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A study on r-regular and l-regular near-rings

Marisetti Sowjanya[1]*, A. Gangadhara Rao[2], P.M. Padmalatha[3] and T. Radharani [4]

Abstract
In this paper, by studying r-regular near-rings and m-regular near-rings, we proved some characterizations of m-regular near-rings, r-regular near-rings with IFP. We introduced the term l-regular near-ring and proved some results.

Keywords
m-regular near-ring, r-regular near-ring, l-regular near-ring, IFP.

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[1] *Department of Mathematics, Eluru College of Engineering & Technology, Eluru-534004, Andhra Pradesh, India.*
[2] *Department of Mathematics, V S R and N V R College, Tenali-522201, Andhra Pradesh, India.*
[3] *Department of Mathematics, JMJ College for Women, Tenali-522202, Andhra Pradesh, India.*
[4] *Department of Mathematics, Lakireddy Balireddy College of Engineering, Mylavaram-521230, Andhra Pradesh, India.*
***Corresponding author**: [1] sowjanyachallari@gmail.com; [2]raoag1967@gmail.com; [3]padmalatha323@gmail.com; [4]radharanitammileti@gmail.com
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Contents

1. Introduction

Development of the concept of near-rings is highly shaped by the inventive research on Ring theory. In ring theory, Roos [14] defined the concept of regularity and this notion was enforced and developed to Near-rings and several mathematicians gave a various characterization of near-rings such as Bell [2], Steve Ligh [7],YV Reddy and CVLN Murthy [13], Ramakotaiah [10, 11], Dheena [5], S Suryanarayanan and N Ganesan [18], Atagün, Akin and Kamacı, Hüseyin and Taştekin, İsmail and SEZGİN, Aslıhan [1]. Yong UK Cho [3] and Christian Lompjerzy Matczuk [8] developed the concept of semicentral idempotents for near-rings and rings. Especially, in ideal theory, Pairote Yiarayong [20] developed a strong relationship on various kinds of prime ideals in near-rings. Wendt Gerhard [19] investigated minimal ideals and primitivity in Right near-rings. Recently, S Ramkumar and T Manikantan [12] established the notion of the extension of a fuzzy soft set over a near-ring.

2. Preliminaries

For necessary definitions and basic results, the author follows [9]. In this Preliminaries section, We recall the required definitions and results as follows.

Definition 2.1. *A triplet* $(\mathfrak{K},+,.)$ *is referred to as The, Right near-ring where*

1. *$\mathfrak{K}$ holds the properties of a "Group" under addition.*
2. *$\mathfrak{K}$ holds the properties of a "Semi-group" under multiplication.*
3. $\left(t^1+q^1\right).s^1 = t^1.s^1+q^1.s^1, \forall\, t^1,q^1,s^1 \in \mathfrak{K}$ *(right distributive law).*

Moreover in this paper, we consider Right near-ring($\mathfrak{K}$, +, .) and we designate a right near-ring as $\mathfrak{K}$ unless and otherwise mentioned. We write t^1s^1 to denote $t^1.s^1$ for any two elements t^1 and s^1 in a near-ring $\mathfrak{K}$.

Example 2.2. *Let* $(\mathfrak{K},+)$ *where* $\mathfrak{K}=\left\{i^1,p^1,q^1,r^1\right\}$ *be a Klein's four group with addition and product tables mentioned below is an example for a near-ring. [see Pilz, p408 (13)(0,7,13,9)]*

Table 1. Addition table

+	i^1	p^1	q^1	r^1
i^1	i^1	p^1	q^1	r^1
p^1	p^1	i^1	r^1	q^1
q^1	q^1	r^1	i^1	p^1
r^1	r^1	q^1	p^1	i^1

Table 2. Product table

.	i^1	p^1	q^1	r^1
i^1	i^1	i^1	i^1	i^1
p^1	i^1	p^1	q^1	r^1
q^1	i^1	i^1	i^1	i^1
r^1	i^1	p^1	q^1	r^1

Definition 2.3. *Let $\mathfrak{K}$ is referred to as "Zero-symmetric near-ring(ZSN)" if $k0 = 0\ \forall k \in \mathfrak{K}$ i.e. $\mathfrak{K} = \mathfrak{K}_o$.*

In the above example 2.2, $(\mathfrak{K}, +, .)$ is a ZSN and we denote it as $\mathfrak{K} \in \eta_0$.

Definition 2.4. *Let $\mathfrak{D}$ be a subgroup of $\mathfrak{K}$ is said to be $\mathfrak{K}$-subgroup ($\mathfrak{K}$-SG) if $\mathfrak{K}\mathfrak{D} \subseteq \mathfrak{D}$.*

If $S, T \subseteq \mathfrak{K}$ then we define $ST = \{st/s \in S, t \in T\}$.
We, now designate a normal subgroup as NSG.

