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Pathos edge semi-middle graph of a tree

K.C. Rajendra Prasad[1*], Venkanagouda M. Goudar[2] and K.M. Niranjan[3]

Abstract
In this communication, the pathos edge semi-middle graph of a tree is introduced. Its study is concentrated only on trees. We present a characterization of those graphs whose pathos edge semi-middle graph of a tree is planar, outerplanar and minimally nonouterplanar. Further, Also we establish a characterization of graphs whose pathos edge semi-middle graph of a trees are noneularian, hamiltonian and the graphs whose crossing number one and two.

Keywords
Crossing number, Middle graph, Planar, Semientire graph.

AMS Subject Classification
05C05, 05C07, 05C10, 05C38, 05C45.

[1] *Research Scholar, UBDT College of Engineering, Davanagere, A Constituent College of Visvesvaraya Technological University, Belagavi. Department of Mathematics, Jain Institute of Technology, Davanagere-577003, Affiliated to Visvesvaraya Technological University, Belagavi, India.*
[2] *Department of Mathematics, Sri Siddhartha Institute of Technology, Tumkur-572105, India.*
[3] *Department of Mathematics, UBDT College of Engineering, Davanagere-577004, India.*
*Corresponding author: [1]rajendraprasadkp@gmail.com; [2]vmgouda@gmail.com; [3]niranjankm64@gmail.com.
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1. Introduction

All graphs consider here are finite, undirected without loops or multiple edges. Any undefined term or notation in this paper may be found in Harary [2].
The idea of pathos of a graph G has been introduced by [1] as a collection of minimum number of edge disjoints open paths whose union is G. The path number of a graph G is the number of path of pathos. Stanton [4] and Harary [2] have calculated the path number of certain classes of graphs like trees and complete graphs.
The crossing number $C_r(G)$ of G is the least number of intersection of pairs of edges in any embedding of G in the plane. Obviously G is planar if and only if $C_r(G) = 0$. The *edgedegree* of an edge uv of a tree T is the sum of the degrees of u and v. The pathoslength is the number of edges that lie on a particular path p_i of pathos of T.

In [5] Venkanagouda introduced the graph valued function, pathos vertex semientire graph of a tree. The present work focuses on the concept of the pathos edge semi-middle graph of a tree in this context. The pathos edge semi-middle graph of a tree denoted by $PM_e(T)$ is the graph whose vertex set is $V(T) \cup E(T) \cup R(T) \cup P_i(T)$ and two vertices of $PM_e(T)$ are adjacent if and only if they corresponds to two adjacent edges of T or one corresponds to a vertex and other to an edge incident with it or one corresponds to edge and other to a region in which edge lie on the region or one corresponds to a vertex and other to a path of pathos in which vertex lies on the path of pathos since the system of pathos for a tree is not unique, the corresponding pathos edge semi-middle graph of a tree is also not unique. The tree T and its pathos edge semi-middle graph of a tree $PM_e(T)$ are shown in Fig.1.

2. Preliminaries.

Theorem 2.1. *[3] For any graph G, $M_e(G)$ is separable if and only if G has a pendant vertex.*

Theorem 2.2. *[3] For any graph G, p vertices, q edges and r regions then $M_e(G)$ has $(p+q+r)$ vertices and $q + \Sigma_{i=1}^{q} \frac{1}{2} d(e_i) + \Sigma_{j=1}^{r} q_{r_j}$ edges. Where $d(e_i)$ is the edgedegree of a edge e_i and q_{r_j} is the number of edges lies on each region.*

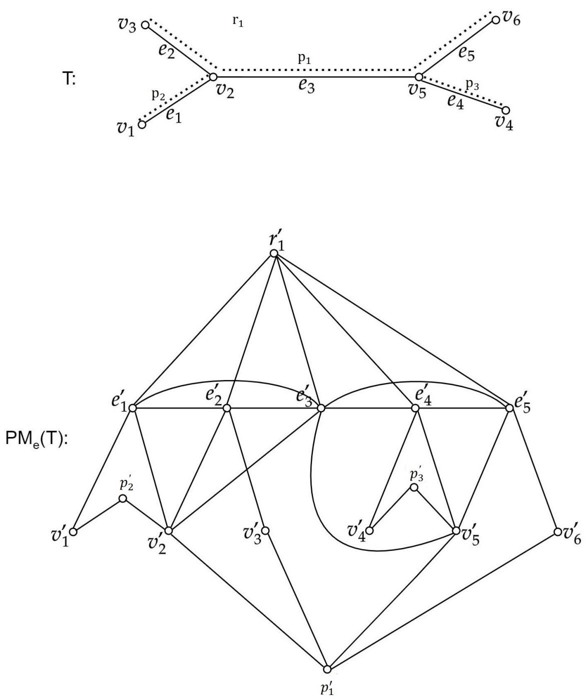

Fig. 1.

