Malaya Journal of Matematik, Vol. 8, No. 4, 2204-2207, 2020

https://doi.org/10.26637/MJM0804/0150

Existence of solutions of nonlocal fractional mixed type integro-differential equations with non-instantaneous impulses in Banach space

M. Mallika Arjunan[1]

Abstract
The key purpose of this manuscript is to examine the existence and uniqueness of PC-mild solution of nonlocal fractional mixed type integro-differential equations with non-instantaneous impulses in Banach space. Based on the general Banach contraction principle, we develop the main results.

Keywords
Fractional differential equations, mild solution, non-instantaneous impulses, fixed point theorem.

AMS Subject Classification
34K30, 35R12, 26A33.

[1] *Department of Mathematics, Vel Tech High Tech Dr. Rangarajan Dr. Sakunthala Engineering College, Avadi-600062, Tamil Nadu, India.*
***Corresponding author**: [1] arjunphd07@yahoo.co.in
Article History: Received **11** October **2020**; Accepted **23** December **2020**

Contents

1. Introduction

Differential equations of fractional order have currently proven to be useful methods for modeling multiple phenomena in different fields of science and engineering [1–3, 5, 7]. Significant advances in fractional differential equations have occurred in recent years; see the monographs by Kilbas et al. [5] and the papers by Zhou and Jio [8, 9] and the references cited therein.

Motivated by [1, 4, 6, 7], in this paper we consider a class of nonlocal fractional order mixed type integro-differential systems with non-instantaneous impulses of the form

$$\begin{aligned} {}^{C}D^{\alpha}x(t) &= f\left(t,x(t),K_1x(t),K_2x(t)\right), \\ & \qquad t\in(s_i,t_{i+1}], i=0,1,2,\dots,m \\ x(t) &= g_i(t,x(t)), \quad t\in(t_i,s_i], i=1,2,\dots,m \\ x(0)+h(x) &= x_0, \end{aligned} \tag{1.1}$$

where ${}^{C}D^{\alpha}$ is the Caputo fractional derivative of order $0<\alpha\le 1, t\in[0,T]; x_0\in X, 0=t_0=s_0<t_1\le s_1<t_2\le s_2<\dots<t_m\le s_m<t_{m+1}=T$ are fixed numbers, $g_i\in C\left((t_i,s_i]\times X;X\right)$, $f:[0,T]\times X^3\to X$ is a nonlinear function, $h:PC(J,X)\to\mathbb{R}$ and the functions K_1 and K_2 are defined by

$$K_1x(t)=\int_0^t u(t,s,x(s))ds \quad \text{and} \quad K_2x(t)=\int_0^T \widetilde{u}(t,s,x(s))ds,$$

$u,\widetilde{u}:\Delta\times X\to X$, where $\Delta=\{(x,s):0\le s\le x\le\tau\}$ are given functions which satisfies assumptions to be specified later on.

The rest of the paper is organized as follows. In Section 2, we present the notations, definitions and preliminary results needed in the following sections. In Section 3 is concerned with the existence results of problem (1.1).

2. Preliminaries

Let us set $J=[0,T], J_0=[0,t_1], J_1=(t_1,t_2],\dots, J_{m-1}=(t_{m-1},t_m], J_m=(t_m,t_{m+1}]$ and introduce the space $PC(J,X):=\{u:J\to X \mid u\in C(J_k,X), k=0,1,2,\dots,m,$ and there exist $u(t_k^+)$ and $u(t_k^-), k=1,2,\dots,m,$ with $u(t_k^-)=u(t_k)\}$. It is clear that $PC(J,X)$ is a Banach space with the norm $\|u\|_{PC}=\sup\{\|u(t)\|:t\in J\}$.

Let us recall the following well-known definitions [5].

Definition 2.1. *A real function $f(t)$ is said to be in the space $C_\alpha, \alpha\in\mathbb{R}$, if there exists a real number $p>\alpha$, such that*

$f(t) = t^p g(t)$, where $g \in C[0,\infty)$ and it is said to be in the space C_α^n if and only if $f^{(n)} \in C_\alpha, n \in \mathbb{N}$.