Definition 2.5. *Let $\mathfrak{I}$ be a NSG of $(\mathfrak{K}, +)$ is referred to as the left ideal of $\mathfrak{K}$, if $\forall\, t, p \in \mathfrak{K}$, $\forall s \in \mathfrak{I}$, $t(p+s) - tp \in \mathfrak{I}$.*

Definition 2.6. *Let $\mathfrak{I}$ be a NSG of $(\mathfrak{K}, +)$ is referred to as the right ideal of $\mathfrak{K}$ if $\mathfrak{I}\mathfrak{K} \subseteq \mathfrak{I}$.*

Definition 2.7. *Let $\mathfrak{I}$ be a NSG of $(\mathfrak{K}, +)$ is referred to as ideal(two-sided ideal)if it satisfies both the definitions of left ideal and a right ideal of $\mathfrak{K}$.*

Proposition 2.8. *[9, proposition 1.34(c)]*
For a $\mathfrak{K} \in \eta_0$, every ideal is a $\mathfrak{K}$-SG of $\mathfrak{K}$.

Definition 2.9. *Assume that* F *is a non-void subset in $\mathfrak{K}$. Then $\{\mathrm{L}_s/s \in I\}$ be the family of all left ideals which contain* F*.$L = \cap_{s\in I} L_s$ is the smallest one among all left ideal containing* F *can be referred as "left ideal generated by* F*".*

Definition 2.10. *Assume that an ideal $\mathfrak{A}$ of $\mathfrak{K}$ is termed to "principal ideal" if $\mathfrak{A}$ is generated by one component.*

If an ideal $\mathfrak{A}$ which is generated by an element 'a', then $\mathfrak{A}$ is symbolized by $\langle a \rangle$.
If a left ideal $\mathfrak{A}$ is generated by a single component 'a', then $\mathfrak{A}$ is symbolized by $\langle a|$.
If the right ideal $\mathfrak{A}$ is generated by a single component 'a', then $\mathfrak{A}$ is symbolized by $|a\rangle$.

Definition 2.11. *The center of a near-ring $\mathfrak{K}$ is defined as $\mathfrak{C} = \{x \in \mathfrak{K}/nx = xn, \forall\, n \in \mathfrak{K}\}$.*

Elements in $\mathfrak{C}$ are said to be central.

Definition 2.12. *A component 'p' is termed as an idempotent element of $\mathfrak{K}$ if $p^2 = p$, for $p \in \mathfrak{K}$.*

Definition 2.13. *A non-zero element 't' in $\mathfrak{K}$ is termed as nilpotent, if $\exists k \in \mathfrak{K}$ which is greater than or equal to 2 such that $t^k = 0$.*

Definition 2.14. *A subset $\mathfrak{S}$ of $\mathfrak{K}$ is referred to as "nil" if for all $t \in \mathfrak{S}$ are nilpotent.*

Definition 2.15. *The set $(0 : \Delta) = \{t \in \mathfrak{K}/tx = 0, \forall\, x \in \Delta\}$, where Δ be a subset of $\mathfrak{K}$, is known as the annihilator of Δ.*

If $\Delta = \{\delta\}$, then $(0 : \Delta)$ is denoted by $(0 : \delta)$.

Corollary 2.16. *[9, corollary 1.43 (a)] For any $\delta \in \mathfrak{K}$, $(0 : \delta)$ is a "left ideal" of $\mathfrak{K}$.*

Corollary 2.17. *[9, corollary 1.43 (b)] If Δ is a $\mathfrak{K}$-SG of Γ, then the annihilator $(0 : \Delta)$ is an ideal in $\mathfrak{K}$.*

According to [2, 5, 9], let $\mathfrak{K}$ is identified as Insertion of Factors Property(IFP), supposing that $ts = 0 \implies tps = 0$, $\forall t, s, p \in \mathfrak{K}$. The above-mentioned near-ring Example2.2 is an example for IFP near-ring.

Proposition 2.18. *[9, proposition 9.3] The following affirmations are equivalent:*

- *$\mathfrak{K}$ has the insertion of factors property(IFP).*
- *$(0 : s)$ is an ideal of $\mathfrak{K}$, $\forall s \in \mathfrak{K}$.*
- *Let $\mathfrak{I} = (0 : \mathfrak{S})$, for all subsets $\mathfrak{S}$ of $\mathfrak{K}$, $\mathfrak{I}$ is an ideal.*

Definition 2.19. *For each component $k \in \mathfrak{K}$, if $k^2 = 0 \Rightarrow k = 0$, then $\mathfrak{K}$ is known as reduced near-ring.*

Lemma 2.20. *[5, lemma 2.8] For each d, l in $\mathfrak{K} \in \eta_0$, which is a reduced near-ring then $dlt = dtl$ where $t^2 = t$, t is in $\mathfrak{K}$.*

Proposition 2.21. *[9, proposition 9.37] If $\mathfrak{K} \in \eta_0$ is having no non-zero nilpotent components, then $\mathfrak{K}$ satisfies the IFP.*

Definition 2.22. *For each component $c \in \mathfrak{K}$, if $\mathfrak{K}c = \mathfrak{K}c^2$ then $\mathfrak{K}$ is known as "left bi potent".*

Definition 2.23. *For each component $k \in \mathfrak{K}$, there is a component l in $\mathfrak{K}$ such that $k = klk$, then $\mathfrak{K}$ is known as "regular near-ring(RN)".*

Definition 2.24. *For each component $p \in \mathfrak{K}$, there is a component l in $\mathfrak{K}$ such that $p = lp^2$, then $\mathfrak{K}$ is known as "left strongly regular near-ring(left SRN)".*

According to [15], for each component $q \in \mathfrak{K}$, there is a component l which is an idempotent in $\mathfrak{K}$ such that $q = ql, l \in \langle q|$, then $\mathfrak{K}$ is known as "r-regular near-ring(r-RN)".

