



# Integral solutions of fractional order mixed type integro-differential equations with non-instantaneous impulses in Banach space

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## Abstract

The main objective of this article is to examine the existence and uniqueness of integral solutions for a class of fractional order mixed type integro-differential equations with non-instantaneous impulses and non-densely defined linear operators in Banach spaces. Based on the Banach contraction principle, we develop the main results.

## Keywords

Fractional differential equations, mild solution, non-instantaneous impulses, fixed point theorem.

## AMS Subject Classification

34K37, 37L05, 47J35, 26A33.

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Article History: Received 23 September 2020; Accepted 10 December 2020

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## 1. Introduction

Various certifiable issues include abrupt shifts in their states; these abrupt changes are known as impulsive impacts on issues. There are two types of impulsive processes in the present hypothesis: the instantaneous impulsive system, and the non-instantaneous impulsive system. In the instantaneous impulsive system, the duration of these abrupt shifts has nothing to do with the period of the whole phase of progression. Such abrupt impulses arise in cardiac pulses, vibrations, and natural events, while in non-instantaneous impulses the duration of these rapid changes proceeds over a short time interval. Hernandez & O'Regan [4] developed a new class of differential equations with non-instantaneous impulses and established the existence of mild and classical solutions. For more details, see for instance [1–3, 6].

Motivated by [1–5], in this paper, we consider a class of fractional mixed type integro-differential systems with non-

instantaneous impulses of the form

$$\begin{aligned}
 {}^c D^\alpha x(t) &= Ax(t) + f(t, x(t), E_1 x(t), E_2 x(t)), \\
 t &\in (s_i, t_{i+1}], i = 0, 1, 2, \dots, m \\
 x(t) &= g_i(t, x(t)), \quad t \in (t_i, s_i], i = 1, 2, \dots, m \quad (1.1) \\
 x(0) &= x_0,
 \end{aligned}$$

where  ${}^c D^\alpha$  is the Caputo fractional derivative of order  $\alpha \in (0, 1)$ ,  $A : D(A) \subset X \rightarrow X$  not necessarily a densely defined closed operator on the Banach space  $(X, \|\cdot\|)$ ,  $0 = t_0 = s_0 < t_1 \leq s_1 < t_2 \leq s_2 < \dots < t_m \leq s_m < t_{m+1} = T$  are fixed numbers,  $g_i \in C\left((t_i, s_i] \times X; \overline{D(A)}\right)$ ,  $f : [0, T] \times X^3 \rightarrow X$  is a non-linear function and the functions  $E_1$  and  $E_2$  are defined by

$$E_1 x(t) = \int_0^t k(t, s, x(s)) ds \quad \text{and} \quad E_2 x(t) = \int_0^T \tilde{k}(t, s, x(s)) ds,$$

$k, \tilde{k} : \Delta \times X \rightarrow X$ , where  $\Delta = \{(x, s) : 0 \leq s \leq x \leq \tau\}$  are given functions which satisfies assumptions to be specified later on.

The rest of the paper is organized as follows. In Section 2, we present the notations, definitions and preliminary results needed in the following sections. In Section 3 is concerned with the existence and uniqueness results of problem (1.1).

## 2. Preliminaries

Let us set  $J = [0, T], J_0 = [0, t_1], J_1 = (t_1, t_2], \dots, J_{m-1} = (t_{m-1}, t_m], J_m = (t_m, t_{m+1}]$  and introduce the space  $PC(J, X) := \{u : J \rightarrow X \mid u \in C(J_k, X), k = 0, 1, 2, \dots, m, \text{ and there exist } x(t_k^+), x(t_k^-), k = 1, 2, \dots, m, \text{ with } x(t_k^-) = x(t_k)\}$ . It is clear that  $PC(J, X)$  is a Banach space with the norm  $\|x\|_{PC} = \sup\{\|x(t)\| : t \in J\}$ .

Let  $X_0 = \overline{D(A)}$  and  $A_0$  be the part of  $A$  in  $\overline{D(A)}$  defined by

$$D(A_0) = \left\{x_1 \in D(A) : Ax_1 \in \overline{D(A)}\right\}, A_0(x_1) = A(x_1)$$

Throughout our analysis, the following hypotheses will be considered:

(H1)  $A : D(A) \subset X \rightarrow X$  satisfies the Hille-Yosida condition, that is, there exist two constants  $\omega \in \mathbb{R}$  and  $\Lambda_0 \geq 0$  such that  $(\omega, \infty) \subset \rho(A)$  and

$$\|(\lambda I - A)^{-n}\|_{L(X)} \leq \frac{\Lambda_0}{(\lambda - \omega)^n}, \text{ for all } \lambda > \omega, n \geq 1$$

(H2) The part  $A_0$  of  $A$  generates a compact  $C_0$ -semigroup  $\{Q(t)\}_{t \geq 0}$  in  $X_0$  which is uniformly bounded, that is, there exists  $\Lambda \geq 1$  such that  $\sup_{t \in [0, \infty)} \|Q(t)\| < \Lambda$ .