Definition 2.2. *The Riemann-Liouville derivative of order $\alpha > 0$ for a function $f \in C_\alpha^n, n \in \mathbb{N}$, is defined as*

$$D_t^\alpha f(t) = D^n D^{\alpha-n} f(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-\alpha-1} f(s)ds, t>0, n-1<\alpha<n.$$

Definition 2.3. *The Caputo fractional derivative of order $\alpha > 0$ for a function $f \in C_\alpha^n, n \in \mathbb{N}$, is defined as*

$$CD_t^\alpha f(t) = D^{\alpha-n} D^n f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1} f^n(s)ds, t>0, n-1<\alpha<n.$$

Lemma 2.4. *Let $f : J \to X$ be a continuous function. A function $x \in C(J,X)$ is a solution of the fractional integral equation*

$$x(t) = x_b - \frac{1}{\Gamma(\alpha)}\int_0^b (b-s)^{\alpha-1} f(s)ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)ds$$

if and only if $x(t)$ is a solution of the following fractional Cauchy problem:

$$CD^\alpha x(t) = f(t), \quad t \in J$$
$$x(b) = x_b, \quad b > 0$$

Now, we recall the following important Lemma which is very useful to prove our main result.

Lemma 2.5. *[7] Let $0 < \rho < 1, \gamma > 0$,*

$$S = \rho^n + D_n^1 \rho^{n-1}\gamma + \frac{D_n^2 \rho^{n-2}\gamma^2}{2!} + \cdots + \frac{\gamma^n}{n!}, \quad n \in \mathbb{N}.$$

Then, for all constant $0 < \xi < 1$ and all real number $s > 1$, we get

$$S \le O\left(\frac{\xi^n}{\sqrt{n}}\right) + O\left(\frac{1}{n^s}\right) = O\left(\frac{1}{n^s}\right), \quad n \to +\infty.$$

In view of the Lemma 2.1, we define the *PC*-mild solution for the given system (1.1).

Definition 2.6. *A function $x \in PC(J,X)$ is a mild solution of the problem (1.1) if $x(0) + h(x) = x_0$, $x(t) = g_i(t,x(t))$, $t \in (t_i,s_i]$, $i = 1,2,\dots,m$ and*

$$x(t) = x_0 - h(x) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,x(s),K_1(x(s)),K_2(x(s)))ds, \quad t \in [0,t_1]$$

and

$$x(t) = g_i(s_i,x(s_i)) - \frac{1}{\Gamma(\alpha)}\int_0^{s_i}(s_i-t)^{\alpha-1} f(s,x(s),K_1(x(s)),K_2(x(s)))ds + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,x(s),K_1(x(s)),K_2(x(s)))ds,$$

where, $t \in (s_i,t_{i+1}]$.

3. Existence and Uniqueness Results

In this section, we present and prove the existence and uniqueness of the system (1.1) under Banach contraction principle fixed point theorem.

To establish our results on the existence of solutions, we consider the following hypotheses:

(A1) The function $f \in C(J \times X^3; X)$ and there exist positive constants $L_{f_k} \in L^1(J,\mathbb{R}^+)$ $(k = 1,2,3)$ such that

$$\|f(t,x_1,x_2,x_3) - f(t,y_1,y_2,y_3)\| \le L_{f_1}(t)\|x_1-y_1\| + L_{f_2}(t)\|x_2-y_2\| + L_{f_3}(t)\|x_3-y_3\|$$

for all $(x_1,x_2,x_3),(y_1,y_2,y_3) \in X$ and every $t \in J$.