Theorem 2.25. *[15, Theorem 2.8] If $\mathfrak{K}$ is r-RN with 1 and has IFP then $a = al$ implies $a = la$, where l is an idempotent in $\mathfrak{K}$, $l \in \langle a|$.*

Theorem 2.26. *[15, Theorem 2.9] Let $\mathfrak{K}$ be a r-RN which satisfies IFP with 1 then $\mathfrak{K}$ is reduced.*

Lemma 2.27. *[9] [17] Let $\mathfrak{K} \in \eta_0$ has IFP if and only if $\mathfrak{H}$ is an ideal where $\mathfrak{H} = (0 : \mathfrak{S})$, for all subsets $\mathfrak{S}$ of $\mathfrak{K}$.*

Lemma 2.28. *[5, lemma 1]*
If a near-ring $\mathfrak{K} \in \eta_0$ is reduced then for any $0 \neq a \in \mathfrak{K}$

1. *$\mathfrak{K} \backslash A(a)$ is reduced and the residue class $\overline{a}$ of a mod$A(a)$ is a nonzero divisor where $A(a) = \{x \in \mathfrak{K}/xa = 0\}$.*
2. *$k_1k_2...k_n = 0$ implies $\langle k_1\rangle\langle k_2\rangle...\langle k_n\rangle = 0$ for any $k_1, k_2, ...k_n$ in $\mathfrak{K}$.*

Theorem 2.29. *[5, Theorem 1]*
Let a near-ring $\mathfrak{K}$ be reduced. If $\mathfrak{S}$ is a nonvoid multiplicative subsemigroup of $\mathfrak{K}$ such that $0 \notin \mathfrak{S}$, then a completely prime ideal $\mathfrak{V}$ exists in $\mathfrak{K}$ such that $\mathfrak{V} \cap \mathfrak{S} = \emptyset$.

3. Characterization of "r-regular near-rings".

The principal object "m -regular near-ring" was cited by G.Gopala Krishna Moorthy, R. Veega, and S. Geetha [6] and proved some results. In this section, with a new idea, we introduced "m-regular near-ring with r-regular near-ring" and gave some characterization.

According to [6] For each component $k \in \mathfrak{K}$, there is a component l in $\mathfrak{K}$ such that $k = kl^mk$ where $m \geq 1$ is a fixed integer, then $\mathfrak{K}$ is known as "m-regular near-ring(m-RN)".

Lemma 3.1. *[6, lemma 3.10] Let $\mathfrak{K}$ be a m-RN, $a \in \mathfrak{K}$ and $a = ab^ma$. Then*

- *The idempotents are ab^m and b^ma.*
- *$ab^m\mathfrak{K} = a\mathfrak{K}$ & $\mathfrak{K}b^ma = \mathfrak{K}a$.*

Let $\mathfrak{D}$ subset of $\mathfrak{K}$ then $\sqrt{\mathfrak{D}}$=$\{x \in \mathfrak{K}/x^k \in \mathfrak{D}, forsome\ k \geq 1\}$

Definition 3.2. *Let $\mathfrak{D}$ be an ideal of $\mathfrak{K}$ is known as Semi-Prime Ideal(S-PI) supposing that for all ideals $\mathfrak{I}$ of $\mathfrak{K}$, $\mathfrak{I}^2 \subseteq \mathfrak{D}$ implies $\mathfrak{I} \subseteq \mathfrak{D}$.*

Theorem 3.3. *Let $\mathfrak{K} \in \eta_0$ be a m-RN, r-RN with unity, and has IFP. Then $\mathfrak{C} = \sqrt{\mathfrak{C}}$ where $\mathfrak{C}$ is $\mathfrak{K}$-SG of $\mathfrak{K}$.*

Proof. Assume that $\mathfrak{C}$ is a $\mathfrak{K}$-SG of $\mathfrak{K}$.
Let $p \in \mathfrak{C}$ implies $p^1 \in \mathfrak{C}$ which implies $p \in \sqrt{\mathfrak{C}}$ hence, we get $\mathfrak{C} \subseteq \sqrt{\mathfrak{C}}$.
Now let $p \in \sqrt{\mathfrak{C}} \Rightarrow p^k \in \mathfrak{C}$.
By using the definition of m-RN, lemma 3.1 and theorem 2.25, we have $p = pl^mp = p(l^mp) = (l^mp)p = l^mp^2$
Now, $p = l^mpp = l^m(l^mp^2)p = l^{2m}p^3 = \cdots = l^{(k-1)m}p^k \subseteq \mathfrak{K}\mathfrak{C} \subseteq \mathfrak{C}$.
Hence, we get $\sqrt{\mathfrak{C}} \subseteq \mathfrak{C}$.
Thus, $\mathfrak{C} = \sqrt{\mathfrak{C}}$ where $\mathfrak{C}$ is $\mathfrak{K}$-SG of a $\mathfrak{K}$. □

Definition 3.4. *For each component p, t in a m-RN $\mathfrak{K}$ is referred to have IFP if $pt = 0$ then $pl^mt = 0$, for some l in $\mathfrak{K}$ and $m \geq 1$ is a fixed integer.*