Let  $(U(t))_{t \geq 0}$  be the integrated semigroup generated by  $A$ . It is to be noted that  $(U'(t))_{t \geq 0}$  is a  $C_0$ -semigroup on  $\overline{D(A)}$  generated by  $A_0$  and  $\|U'(t)\| \leq \Lambda e^{\omega t}, t \geq 0, \Lambda$  and  $\omega$  are the constants used in the Hille-Yosida condition.

Let  $B_\lambda = \lambda R(\lambda, A) := \lambda(\lambda I - A)^{-1}$ . Then for all  $x_1 \in X_0, B_\lambda x_1 \rightarrow x_1$  as  $\lambda \rightarrow \infty$ . Also from Hille-Yosida condition, it is clear that  $\lim_{\lambda \rightarrow \infty} \|B_\lambda\| \leq \Lambda_0$ .

Based on the above discussion along with [2], we define the integral solution of the given system (1.1).

**Definition 2.1.** [2] A function  $x \in PC(J, X)$  is said to be an integral solution of the Cauchy problem (1.1) if it satisfies  $x(0) = x_0 \in X_0, x(t) = g_i(t, x(t))$  for all  $t \in (t_i, s_i], i = 1, 2, \dots, m$ :

$$\begin{aligned} x(t) &= U_\alpha(t)x_0 + \lim_{\lambda \rightarrow \infty} \int_0^t V_\alpha(t-s)B_\lambda \\ &\quad f(s, x(s), E_1x(s), E_2x(s))ds, \quad t \in [0, t_1] \text{ and} \\ x(t) &= U_\alpha(t-s_i)g_i(s_i, x(s_i)) + \lim_{\lambda \rightarrow \infty} \int_{s_i}^t V_\alpha(t-s)B_\lambda \\ &\quad f(s, x(s), E_1x(s), E_2x(s))ds, \quad t \in [s_i, t_{i+1}], \\ &\quad i = 1, 2, \dots, m, \end{aligned}$$

where

$$\begin{aligned} U_\alpha(t) &= I_{0+}^{1-\alpha} V_\alpha(t), V_\alpha(t) = t^{\alpha-1} W_\alpha(t), \\ W_\alpha(t) &= \int_0^\infty \alpha \theta \Lambda_\alpha(\theta) Q(t^\alpha \theta) d\theta. \end{aligned}$$

**Remark 2.2.** For any fixed  $t > 0, V_\alpha(t)$  and  $U_\alpha(t)$  are linear operators, and for any  $x_1 \in X_0$ ,

$$\|V_\alpha(t)x_1\| \leq \frac{\Lambda t^{\alpha-1}}{\Gamma(\alpha)} \|x_1\| \text{ and } \|U_\alpha(t)x_1\| \leq \Lambda \|x_1\|.$$

## 3. Existence and Uniqueness Results

In this section, we present and prove the existence and uniqueness of the system (1.1) under Banach contraction principle fixed point theorem.

To establish our results on the existence of solutions, we consider the following hypotheses:

Here, we take  $a = \frac{\alpha-1}{1-\alpha_1} \in (-1, 0)$ .

(H(f)) The function  $f \in C(J \times X^3; X)$  and there exists a constant  $\alpha \in (0, \alpha)$  and a function  $L_f \in L^{\frac{1}{\alpha_1}}(J, \mathbb{R}^+)$  such that

$$\begin{aligned} &\|f(t, x_1, x_2, x_3) - f(t, y_1, y_2, y_3)\| \\ &\leq L_f(t) [\|x_1 - y_1\| + \|x_2 - y_2\| + \|x_3 - y_3\|] \end{aligned}$$

for all  $(x_1, x_2, x_3), (y_1, y_2, y_3) \in X$  and every  $t \in J$ .

(H(k,  $\tilde{k}$ )) The functions  $k, \tilde{k} : \Delta \times X \rightarrow X$  are continuous and there exist constants  $L_k, L_{\tilde{k}} > 0$  such that

$$\left\| \int_0^t [k(t, s, x(s)) - k(t, s, y(s))] ds \right\| \leq L_k \|x - y\|,$$

for all  $x, y \in X$ ; and

$$\left\| \int_0^T [\tilde{k}(t, s, x(s)) - \tilde{k}(t, s, y(s))] ds \right\| \leq L_{\tilde{k}} \|x - y\|,$$

for all  $x, y \in X$ ;

(H(g)) For  $i = 1, 2, \dots, m$ , the functions  $g_i \in C((t_i, s_i] \times X; X_0)$  and there exists  $L_{g_i} \in C(J, \mathbb{R}^+)$  such that

$$\|g_i(t, x) - g_i(t, y)\| \leq L_{g_i} \|x - y\|$$

for all  $x, y \in X$  and  $t \in (t_i, s_i]$ .