(A2) The functions $u, \widetilde{u} : \Delta \times X \to X$ are continuous and there exist constants $L_u, L_{\widetilde{u}} > 0$ such that

$$\left\|\int_0^t [u(t,s,x(s)) - u(t,s,y(s))]ds\right\| \le L_u\|x-y\|,$$

for all, $x,y \in X$; and

$$\left\|\int_0^T [\widetilde{u}(t,s,x(s)) - \widetilde{u}(t,s,y(s))]ds\right\| \le L_{\widetilde{u}}\|x-y\|,$$

for all, $x,y \in X$;

(A3) For $i = 1,2,\dots,m$, the functions $g_i \in C((t_i,s_i] \times \mathbb{R};\mathbb{R})$ and there exists $L_{g_i} \in C(J,\mathbb{R}^+)$ such that

$$|g_i(t,x) - g_i(t,y)| \le L_{g_i}|x-y|$$

for all $x,y \in X$ and $t \in (t_i,s_i]$.

(A4) $h : PC(J,X) \to X$ is continuous and there exists a positive constant $L_h > 0$ such that

$$\|h(x) - h(y)\| \le L_h\|x-y\|_{PC}, \quad \text{for all} \quad x,y \in PC(J,X).$$

Theorem 3.1. *If hypotheses $(A1)-(A4)$ hold and $0 \le \Lambda < 1$ $(\Lambda = \max\{L_h, L_{g_i}\})$, then problem (1.1) has a unique PC-mild solution $x^* \in PC(J,X)$.*

Proof. From Definition 2.4, we define an operator $\Upsilon : PC(J,X) \to PC(J,X)$ as $(\Upsilon x)(t) = (\Upsilon_1 x)(t) + (\Upsilon_2 x)(t)$, where

$$(\Upsilon_1 x)(t) = \begin{cases} x_0 - h(x), & t \in [0,t_1] \\ g_i(t,x(t)), & t \in (t_i,s_i] \\ g_i(s_i,x(s_i)), & t \in (s_i,t_{i+1}], \end{cases} \tag{3.1}$$

and

$$(\Upsilon_2 x)(t) = \begin{cases} \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,x(s),K_1(x(s)),K_2(x(s)))ds, \\ t \in [0,t_1] \\ 0, \\ t \in (t_i,s_i] \\ -\frac{1}{\Gamma(\alpha)}\int_0^{s_i}(s_i-t)^{\alpha-1} f(s,x(s),K_1(x(s)),K_2(x(s)))ds \\ +\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,x(s),K_1(x(s)),K_2(x(s)))ds, \\ t \in (s_i,t_{i+1}]. \end{cases} \tag{3.2}$$

For any $x,y \in PC(J,X)$, by (3.1) we sustain

$$\|(\Upsilon_1 x)(t)-(\Upsilon_1 y)(t)\| \leq \begin{cases} \Lambda\|x-y\|_{PC}, & t\in[0,t_1] \\ \Lambda\|x-y\|_{PC}, & t\in(t_i,s_i], i=1,2,\ldots,m, \\ \Lambda\|x-y\|_{PC}, & t\in(s_i,t_{i+1}], i=1,2,\ldots,m, \end{cases} \tag{3.3}$$

which means

$$\|(\Upsilon_1 x)(t)-(\Upsilon_1 y)(t)\| \leq \Lambda\|x-y\|_{PC},$$

where $t\in[0,t_1]\cup(t_i,s_i]\cup(s_i,t_{i+1}], i=1,2,\ldots,m$. Then we obtain

$$\left\|\left(\Upsilon_1^2 x\right)(t)-\left(\Upsilon_1^2 y\right)(t)\right\| \leq \Lambda^2\|x-y\|_{PC},$$

where $t\in[0,t_1]\cup(t_i,s_i]\cup(s_i,t_{i+1}], i=1,2,\ldots,m$. It is clear that, we have

$$\|(\Upsilon_1^n x)(t)-(\Upsilon_1^n y)(t)\| \leq \Lambda^n\|x-y\|_{PC}, \tag{3.4}$$

where $t\in[0,t_1]\cup(t_i,s_i]\cup(s_i,t_{i+1}], i=1,2,\ldots,m$.