Theorem 3.5. *If $\mathfrak{K} \in \eta_0$ be a m-RN, r-RN in which all the idempotents are central then $\mathfrak{K}$ is reduced.*

Proof. Suppose $p \in \mathfrak{K}$ such that $p^2 = 0$.
By using the definition of m-RN, and lemma 3.1, $p = pl^mp = l^mp^2 = l^m0 = 0$.
Therefore, $\mathfrak{K}$ is reduced. □

Theorem 3.6. *If $\mathfrak{K} \in \eta_0$ be a m-RN, r-RN in which all the idempotents are central then $\mathfrak{K}$ satisfies IFP.*

Proof. Let $t, p \in \mathfrak{K}$ such that $tp = 0$.
Now,$(pt)^2 = (pt)(pt) = p(tp)t = p0 = 0$.
By the theorem 3.5, $pt = 0$.
For $m \geq 1$, a fixed integer, consider $(tl^mp)^2 = (tl^mp)(tl^mp) = tl^m(pt)l^mp = tl^m0 = 0$.
By the theorem 3.5, $tl^mp = 0$.
Hence $\mathfrak{K}$ has IFP. □

Theorem 3.7. *If $\mathfrak{K} \in \eta_0$ be a m-RN, r-RN in which all the idempotents are central then every $\mathfrak{K}$-SG is an ideal.*

Proof. Let $\mathfrak{K}$ be r-RN in which all idempotents are central.
By the definition of r-RN and By the theorem2.25, we have $a = ea, e^2 = e, e \in \langle a|$.
Let $a \in \mathfrak{K}$, Since, by the definition of m-RN, we have $a = ab^ma$ where $m \geq 1$, a fixed integer and By the lemma 3.1, b^ma is idempotent.
Let $b^ma = e$ then by using the lemma 3.1, $\mathfrak{K}e = \mathfrak{K}b^ma = \mathfrak{K}a$.
Let $\mathfrak{F} = \{c - ce/c \in \mathfrak{K}\}$.
Claim: $(0 : \mathfrak{F}) = \{y \in \mathfrak{K}/sy = 0 \forall s \in \mathfrak{F}\} = \mathfrak{K}e$.
Now, $(c - ce)e = ce - ce^2 = ce - ce = 0\ \forall c \in \mathfrak{K}$.
By the theorem 3.6, $\mathfrak{K}$ has IFP, $(c - ce)\mathfrak{K}e = 0\ \forall c \Rightarrow \mathfrak{K}e \in (0 : \mathfrak{F})$.
Let $y \in (0 : \mathfrak{F}) \Rightarrow sy = 0, forall\ s \in \mathfrak{F}$.
$\Rightarrow syx^my = 0$.
Now, $yx^m - (yx^m)e \in \mathfrak{F} \Rightarrow [yx^m - (yx^m)e]y = 0$.
$\Rightarrow yx^my - yx^mey = 0, forall\ e \in \langle y|$.
$\Rightarrow y - ye = 0 \Rightarrow y = ye \in \mathfrak{K}e$.
$\Rightarrow (0 : \mathfrak{F}) \subseteq \mathfrak{K}e$.
Therefore, $(0 : \mathfrak{F}) = \mathfrak{K}e = \mathfrak{K}b^ma = \mathfrak{K}a$.
By the lemma 2.27, $(0 : \mathfrak{F})$ become an ideal, for any subset of $\mathfrak{F}$ of $\mathfrak{K}$.
$\Rightarrow \mathfrak{K}a$ become an ideal.
Thus, every $\mathfrak{K}$-SG is an ideal of $\mathfrak{K}$. □

Theorem 3.8. *If $\mathfrak{K} \in \eta_0$ be a m-RN, r-RN in which all the idempotents are central then $\mathfrak{K}$ is semi-prime near-ring.*

Proof. Let us define an ideal $\mathfrak{D}$ in $\mathfrak{K}$ such that $pt \in \mathfrak{D}$ for $p, t \in \mathfrak{K}$.
Let $\mathfrak{F}$ be $\mathfrak{K}$-SG of $\mathfrak{K}$.
Then by the theorem 3.7, $\mathfrak{F}$ is an ideal of $\mathfrak{K}$ and suppose that $\mathfrak{D}^2 \subseteq \mathfrak{F}$.
Since $\mathfrak{K}$ is zero-symmetric, $\mathfrak{K}\mathfrak{D} \subseteq \mathfrak{D}$.
If $p \in \mathfrak{D}$, then $p = pt^mp \in \mathfrak{D}\mathfrak{K}\mathfrak{D} \subseteq \mathfrak{D}\mathfrak{D} \subseteq \mathfrak{D}^2 \subseteq \mathfrak{F}$.
$\Rightarrow \mathfrak{D} \subseteq \mathfrak{F}$.
So, any $\mathfrak{K}$-SG of $\mathfrak{K}$ is a S-PI.

Specifically, $\{0\}$ is a S-PI and hence $\mathfrak{K}$ is a semi-prime near-ring. □

Example 3.9. *Let us define $\mathfrak{K}$ on $Z_6 = \{0,1,2,3,4,5\}$ with addition and product tables.[see Pilz, p409 (24)(3, 5, 5, 3, 1, 1)]*
Addition is modulo 6.