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Theorem 2.3. *[3] For any graph G, $M_e(G)$ is planar if and only if G satiesfies the following conditions. i) $\triangle(G) \leq 3$. ii) if $deg(v) = 3$, then v is a cutvertex.*

Theorem 2.4. *[3] For any graph G, $M_e(G)$ is outerplanar if and only if G is a path P_n.*

3. Pathos edge semi-middle graph of a tree

We begin with the following observations.

Observation 3.1. If v is a pendant vertex of a tree T, then the degree of a corresponding vertex v' in $PM_e(T)$ is even.

Observation 3.2. For any edge e_i in T with edgedegree n, the degree of the vertex e_i' which corresponds to e_i in $PM_e(T)$ is always $(n+1)$.

Observation 3.3. If the pathos length of the path of pathos p_i in T is n, then the degree of the corresponding pathosvertex in $PM_e(T)$ is $(n+1)$.

Observation 3.4. The $PM_e(T)$ is 5-minimally nonouterplanar if and only if $T = K_{1,3}$.

Theorem 3.1. *For any tree T, $PM_e(T)$ is always non-separable.*

Proof. Suppose T be any tree. Let $T : v_1, v_2, v_3, \ldots v_n$. Further, $V[PM_e(T)] = \{v_1', v_2', v_3' \ldots v_n', e_1', e_2', e_3' \ldots e_{n-1}', r_1', p_1', p_2', \ldots p_n'\}$. By Theorem 2.1, $M_e(G)$ is separable. In $PM_e(T)$, the pathosvertices are adjacent to $v_1', v_2', v_3' \ldots v_n'$. Clearly $PM_e(T)$ has no cutertex. Thus $PM_e(T)$ is non-separable. Hence the proof. □

Theorem 3.2. *$T(p,q)$ be any tree with r regions and k path of pathos, then $PM_e(T)$ has $(p+q+r+\Sigma_{l=1}^{k} p_l)$ vertices and $q + \Sigma_{i=1}^{q} \frac{1}{2} d(e_i) + \Sigma_{j=1}^{r} q_{r_j} + \Sigma_{l=1}^{k} p_{v_l}$ edges. Where $d(e_i)$ is the edgedegree of a edge e_i, q_{r_j} is the number of edges lies on each region and p_{v_l} is the number of vertices which lies on the path of pathos.*

Proof. By the definition of $PM_e(T)$, $V[PM_e(T)] = (p+q+r+\Sigma_{l=1}^{k} p_l)$. Further by Theorem 2.2, $E[M_e(G)] = q + \Sigma_{i=1}^{q} \frac{1}{2} d(e_i) + \Sigma_{j=1}^{r} q_{r_j}$. The degree of a pathosvertex is the sum of the number of vertices lies on the each path of pathos in T which is $\Sigma_{l=1}^{k} p_{v_l}$. The number of edges in $PM_e(T)$ is equal to the sum of edges in $M_e(G)$ and $\Sigma_{l=1}^{k} p_{v_l}$. Hence $E[PM_e(T)] = q + \Sigma_{i=1}^{q} \frac{1}{2} d(e_i) + \Sigma_{j=1}^{r} q_{r_j} + \Sigma_{l=1}^{k} p_{v_l}$. □

Theorem 3.3. *For any tree T, $PM_e(T)$ is planar if and only if T is a path or $K_{1,3}$.*