For any real number $0<\varepsilon<1$, there exists a continuous function $\phi(s)$ such that $\frac{1}{\Gamma(\alpha)}\int_0^T (t-s)^{\alpha-1} \mid \ell(s)-\phi(s) \mid ds<\varepsilon$, where $\ell(s)=[L_{f_1}(s)+L_{f_2}(s)L_u+L_{f_3}(s)L_{\tilde{u}}]$ is a Lebesgue integrable function. For any $t\in[0,t_1], x,y\in PC(J,X)$ and by (3.2), we obtain

$$\begin{aligned}
&\|(\Upsilon_2 x)(t)-(\Upsilon_2 y)(t))\| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,x(s),K_1(x(s)),K_2(x(s))) \\
&-f(s,y(s),K_1(y(s)),K_2(y(s)))\|ds \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[L_{f_1}(s)+L_{f_2}(s)L_k+L_{f_3}(s)L_{\tilde{k}}] \\
&\quad \|x(s)-y(s)\|ds \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\ell(s)ds\|x-y\|_{PC} \\
&\leq \left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|\ell(s)-\phi(s)|ds\right. \\
&\left.+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|\phi(s)|ds\right)\|x-y\|_{PC} \\
&\leq (\varepsilon+\lambda t)\|x-y\|_{PC} \\
&= \left(D_1^0\varepsilon^1+D_1^1\frac{(\lambda t)^1}{1!}\right)\|x-y\|_{PC},
\end{aligned}$$

where $\max_{t\in J}\frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}|\phi(t)|=\lambda$.

Assume that, for any natural number k, we get

$$\begin{aligned}
&\|(\Upsilon_2^k x)(t)-(\Upsilon_2^k y)(t))\| \\
&\leq \left(D_k^0\varepsilon^k+D_k^1\varepsilon^{k-1}\frac{(\lambda t)^1}{1!}+\cdots+D_k^k\varepsilon^{k-k}\frac{(\lambda t)^k}{k!}\right)\|x-y\|_{PC}.
\end{aligned}$$

From the above inequality and the formula $D_{k+1}^m=D_k^m+D_k^{m-1}$, we obtain

$$\begin{aligned}
&\|(\Upsilon_2^{k+1} x)(t)-(\Upsilon_2^{k+2} y)(t))\| \\
&\leq \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[L_{f_1}(s)+L_{f_2}(s)L_u+L_{f_3}(s)L_{\tilde{u}}]\| \\
&\quad (\Upsilon_2^k x)(s)-(\Upsilon_2^k)y(s)\|ds \\
&=\frac{1}{\Gamma(\alpha)}\int_0^t \ell(s)\|(\Upsilon_2^k x)(s)-(\Upsilon_2^k)y(s)\|ds \\
&\leq \left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|\ell(s)-\phi(s)|\left(D_k^0\varepsilon^k+D_k^1\varepsilon^{k-1}\frac{(\lambda s)^1}{1!}\right.\right. \\
&\left.\left.+\cdots+D_k^k\varepsilon^{k-k}\frac{(\lambda s)^k}{k!}\right)ds\right)\|x-y\|_{PC} \\
&\quad+\left(\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|\phi(s)|\left(D_k^0\varepsilon^k+D_k^1\varepsilon^{k-1}\frac{(\lambda s)^1}{1!}\right.\right. \\
&\left.\left.+\cdots+D_k^k\varepsilon^{k-k}\frac{(\lambda s)^k}{k!}\right)ds\right)\|x-y\|_{PC} \\
&\leq \varepsilon\left(D_k^0\varepsilon^k+D_k^1\varepsilon^{k-1}\frac{(\lambda t)^1}{1!}+\cdots+D_k^k\varepsilon^{k-k}\frac{(\lambda t)^k}{k!}\right)\|x-y\|_{PC} \\
&\quad+\lambda\int_0^t\left(D_k^0\varepsilon^k+D_k^1\varepsilon^{k-1}\frac{(\lambda s)^1}{1!}+\cdots+D_k^k\varepsilon^{k-k}\frac{(\lambda s)^k}{k!}\right)ds \\
&\quad\|x-y\|_{PC} \\
&\leq \left(D_{k+1}^0\varepsilon^{k+1}+D_{k+1}^1\varepsilon^k\frac{(\lambda t)^1}{1!}+\cdots\right. \\
&\left.+D_{k+1}^{k+1}\varepsilon^{(k+1)-(k+1)}\frac{(\lambda t)^{k+1}}{(k+1)!}\right)\|x-y\|_{PC}.
\end{aligned}$$