Table 3. Product table

.	0	1	2	3	4	5
0	0	0	0	0	0	0
1	3	5	5	3	1	1
2	0	4	4	0	2	2
3	3	3	3	3	3	3
4	0	2	2	0	4	4
5	3	1	1	3	5	5

Then $(\mathfrak{K},+,.)$ is a r-RN and also m-RN.

4. Characterization of "l-regular near-rings".

On studying the concepts of r-regular near-ring in [15, 16], the term l-regular near-ring was introduced. Yong Uk Cho [4] introduced semicentral idempotents and developed some results in the concept of reducibility in near-ring and we extended this notion of semicentral idempotent to the generalized regular near-rings namely r-regular near-ring(r-RN) and l-regular near-rings(l-RN).
We introduce the term "l-regular near-ring(l-RN)" as follows:

Definition 4.1. *For each element $q \in \mathfrak{K}$, there is a component l which is an idempotent in $\mathfrak{K}$ such that $q = lq, l \in |q\rangle$, then $\mathfrak{K}$ is known as "l-regular near-ring(l-RN)".*

Definition 4.2. *For each element $p^2 = p \in \mathfrak{K}$ is referred to be left semicentral idempotent(left-SCI) if $\mathfrak{K}p = p\mathfrak{K}p$.*

Definition 4.3. *For each element $q^2 = q \in \mathfrak{K}$ is referred to be right semicentral idempotent(right-SCI) if $q\mathfrak{K} = q\mathfrak{K}q$.*

Definition 4.4. *For each element $e^2 = e \in \mathfrak{K}$ is referred to be central idempotent(CI) if $ek = ke$ for all $k \in \mathfrak{K}$.*

Theorem 4.5. *Let $\mathfrak{K} \in \eta_0$, r-RN with 1 and has IFP. Then every left-SCI is right-SCI.*

Proof. Since by the theorem 2.25, $q = qe$ implies $q = eq$ for all $q \in \mathfrak{K}$.
Let $\mathfrak{K} \in \eta_0$,r-RN with 1 and has IFP.
Now for each $q \in \mathfrak{K}\ \exists\ e^2 = e \in \mathfrak{K}$ such that $q = qe, e \in \langle q| \subseteq \langle q\rangle$.
Since $(1-e)e = 0 \implies (1-e)qe = 0\ \forall q \in \mathfrak{K}$.
$\implies qe - eqe = 0 \implies qe = eqe \implies$ e is left-SCI .
By the theorem 2.25, $qe = eqe = eq \implies eqe = eq \implies e$ is right-SCI.
Thus, every left-SCI is right-SCI. □

Corollary 4.6. *Let $\mathfrak{K} \in \eta_0$, r-RN with 1 and has IFP. Then $\mathfrak{K}$ is central.*

Theorem 4.7. *Let $\mathfrak{K} \in \eta_0$ be l-RN with 1 and has IFP. Then for any idempotent is left-SCI.*

Proof. Let $\mathfrak{K} \in \eta_0$, l-RN with 1 and has IFP.
Now for each $q \in \mathfrak{K}\ \exists\ e^2 = e \in \mathfrak{K}$ such that $q = eq, e \in |q\rangle \subseteq \langle q\rangle$.
Since $(1-e)e = 0 \implies (1-e)qe = 0\ \forall q \in \mathfrak{K}$.
$\implies qe - eqe = 0 \implies qe = eqe \implies$ e is left-SCI.
Thus, for any idempotent is left semicentral idempotent(left-SCI). □

In the above theorems 4.5, 4.7 and corollary 4.6, the concepts of unity and reducibility is essential.

Example 4.8. *Consider a near-ring on the group $Z_6 = \{0,1,2,3,4,5\}$ with addition and product table given below.[see Pilz, p410 (53)(0, 1, 4, 3, 4, 1)]*
Addition is modulo 5.

Table 4. Product table

.	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	4	3	4	1
2	0	2	2	0	2	2
3	0	3	0	3	0	3
4	0	4	4	0	4	4
5	0	5	2	3	2	5

This near-ring is r-RN and also l-RN.
This near-ring is ZSN, reduced without unity.
It is clear that the idempotent elements 2 and 5 are not central.
This near-ring $\mathfrak{K}$ is right-SCI but not left-SCI. (for an element $1 \in \mathfrak{K}$ such that $2.1 \neq 1.2.1$).

Example 4.9. *Any regular near-ring(RN) is r-RN and l-RN. Let us consider $\mathfrak{K}$ on the group $Z_5 = \{0,1,2,3,4\}$ with addition and product tables. [see Pilz, p408, (7)(0, 1, 4, 1, 4)]*
Addition is modulo 5.

Table 5. Product table

.	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	4	3	2	1
3	0	1	2	3	4
4	0	4	3	2	1

Then $(\mathfrak{K},+,.)$ is a RN.

