Malaya Journal of Matematik, Vol. 8, No. 4, 2253-2258, 2020

https://doi.org/10.26637/MJM0804/0158

Laplace-Carson transform of fractional order

T. G. Thange[1] and A. R. Gade[2*]

Abstract

In this paper, we proposed new generalized Laplace-Carson transform of fractional order called Fractional Laplace-Carson transform of order $0 < \alpha < 1$. This transform is applying for functions which are differentiable but by fractional order. By using the definition of fractional order Laplace-Carson transform we prove fundamental properties of this integral transform. Finally, we have obtained convolution and inversion.

Keywords

Laplace-Carson transform, Laplace transform, Mittag-Leffler function, Generalization function, Fractional Derivative and Fractional Integration.

[1] *Department of Mathematics, Yogeshwari Mahavidyalaya Ambajogai, Beed, MH, India.*

[2] *Department of Mathematics, Arts Commerce and Science College Kille-Dharur, Beed, MH, India.*

*__Corresponding author__: [2] ananthagade45@gmail.com

Article History: Received **01** October **2020**; Accepted **22** December **2020**

Contents

1. Introduction

We all are familiar about the application of integral transform for the solution of different differential and integral equations [1,2]. It is the best tool for finding the solutions of many of this problem. Laplace-Carson transform is the Laplace type integral transform but it is generalization of Laplace transform [3,4] which is widely used for solving differential equation [7,8] with efficient and more convenient way. If $\Omega(\zeta)$ is continuous and continuously differentiable then by using regular definitions of different integral transform we solve differential equations of function $\Omega(\zeta)$ but if $\Omega(\zeta)$ is continues but differentiable by fractional order α, then this definitions does not work, in that case we use the definition of fractional order Laplace-Carson transform for finding the solution of differential equations in particular fractional order differential equations of function $\Omega(\zeta)$.

Authors work on different fractional order integral transforms & generalized integral transforms[8,10,16] to solve many real life problems in different fields.

In this Paper firstly we study the basic definitions, like definition of Laplace-Carson transform, Mittag-Leffler function, fractional derivative and so on. After that we define fractional order Laplace-Carson transform and prove some important results and properties and in the final part we give proof of convolution theorem and inversion formula.

2. Basics of Laplace-Carson Transform and Fractional Derivatives

First, we summaries definitions of Laplace, Laplace-Carson transform, fractional order derivative in the finite difference form and other related definitions.

Definition 2.1. *Let $\Omega(\zeta)$ is continuous real valued function for $\zeta \in \mathbb{R}^+ = [0,\infty)$, and $|\,\Omega(\zeta)\,| \leq Pe^{Q\zeta}\,(\zeta \geq 0)$ for constants P and Q. Then for $\eta \in \mathbb{C}, (Re(\eta)) \geq Q$, define the Laplace transform [9] is,*

$$\mathscr{L}[\Omega(\zeta)](\eta) = \int_0^\infty exp(-\eta\zeta)\Omega(\zeta)d\zeta. \tag{2.1}$$

Definition 2.2 (Laplace-Carson transform [3,4])**.** *Let $\Omega(\zeta)$ is continuous real valued function for $\zeta \in \mathbb{R}^+ = [0,\infty)$, and $\mid \Omega(\zeta) \mid \leq Pe^{Q\zeta}(\zeta \geq 0)$ for constants P and Q. Then for $\eta \in \mathbb{C}, (Re(\eta)) \geq Q$. i.e.*

$$\Delta = \{\Omega(\zeta) \mid \exists P, Q > 0 \mid \Omega(\zeta) \mid \leq Pe^{Q\zeta}, \text{ where } \zeta \in \mathbb{R}^+ = [0,\infty]\},$$

define the Laplace-Carson transform the integral as,

$$\mathscr{LC}[\Omega(\zeta)](\eta) = \Omega^{'}(\eta) = \int_0^\infty \eta exp(-\eta\zeta)\Omega(\zeta)d\zeta \quad (2.2)$$

$$\mathscr{LC}[\Omega(\zeta)](\eta) = \Omega^{'}(\eta) = \lim_{z\to\infty}\int_0^z \eta exp(-\eta\zeta)\Omega(\zeta)d\zeta.$$