By mathematical methods of induction, for any natural number n, we get

$$\begin{aligned}
&\|\Upsilon_2^n x-\Upsilon_2^n y\|_{PC} \\
&\leq \left(D_n^0\varepsilon^n+D_n^1\varepsilon^{n-1}\frac{\zeta^1}{1!}+\cdots+D_n^n\varepsilon^{n-n}\frac{\zeta^n}{n!}\right)\|x-y\|_{PC},
\end{aligned}$$

where $\zeta=\lambda T$. By Lemma 2.2, we have

$$\begin{aligned}
\|\Upsilon_2^n x-\Upsilon_2^n y\|_{PC} &\leq \left[O\left(\frac{\eta^n}{\sqrt{n}}\right)+O\left(\frac{1}{h^\mu}\right)\right]\|x-y\|_{PC} \\
&=O\left(\frac{1}{n^\mu}\right)\|x-y\|_{PC}, \quad (n\to+\infty),
\end{aligned} \tag{3.5}$$

where $0<\eta<1, \mu>1$. It is easy to see that the above equation (3.5) holds for $t\in(s_i,t_{i+1}], i=1,2,\ldots,m$. By (3.4) and (3.5), we obtain

$$\|\Upsilon^n x-\Upsilon^n y\|_{PC} \leq \left(\Lambda^n+O\left(\frac{1}{n^\mu}\right)\right)\|x-y\|_{PC}, \quad \forall n>n_0.$$

Thus, for any fixed constant $\mu>1$, we can find a positive integer n_0 such that, for any $n>n_0$, we get $0<\Delta^n+\frac{1}{n^\mu}<1$. Therefore, for any $x,y\in PC(J,X)$, we have

$$\|\Upsilon^n x-\Upsilon^n y\|_{PC} \leq \left(\Lambda^n+\frac{1}{n^\mu}\right)\|x-y\|_{PC} \leq \|x-y\|_{PC}, \quad \forall n>n_0.$$

By the general Banach contraction mapping principle, we get that the operator Υ has a unique fixed point $x^* \in PC(J,X)$, which means that problem (1.1) has a unique *PC*-mild solution.

□

References

[1] J. Borah and S. N. Bora, Existence of mild solution of a class of a class of nonlocal fractional order differential equation with not instantaneous impulses, *Fractional Calculus & Applied Analysis*, 22(2)(2019), 495–508.

[2] X. Fu, X. Liu and B. Lu, On a new class of impulsive fractional evolution equations, *Advances in Difference Equations*, (2015) 2015:227.

[3] Ganga Ram Gautham and Jeydev Dabas, Mild solution for nonlocal fractional functional differential equation with not instantaneous impulse, *International Journal of Nonlinear Science*, 21(3)(2016), 151–160.

[4] E. Hernández and D. O'Regan, On a new class of abstract impulsive differential equations, *Proc.Amer. Math. Soc.,* 141 (2013), 1641–1649.

[5] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, vol. 204. Elsevier, Amsterdam, 2006.

[6] M. Mallika Arjunan, Existence results for nonlocal fractional mixed type integro-differential equations with non-instantaneous impulses in Banach space, *Malaya Journal of Matematik*, 7(4)(2019), 837–840.

[7] B. Zhu, B. Han, L. Liu and W. Yu, On the fractional partial integro-differential equations of mixed type with non-instantaneous impulses, *Boundary Value Problems*, (2020), 2020:154

[8] Y. Zhou and F. Jiao, Nonlocal Cauchy problem for fractional evolution equations, *Nonlinear Anal., Real World Appl.* 11(2010), 4465–4475.

[9] Y. Zhou and F. Jiao, Existence of mild solutions for fractional neutral evolution equations, *Comput. Math. Appl.,* 59(2010), 1063–1077.

⋆⋆⋆⋆⋆⋆⋆⋆⋆
ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
⋆⋆⋆⋆⋆⋆⋆⋆⋆