Proof. Suppose $PM_e(T)$ is planar. Consider the star, $K_{1,4}$: v_1, v_2, v_3, v_4, v_5 and $\deg(v_1) = 4$. Further $V[PM_e(T)] = \{v_1', v_2', v_3', v_4', v_5', e_1', e_2', e_3', e_4', r_1', p_1', p_2'\}$. By Theorem 2.3, $M_e(K_{1,4})$ is non-planar. Clearly $PM_e(T)$ is also non-planar, a contradiction.
Conversely,
Case 1. Suppose $T = P_n : v_1, v_2, v_3, \ldots v_n, n \geq 2$. Further, $V[PM_e(T)] = \{v_1', v_2', v_3', \ldots v_n', e_1', e_2', e_3' \ldots e_{n-1}', r_1', p_1'\}$. By Theorem 2.4, $M_e(P_n)$ is outerplanar. In $PM_e(P_n)$, p_1' is adjacent to $v_1', v_2', v_3' \ldots v_n'$ of $M_e(P_n)$. Clearly $PM_e(P_n)$ is planar.
Case 2. Suppose $T = K_{1,3} : v_1, v_2, v_3, v_4$ and $\deg(v_1) = 3$. Further $V[PM_e(K_{1,3})] = \{v_1', v_2', v_3', v_4', e_1', e_2', e_3', r_1', p_1', p_2'\}$. By Theorem 2.3, $M_e(K_{1,3})$ is planar. By Observation 3.4, $PM_e(K_{1,3})$ is 5 minimally non-outerplanar.Clearly $PM_e(K_{1,3})$ is planar. □

Proposition 3.1. The $PM_e(T)$ of a T is 1-minimally non-outerplanar if and only if $T = P_3$.

Theorem 3.4. *For any tree T, $PM_e(T)$ is outerplanar if and only if T is a path P_2.*

Proof. Consider a tree T is not a path P_2. Let $T = P_3$: v_1, v_2, v_3. Further, $V[PM_e(P_3)] = \{v_1', v_2', v_3', e_1', e_2', r_1', p_1'\}$. By Proposition 3.1, $PM_e(P_3)$ is 1-minimally nonouterplanar, a contradiction.
Conversely, Suppose $T = P_2$, then $PM_e(P_2) = C_4(P_{n_1})$. Since $C_4(P_{n_1})$ is outerplanar. It follows that $PM_e(P_2)$ is outerplanar. □

Proposition 3.2. The $PM_e(T)$ of a T is 3-minimally non-outerplanar if and only if $T = P_4$.

Theorem 3.5. *$PM_e(T)$ of a connected graph T is $(2k-1)$ minimally non-outerplanar $k \geq 1$ if and only if T is P_{k+2}.*

Proof. Suppose T is $P_{k+2}, k \geq 1$ to establish the result, we apply mathematical induction on k. Consider $k = 1$ then by Proposition 3.1, is 1-minimally non-outerplanar. Consider the result is valid for $k = m$, therefore if T is P_{m+2} then $PM_e(T)$ is $(2m-1)$-minimally non-outerplanar. Suppose $k = m+1$ then T is P_{m+3}. We now prove that $PM_e(T)$ is $2(m+1)-1$ minimally non-outerplanar. Let $T = P_{m+3}$ and v_1 be an end vertex of T. Let $T_1 = T - v_1 = P_{m+2}$. By inductive hypothesis, $PM_e(T_1)$ is $(2m-1)$-minimally non-outerplanar. Let $e_i = (v_i, v_j)$ be an end edge, r_i be the region and p_1 be the pathosvertex of T_1. Then e_i is an end edge incident with the cutvertex v_i. The vertices $e_i^{'}, r_i^{'}, v_j^{'}$ and $p_i^{'}$ in $PM_e(T_1)$ are on the boundary of the exterior region. Now join the vertex v_1 to the vertex v_j of T_1 such that the resulting graph is T. Let $e_j = (v_j, v_1)$ be an endedge, p_i be the pathosvertex and r_i be the region of T. The formation of $PM_e(T)$ is an extension of $PM_e(T_1)$ with additional vertices e_j and v_1 such that $e_j^{'}$ adjacent with $e_i^{'}, v_j^{'}, v_1^{'}$ and $r_1^{'}$. Similarly $p_i^{'}$ is adjacent with $v_i^{'}, v_j^{'}$ and $v_1^{'}$. Clearly $v_j^{'}$ is an inner vertex of $PM_e(T)$, but it not an inner vertex of $PM_e(T_1)$.Thus $PM_e(T)$ is $[2(m+1)-1]$-minimally non-outerplanar.

Conversely, assume $PM_e(T)$ is $(2k-1)$ minimally non-outerplanar then by Theorem 3.3, $PM_e(T)$ is planar. Thus T is a path. Suppose T is a path. We obtain the following cases.