Remark 4.10. *In the above mentioned example 4.9, the near-ring $\mathfrak{K}$ is left-SCI but not right-SCI(for an element $1 \in \mathfrak{K}$ such that $2.1 \neq 2.1.2$).*

Theorem 4.11. *For a near-ring $\mathfrak{K}$ is l-RN then $\mathfrak{K} = \mathfrak{K}l$.*

Proof. By the definition of l-RN, then $l = el$, since $e^2 = e$,$e \in |l\rangle$.
$\implies l \in \mathfrak{K}l\ \forall l \in \mathfrak{K}$.
Therefore $\mathfrak{K} = \mathfrak{K}l$. □

Theorem 4.12. *For a near-ring $\mathfrak{K}$ is l-RN then $(0:u) = (0:\mathfrak{K}u) = (0:\mathfrak{K}), \forall u \in \mathfrak{K}$*

Proof. Since $\mathfrak{K}$ is l-RN, $u \in \mathfrak{K}u$.
Let $x \in (0:\mathfrak{K}u)$.
Now $x\mathfrak{K}u = 0 \implies xu = 0 \implies (0:\mathfrak{K}u) \subseteq (0:u)$.
Let $x \in (0:u)$ then $xu = 0$
$\implies x\mathfrak{K}u = 0 \implies x \in (0:\mathfrak{K}u) \implies (0:u) \subseteq (0:\mathfrak{K}u)$.
Therefore $(0:u) = (0:\mathfrak{K}u)$.
By the theorem 4.11, $(0:u) = (0:\mathfrak{K}u) = (0:\mathfrak{K})$. □

Theorem 4.13. *Let a near-ring $\mathfrak{K}$ be l-RN. Then every principal ideal is generated by an idempotent.*

Proof. Let $c \in \mathfrak{K}$. Consider a principal ideal generated by c, $\langle c\rangle$.
If $\mathfrak{K}$ is l-RN,$c = uc, u^2 = u, u \in |c\rangle \subseteq \langle c\rangle \implies \langle u\rangle \subseteq \langle c\rangle$.
$c = uc \in \langle u\rangle \implies \langle c\rangle \subseteq \langle u\rangle$.
Therefore $\langle c\rangle = \langle u\rangle$. □

Example 4.14. *Let us consider $\mathfrak{K}$ on $Z_6 = \{0,1,2,3,4,5\}$ with addition and product table given below.[see Pilz, p409 (24)(3, 5, 5, 3, 1, 1)]*
Addition is modulo 6.

Table 6. Product table

.	0	1	2	3	4	5
0	0	0	0	0	0	0
1	3	5	5	3	1	1
2	0	4	4	0	2	2
3	3	3	3	3	3	3
4	0	2	2	0	4	4
5	3	1	1	3	5	5

The only ideals of $\mathfrak{K}$ are $\{0\}$, $\{0,2,4\}$ and $\{0,1,2,3,4,5\}$. This near-ring $(\mathfrak{K},+,.)$ is both r-RN and l-RN.

Theorem 4.15. *Let a near-ring $\mathfrak{K}$ be l-RN. Then $\mathfrak{K}$ has no nonzero nil ideals.*

Proof. Suppose A be a nonzero nil ideal in $\mathfrak{K}$.
Let $0 \neq a \in A$ and $a = ea, e \in |a\rangle, e^2 = e$.
By the theorem 4.13, $e \in \langle e\rangle = \langle a\rangle \subseteq A$.
$\implies$ $'e'$ is nilpotent, which is a conflict to 'e' is idempotent.
Thus, $\mathfrak{K}$ has no nonzero nil ideals. □

Theorem 4.16. *For a near-ring $\mathfrak{K} \in \eta_0$ is l-RN and every $\mathfrak{K}$-subgroup is an ideal of $\mathfrak{K}$ then $\mathfrak{K}$ is left SRN.*

Proof. Suppose that $\mathfrak{K}$ is l-RN and every $\mathfrak{K}$-subgroup is an ideal of $\mathfrak{K}$.
By proposition 2.8, $a = ea, e^2 = e, e \in |a\rangle \subseteq \langle a\rangle = \mathfrak{K}a$.
$\implies e = na, for some\ n \in \mathfrak{K}$.
Therefore $a = ea = naa = na^2$ for some $n \in \mathfrak{K}$.
Hence $\mathfrak{K}$ is left SRN. □

Theorem 4.17. *For a near-ring $\mathfrak{K} \in \eta_0$ is l-RN with 1 then $\mathfrak{K}$ is reduced.*

Proof. Let $t \in \mathfrak{K}$ and $t^2 = 0 \implies t \in (0:t) \implies \langle t\rangle \subseteq (0:t)$.
Suppose $\mathfrak{K}$ is l-RN, then $t = et, e^2 = e, e \in |t\rangle \subseteq \langle t\rangle \subseteq (0:t) \implies et = 0$.
Therefore $t = 0$.
Hence $\mathfrak{K}$ is reduced. □

Theorem 4.18. *For a near-ring $\mathfrak{K} \in \eta_0$ is l-RN with 1 and has IFP then $d = ed$ implies $d = de$ where 'e' is an idempotent.*

Proof. Suppose $\mathfrak{K}$ is l-RN with 1 and has IFP.
Now $d \in \mathfrak{K}\ \exists e^2 = e \in \mathfrak{K} \ni d = ed, e \in |d\rangle \subseteq \langle d\rangle$.
Since $(1-e)e = 0 \implies (1-e)de = 0\ \forall d \in \mathfrak{K} \implies de - ede = 0 \implies de = ede = ed = d$ [by the lemma 2.20].
Therefore $d = ed$ implies $d = de$. □

