and $(L_2, \leq_2)$ be isomorphic lattices with lattice isomorphism g. Let $a, b \in L_1$ such that $a \leq_1 b$. Then v_a and v_b are adjacent in $G(L_1)$. By the definition of isomorphism of lattices, $g(a) \leq_2 g(b) \in L_2$. Consequently, $v_{g(a)}$ and $v_{g(b)}$ are adjacent in $G(L_2)$. Thus, v_a and v_b are adjacent in $G(L_1)$ iff $a \leq_1 b$ iff $g(a) \leq_2 g(b)$ iff $v_{g(a)}$ and $v_{g(b)}$ are adjacent in $G(L_2)$. Hence the theorem. □

Observation 3.7. *There exist non isomorphic lattices with isomorphic relation graphs.*

There exist non isomorphic lattices with isomorphic relation graphs.

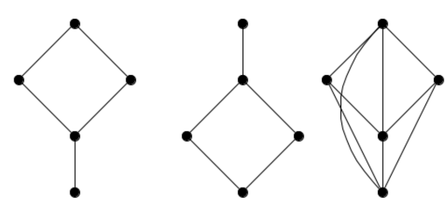

Fig.2:(a)L_1 (b)L_2 (c)$G(L)$

Theorem 3.8. . *If the relation graphs of two lattices are isomorphic, then they are either isomorphic or dually isomorphic.*

Proof. Let Let $(L_1, \leq_1)$ and $(L_2, \leq_2)$ be two lattices with elements $\{a_1, a_2, a_3, \ldots, a_n\}$ and $\{b_1, b_2, b_3, \ldots, b_n\}$ respectively such that $G(L_1) \simeq G(L_2)$. Let the vertices of these graphs be respectively labelled $v_{a_1}, v_{a_2}, \ldots, v_{a_n}$ and $v_{b_1}, v_{b_2}, \ldots, v_{b_n}$ such that

$$(v_{a_i} \,\mathrm{adj}\, v_{a_j}) \iff (v_{b_i} \,\mathrm{adj}\, v_{b_j}) \tag{3.1}$$

Now,

$$\begin{aligned} a_i \leq_1 a_j \iff v_{a_i} \,\mathrm{adj}\, v_{a_j} \iff & \; v_{b_i} \,\mathrm{adj}\, v_{b_j} \quad \text{by}(3.1) \\ \iff & \; b_i \leq_2 b_j \,\text{or}\, b_j \leq_2 b_i \end{aligned}$$

by definition of adjacency in the relation graph. Then by definition this means that $(L_1, \leq_1)$ and $(L_2, \leq_2)$ are either isomorphic or dually isomorphic. □

4. Conclusion

In this paper, we have defined the relation graph of a given lattice and gone on to prove some of their properties, especially with regard to isomorphism. The authors intend to explore more features of this class of graphs, such as the radius, diameter, girth, connectivity and chromatic number, to name a few.

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