Case 1. Suppose $T = P_{k+1}, k \geq 1$. In particular if $k = 1$ then $T = P_2$ by the Theorem 3.4, $PM_e(P_2)$ is outerplanar, a contradiction.

Case 2. Suppose $T = P_{k+3}$, in particular if $k = 1$ then $T = P_4$ by Proposition 3.2, $PM_e(P_4)$ is 3-minimally non-outerplanar, a contradiction. Hence T is P_{k+2}. □

Theorem 3.6. *For any tree T, $PM_e(T)$ has crossing number one if and only if T is $K_{1,3}(P_{n_1}, P_{n_2})$ where $n_1, n_2 \geq 1$.*

Proof. Suppose $PM_e(T)$ has crossing number one, then $PM_e(T)$ is non-planar. By the Theorem 3.3 we have $T = K_{1,n}$, $n \geq 4$. We now consider the following cases.

Case 1. Assume that $T = K_{1,n}$ for $n = 4 : v_1, v_2, v_3, v_4, v_5$ and $\deg(v_i) = 4$. Further, $V[PM_e(K_{1,n})] = \{v_1^{'}, v_2^{'}, v_3^{'}, v_4^{'}, v_5^{'}, e_1^{'}, e_2^{'}, e_3^{'}, e_4^{'}, r_1^{'}, p_1^{'}, p_2^{'}\}$. In $M_e(K_{1,n})$, $C_r[M_e(K_{1,4})] = 2$. Hence $PM_e(K_{1,n})$ has crossing number at least two, a contradiction.

Case 2. Assume that $T = K_{1,3}(P_{n_1}, P_{n_2}, P_{n_3})$, $n_1, n_2, n_3 \geq 1$. By Theorem 3.3, $PM_e(K_{1,3})$ is non-planar. The graph T contains two path of pathos and their corresponding to two pathosvertices $p_1^{'}, p_2^{'}$ in $PM_e(T)$. These two vertices also joined by the vertices and gives crossing number two, a contradiction.

Conversely, Suppose $T = K_{1,3}(P_{n_1}, P_{n_2})$, $n_1, n_2 \geq 1$. By Theorem 3.3, $PM_e(T)$ is non-planar. $K_{1,3}(P_{n_1}, P_{n_2})$ contains two path of pathos p_1 and p_2 such that p_1 lies in the interior region and p_2 lies in the exterior region. In $PM_e(T)$, two pathosvertices joined by the vertices gives crossing number one. Hence $PM_e(T)$ has crossing number one. □

Theorem 3.7. *For any tree T, $PM_e(T)$ has crossing number two if and only if T is $K_{1,3}(P_{n_1}, P_{n_2}, P_{n_3})$, where $n_1, n_2, n_3 \geq 1$.*

Proof. Suppose $PM_e(T)$ has crossing number two. Assume that $T = K_{1,4} : v_1, v_2, v_3, v_4, v_5$ and $\deg(v_1) = 4$. Further, $V[PM_e(T)] = \{v_1^{'}, v_2^{'}, v_3^{'}, v_4^{'}, v_5^{'}, e_1^{'}, e_2^{'}, e_3^{'}, e_4^{'}, r_1^{'}, p_1^{'}, p_2^{'}\}$. In $M_e(T)$, $C_r[M_e(K_{1,4})] = 2$. In $PM_e(T)$, pathosvertices joined by $v_1^{'}, v_2^{'}, v_5^{'}$ for $p_1^{'}$ and $v_2^{'}, v_3^{'}, v_4^{'}$ for $p_2^{'}$, gives crossing number three. Which is a contradiction.

Conversely, suppose $T = K_{1,3}(P_{n_1}, P_{n_2}, P_{n_3})$, $n_1, n_2, n_3 \geq 1 : v_1, v_2, v_3, ...v_n$. Further, $V[PM_e(T)] = \{v_1^{'}, v_2^{'}, v_3^{'}...v_n^{'}, e_1^{'}, e_2^{'}, e_3^{'}...e_{n-1}^{'}, p_1^{'}, p_2^{'}, r_1^{'}\}$. By Theorem 3.6, Case 2, $C_r[PM_e(T)] = 2$. Hence $PM_e(T)$ has crossing number two. □

Theorem 3.8. *For any tree T, $PM_e(T)$ is always noneulerian.*

Proof. Let T be a non-trivial tree. We consider the following cases.