Definition 4.19. *Let $\mathfrak{K}$ is referred to as weakly regular near-ring(WRN) if $A^2 = A$ for every ideal A of $\mathfrak{K}$.*

Definition 4.20. *Let an ideal $\mathfrak{D}$ of $\mathfrak{K}$ is referred to as "Completely Prime Ideal(CPI) if $kl \in \mathfrak{D}$ implies $k \in \mathfrak{D}$ or $l \in \mathfrak{D}$.*

Definition 4.21. *Let an ideal $\mathfrak{D}$ of $\mathfrak{K}$ is referred to as "3-Prime Ideal(3-PI) if $kn^1l \in D$ implies $k \in D$ or $l \in D$ for every $n^1 \in \mathfrak{K}$.*

Theorem 4.22. *Let a near-ring $\mathfrak{K}$ be l-RN. Then $\mathfrak{K}$ is WRN.*

Proof. Let $\mathfrak{D}$ be an ideal of $\mathfrak{K}$ and $a \in \mathfrak{D}$.
$a = ea, e^2 = e, e \in |a\rangle \subseteq \langle a\rangle \subseteq \mathfrak{D} \subseteq \mathfrak{D}.\mathfrak{D} = \mathfrak{D}^2$.
But $\mathfrak{D}^2 \subseteq \mathfrak{D}$, therefore $\mathfrak{D} = \mathfrak{D}^2$.
Thus, $\mathfrak{K}$ is WRN. □

Theorem 4.23. *Let a near-ring $\mathfrak{K}$ be l-RN. Then $\mathfrak{K}$ has no nonzero nilpotent ideal.*

Proof. Suppose J be a nonzero nilpotent ideal in $\mathfrak{K}$.
Then $J^k = (0)$ for some k which is greater than or equal to 2.
By the theorem 4.22, every ideal in a $\mathfrak{K}$ is idempotent i,e., $J = J^2$.
$J^k = J^{k-2}J = J^{k-4}J^2J = J^{k-4}JJ = J^{k-4}J^2 = J^{k-4}J = ...$
Continuing in this way we get $J = (0)$.
It is a contradiction.
Thus $\mathfrak{K}$ has no nonzero nilpotent ideal. □

Theorem 4.24. *Let a near-ring $\mathfrak{K}$ be l-RN with left unity then every CPI is a maximal.*

Proof. Let $\mathfrak{C}$ be a CPI of $\mathfrak{K}$
Suppose $\mathfrak{C} \subset \mathfrak{M} \subseteq \mathfrak{K}$ then $\exists a \in \mathfrak{M} \setminus \mathfrak{C}$
Now $a = ea, e^2 = e, e \in |a\rangle \subseteq \langle a \rangle \subseteq \mathfrak{M} \implies e \in \mathfrak{M}$.
$(1-e)a = 0 \in \mathfrak{C} \implies 1-e \in \mathfrak{C} \subset \mathfrak{M} \implies 1-e \in \mathfrak{M}$.
Let $c \in \mathfrak{K}$ then $c = 1.c = (1-e+e)c = (1-e)c + ec \in \mathfrak{M}$.
Therefore $\mathfrak{K} = \mathfrak{M}$.
Hence $\mathfrak{C}$ is a maximal ideal. □

Theorem 4.25. *Let $\mathfrak{K}$ be a l-RN with left unity and has IFP then every 3-PI is maximal.*

Proof. Let $\mathfrak{C}$ be a 3-PI of $\mathfrak{K}$.
Assume $\mathfrak{C} \subset \mathfrak{M} \subseteq \mathfrak{K}$.
Let $c \in \mathfrak{M} \setminus \mathfrak{C}$.
Now $c = ec, e^2 = e, e \in |c\rangle \subseteq \langle c \rangle$.
$(1-e)c = 0$.
Since $\mathfrak{K}$ has IFP, $(1-e)nc = 0 \forall n \in \mathfrak{K}$.
$(1-e)\mathfrak{K}c = 0 \subseteq \mathfrak{C} \implies 1-e \in \mathfrak{C} \subseteq \mathfrak{M} \implies 1-e \in \mathfrak{M}$.
For any x in $\mathfrak{K}, x = ex + (1-e)x \in \mathfrak{M}$.
Therefore $\mathfrak{K} = \mathfrak{M}$.
Thus $\mathfrak{C}$ is maximal ideal. □

Theorem 4.26. *If a near-ring $\mathfrak{K}$ is l-RN then every ideal I of $\mathfrak{K}$ is l-RN.*

Proof. Suppose $\mathfrak{K}$ is l-RN, then $a = ea, e^2 = e, e \in |a\rangle$.
Assume that I is an ideal $\mathfrak{K}$.
Let $a \in I$ then $a = ea$, $e \in |a\rangle \subseteq I$.
Therefore I is l-RN. □

Theorem 4.27. *For a near-ring $\mathfrak{K} \in \eta_0$ with identity,*

1. *$\mathfrak{K}$ is l-RN and has IFP.*
2. *$\mathfrak{K}$ is reduced and every CPI is maximal.*

are equivalent.