**Theorem 3.1.** Assume the hypotheses  $H(f), H(k, \tilde{k}), H(g)$  to hold and

$$\begin{aligned} C &= \max \left\{ \max_{1 \leq i \leq m} \left\{ \Lambda L_{g_i} + \frac{\Lambda \Lambda_0 (t_{i+1} - s_i)^{(1+a)(1-\alpha_1)}}{\Gamma(\alpha) (1+a)^{1-\alpha_1}} \right. \right. \\ &\quad \left. \left. \|L_f\|_{L^{\alpha_1}([s_i, t_{i+1}])} [1 + L_k + L_{\tilde{k}}] \right\}, \right. \\ &\quad \left. \frac{\Lambda \Lambda_0 t_1^{(1+a)(1-\alpha_1)}}{\Gamma(\alpha) (1+a)^{1-\alpha_1}} \|L_f\|_{L^{\alpha_1}([0, t_1])} [1 + L_k + L_{\tilde{k}}] \right\} < 1. \end{aligned}$$

Then there exists a unique integral solution in  $PC(J, X)$  of the system (1.1) provided  $x_0 \in X_0$ .

*Proof.* Define the operator  $\Upsilon : PC(J, X) \rightarrow PC(J, X)$  by  $\Upsilon x(0) = x_0, \Upsilon x(t) = g_i(t, x(t))$  for  $t \in (t_i, s_i]$  and

$$\begin{aligned} \Upsilon x(t) &= U_\alpha(t-s_i)g_i(s_i, x(s_i)) + \lim_{\lambda \rightarrow \infty} \int_{s_i}^t V_\alpha(t-s)B_\lambda \\ &\quad f(s, x(s), E_1x(s), E_2x(s))ds, \quad t \in [s_i, t_{i+1}], i \geq 0. \end{aligned}$$



Let  $x, y \in PC(J, X)$ . For  $t \in (s_i, t_{i+1}]$ ,  $i = 1, 2, \dots, m$ , we obtain

$$\begin{aligned} & \|Yx(t) - Yy(t)\| \\ & \leq \|U_\alpha(t - s_i)g_i(s_i, x(s_i)) - U_\alpha(t - s_i)g_i(s_i, y(s_i))\| \\ & + \lim_{\lambda \rightarrow \infty} \int_{s_i}^t \|V_\alpha(t - s)B_\lambda f(s, x(s), E_1x(s), E_2x(s)) \\ & - V_\alpha(t - s)B_\lambda f(s, y(s), E_1y(s), E_2y(s))\| ds \\ & \leq \Lambda L_{g_i} \|x - y\|_{PC} + \frac{\Lambda \Lambda_0}{\Gamma(\alpha)} \int_{s_i}^t (t - s)^{\alpha-1} L_f(s) [1 + L_k + L_{\tilde{k}}] ds \|x - y\|_{PC} \\ & \leq \Lambda L_{g_i} \|x - y\|_{PC} + \frac{\Lambda \Lambda_0}{\Gamma(\alpha)} \left[ \int_{s_i}^t ((t - s)^{\alpha-1})^{\frac{1}{1-\alpha_1}} ds \right]^{1-\alpha_1} \\ & \|L_f\|_{L^{\alpha_1}([s_i, t_{i+1}])} [1 + L_k + L_{\tilde{k}}] \|x - y\|_{PC} \\ & \leq \left[ \Lambda L_{g_i} + \frac{\Lambda \Lambda_0}{\Gamma(\alpha)} \frac{(t_{i+1} - s_i)^{(1+a)(1-\alpha_1)}}{(1+a)^{1-\alpha_1}} \|L_f\|_{L^{\alpha_1}([s_i, t_{i+1}])} [1 + L_k + L_{\tilde{k}}] \right] \\ & \|x - y\|_{PC}. \end{aligned}$$

For  $t \in [0, t_1]$

$$\begin{aligned} & \|Yx(t) - Yy(t)\| \\ & \leq \left[ \frac{\Lambda \Lambda_0}{\Gamma(\alpha)} \frac{t_1^{(1+a)(1-\alpha_1)}}{(1+a)^{1-\alpha_1}} \|L_f\|_{L^{\alpha_1}([0, t_1], \mathbb{R}^+)} [1 + L_k + L_{\tilde{k}}] \right] \|x - y\|_{PC}. \end{aligned}$$

For  $t \in (t_i, s_i]$ ,  $i = 1, 2, \dots, m$ , we have

$$\|Yx(t) - Yy(t)\| \leq L_{g_i} \|x - y\|_{PC} \leq \Lambda L_{g_i} \|x - y\|_{PC}.$$

From above, we observe that

$$\|Y(x) - Y(y)\|_{PC} \leq C \|x - y\|_{PC}$$

which implies that  $Y(\cdot)$  is a contraction and there exists a unique integral solution of the system (1.1). □

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 ISSN(P):2319 – 3786  
 Malaya Journal of Matematik  
 ISSN(O):2321 – 5666  
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