Case 1. Suppose T be a path. If $n = 3$, both edges having edgedegree odd, by Observation 3.2, both vertices have even degree in $PM_e(T)$. But the pathosvertex $p_1^{'}$ is adjacent to $v_1^{'}, v_2^{'}, v_3^{'}$ and $v_2^{'}, p_1^{'}$ to get odd degree. Hence $PM_e(T)$ is noneulerian. If $n \geq 3$, then the internal edges having edgedegree even. By Observation 3.2, the corresponding vertices in $PM_e(T)$ have odd degree. Then $PM_e(T)$ is noneulerian.

Case 2. Suppose $T = K_{1,n} : v_1, v_2, v_3, ...v_n$. Further $V[PM_e(K_{1,n})] = \{v_1^{'}, v_2^{'}, v_3^{'}, ...v_n^{'}, e_1^{'}, e_2^{'}, e_3^{'}, ...e_{n-1}^{'}, r_1^{'}, p_1^{'}, p_2^{'}, ...p_n^{'}\}$. If n is odd then each edge having edgedegree even. In $PM_e(T)$, the corresponding vertices having degree odd, which is noneulerian. If n is even then each edge having edgedegree odd. By observation 3.2, In $PM_e(T)$ the corresponding vertices $e_i^{'}$ having even degree. By definition of $PM_e(T)$, $e_1^{'}, e_2^{'}, e_3^{'}, ...e_{n-1}^{'}$ is adjacent to $r_1^{'}$ gives a vertex $r_1^{'}$ having even degree. Also $v_1^{'}, v_2^{'}, v_3^{'}, ...v_n^{'}$ adjacent to $p_1^{'}, p_2^{'}$ gives a vertices $p_1^{'}, p_2^{'}$ having odd degree or vice-versa. Hence $PM_e(T)$ is noneulerian. □

Theorem 3.9. *For any tree T, $PM_e(T)$ is hamiltonian if and only if T is $K_{1,3}(P_{n_1}, P_{n_2}, P_{n_3})$ where $n_1, n_2, n_3 \geq 0$ or $B_{2,2}$ or subdivision of any edge in $B_{2,2}$.*

Proof. Let T be any tree. We have the following cases.

Case 1. Suppose $T = K_{1,3}(P_{n_1}, P_{n_2}, P_{n_3})$, $n_1, n_2, n_3 \geq 0 : v_1, v_2, v_3, ...v_n$. Further, $V[PM_e(T)] = \{v_1^{'}, v_2^{'}, v_3^{'}, ...v_n^{'}, e_1^{'}, e_2^{'}, e_3^{'}, ...e_{n-1}^{'}, p_1^{'}, p_2^{'}, r_1^{'}\}$. Then there exists a cycle $r_1^{'}, e_1^{'}, v_1^{'}, p_1^{'}, ...e_{n-1}^{'}, v_n^{'}, p_2^{'}, r_1^{'}$. Which includes all the vertices of $PM_e(T)$. Hence $PM_e(T)$ is hamiltonian.

Case 2. suppose $T = B_{2,2}$ or subdivision of any edge in $B_{2,2} : v_1, v_2, v_3, ...v_n$. Further $V[PM_e(T)] = \{v_1^{'}, v_2^{'}, v_3^{'}, ...v_n^{'}, e_1^{'}, e_2^{'}, e_3^{'}, ...e_{n-1}^{'}, r_1^{'}, p_1^{'}, p_2^{'}, p_3^{'}\}$. By Theorem 3.1, $PM_e(T)$ is always nonseparable. Then there exists a hamiltonian cycle. Which includes all the vertices of $PM_e(T)$. Hence $PM_e(T)$ is hamiltonian.

Conversely, Suppose $T = P_n : v_1, v_2, v_3, ...v_n$. Further, $V[PM_e(T)] = \{v_1^{'}, v_2^{'}, v_3^{'}, ...v_n^{'}, e_1^{'}, e_2^{'}, e_3^{'}...e_{n-1}^{'}, r_1^{'}, p_1^{'}\}$. By Theorem 3.1, $PM_e(T)$ is always non-separable. Then there exists a hamiltonian path. Hence $PM_e(T)$ is nonhamiltonian. □

4. Conclusions

In this paper we obtained the new graph valued function called pathos edge semi-middle graph of a tree. We studied the characterization of graphs whose pathos edge semi-middle graph of a tree is planar, outerplanar, crossing number one and two. Further, we obtain $PM_e(T)$ is noneulerian and hamiltonian.

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