Proof. (1) $\implies$ (2)
Suppose $\mathfrak{K}$ is l-RN.
By theorem 4.17, $\mathfrak{K}$ is reduced and by theorem 4.24, it is proved.
(2) $\implies$ (1)
Suppose $\mathfrak{K} \in \eta_0$ is reduced and every CPI is maximal.
Since $\mathfrak{K} \in \eta_0$ is reduced, $ab = 0 \implies ba = 0$.
Consider $nba = n(ba) = n0 = 0 \implies (nb)a = 0 \implies anb = 0 \ \forall n \in \mathfrak{K}$.
Therefore $\mathfrak{K}$ has IFP.
Let $0 \neq a \in \mathfrak{K}$, by the lemma 2.28, $\overline{K} = \mathfrak{K} \setminus A(a)$ is reduced and $\overline{a}$ is not a zero divisor.
Also, every CPI of $\overline{K}$ is a maximal ideal in $\overline{K}$.
Let Q be the multiplicative subsemigroup generated by an element $\overline{a} - \overline{t}\,\overline{a}$ where $\overline{t} \in |a\rangle$.
If not, by the theorem 2.29, there exists a CPI $\overline{P}$ with $\overline{P} \cap Q = \emptyset$.
Suppose $|a\rangle \subseteq \overline{P}$ then $\overline{a} \in \overline{P}$.
$\implies \overline{a} - \overline{t}\overline{a} \in \overline{P}$.
$\implies \overline{a} - \overline{t}\,\overline{a} \in \overline{P} \cap Q$, it is a contradiction to the fact that $\overline{P} \cap Q = \emptyset$.
Suppose $|a\rangle \not\subseteq \overline{P}$ and $\overline{P}$ is maximal, we have $\overline{K} = \overline{P} + |a\rangle$.
$\overline{1} = \overline{\alpha} + \overline{t}$ where $\overline{\alpha} \in \overline{P}, \overline{t} \in |a\rangle$.
$\overline{a} = \overline{\alpha}\,\overline{a} + \overline{t}\,\overline{a}$.
$\implies \overline{a} - \overline{t}\,\overline{a} = \overline{\alpha}\,\overline{a} \in \overline{P}$.
$\implies \overline{a} - \overline{t}\,\overline{a} \in \overline{P} \cap Q$, it is a contradiction to the fact, $\overline{P} \cap Q = \emptyset$.
Thus $\overline{0} \in Q$.
Now $\overline{0} = (\overline{a} - \overline{t_1}\,\overline{a})(\overline{a} - \overline{t_2}\,\overline{a}) \cdots (\overline{a} - \overline{t_n}\,\overline{a}) = (\overline{1} - \overline{t_i})\overline{a}$, $\overline{t_i} \in |a\rangle$
Since $\overline{a}$ is not zero divisor, $(\overline{1} - \overline{t_i}) = 0 \implies \overline{1} = \overline{t_i}, t \in |a\rangle$.
Hence $(1-t) \in A(a) \implies (1-t)a = 0, t \in |a\rangle, t^2 = t \implies a = ta, t^2 = t, t \in |a\rangle$.
Therefore $\mathfrak{K}$ is l-RN. □

Definition 4.28. *Let a near-ring $\mathfrak{K}$ is referred to as "Left Quasi Duo(LQD)" if every maximal left ideal of $\mathfrak{K}$ is two-sided ideal.*

Theorem 4.29. *For a near-ring $\mathfrak{K} \in \eta_0$ is the LQD with left unity 1, $\mathfrak{K}$ is l-RN then $\mathfrak{K} = \langle q \rangle + (0:q)$.*

Proof. Since $\mathfrak{K}$ is l-RN, then $q = tq$, $t^2 = t$, $t \in |q\rangle \subseteq \langle q \rangle$.
$\implies q \in \langle q \rangle q$.
Then $\mathfrak{K}q \subseteq \mathfrak{K}\langle q \rangle q \subseteq \langle q \rangle q$.
And we have $\langle q \rangle q \subseteq \mathfrak{K}q$.
Therefore $\mathfrak{K}q = \langle q \rangle q$.
Suppose that $\mathfrak{K} \neq \langle q \rangle + (0:q)$.
Then there exists a maximal left ideal $\mathfrak{C}$ such that $\langle q \rangle + (0:q) \subseteq \mathfrak{C}$.
Since $\mathfrak{K}$ is LQD, $\mathfrak{C}$ is a two-sided ideal.
Since $q \in \mathfrak{C}$, $\langle q \rangle q \subseteq \mathfrak{C}q \subseteq \mathfrak{K}q = \langle q \rangle q$.
Therefore $\mathfrak{C}q = \langle q \rangle q$.
Therefore $\mathfrak{K}q = \langle q \rangle q = \mathfrak{C}q$.
Therefore $s \in \langle q \rangle$ such that $q = sq, s \in \langle q \rangle$
$\implies (1-s)q = 0 \implies 1-s \in (0:q)$.
Therefore $1 = s + (1-s) \in \langle q \rangle + (0:q) \subseteq \mathfrak{C}$.
It is a contradiction.
Therefore $\mathfrak{K} = \langle q \rangle + (0:q)$. □

5. Conclusion

In mathematics, several researchers are working on algebra. Recently as an application of near-rings, mathematicians used planar near-rings, near-rings of polynomials, and other near-rings to expand designs and codes. In this publication, we made an effort to develop the concept of regular near-rings and generalized regular near-rings.

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