The Inverse Laplace-Carson integral transform [3] is defined as,

$$\mathscr{LC}^{-1}[\Omega^{'}(\zeta)](\eta) = \Omega(\zeta) = \frac{1}{2\pi i}\int_{\mu-i\infty}^{\mu+i\infty}\frac{1}{\eta}e^{(\eta\zeta)}\Omega^{'}(\eta)d\eta, \quad (2.3)$$

where $\zeta > 0$ and μ is real number such that the counter path of integration is in the region of convergence of $\Omega^{'}(\eta)$.

Definition 2.3 (Derivative of Laplace-Carson integral transform [4,7])**.** *If the function $\Omega^{(n)}(\zeta)$ is the n^{th} derivative of the function $\Omega(\zeta)$ with respect to ζ then it's Laplace-Carson integral transform is defined as,*

$$\mathscr{LC}[\Omega^{(n)}(\zeta)] = \eta^{(n)}\Omega^{'}(\eta) - \sum_{k=0}^{n-1}(\eta)^{(n-k)}.\Omega^{(k)}(0), \text{ where } n \geq 1. \quad (2.4)$$

Definition 2.4 (Laplace-Carson transform of Mittag-Leffler function [9,10,11])**.**

$$E_{\alpha,\beta}(\zeta) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}$$

$\alpha, \beta \in \mathbb{C}$ and $Re(\alpha)$, $Re(\beta) > 0$ is given by,

$$\mathscr{LC}[\zeta^{\tau-1}E_{\alpha,\beta}(\varepsilon\zeta^\alpha)] = \eta^\beta(1-\varepsilon(\eta)^\alpha)^{-\tau}, \quad (2.5)$$

where $Re(\alpha)$, $Re(\beta)$, $Re(\tau) > 0$ and $\varepsilon \in \mathbb{C}$.

Definition 2.5 ([11,12])**.** *$\Gamma(z)$ is Euler Gamma function, which is generalization of factorial function from set of integers to the set of complex numbers. Defined as,*

$$\Gamma(\zeta) = \int_0^\infty t^{z-1}e^{-t}dt, z \in \mathbb{C},$$

with $Re(z > 0)$, with $\Gamma(\zeta+1) = \zeta\Gamma(\zeta)$, where $\zeta \in \mathbb{R}^+$ and $\Gamma(\zeta) = (\zeta-1)!$ where $\zeta \in \mathbb{R}^+$.

Definition 2.6 (Definition of Fractional order derivative in finite Difference form [9,12])**.** *Let $\Omega : \mathbb{R} \to \mathbb{R}$ denotes a continuous function and $h > 0$ denote content discretization span then, Define the forward operator FW(h) by the equality, $FW(h)(\Omega(\zeta)) = \Omega(\zeta+h)$. Then fractional order derivative of order α, where $0 < \alpha < 1$ of $\Omega(\zeta)$ is,*

$$\Delta^\alpha\Omega(\zeta) = (FW-1)^\alpha = \sum_{k=0}^{\infty}(-1)^k C_k^\alpha\Omega(\zeta+(\alpha-k)h). \quad (2.6)$$

Fractional derivative of order α is the limit,

$$\Omega^{(\alpha)}(\zeta) = \lim_{h\to 0}\frac{\Delta^\alpha\Omega(\zeta)}{h^\alpha}. \quad (2.7)$$

3. Main Result

Definition 3.1. *Laplace-Carson transform of fractional order α of non-negative function $\Omega(\zeta)$ denoted by $\mathscr{LC}_\alpha\,[\Omega(\zeta)]$, and defined as,*

$$\mathscr{LC}_\alpha[\Omega(\zeta)] = \Omega_\alpha^{'}(\eta) = \eta^\alpha\int_0^\infty E_\alpha(-\eta^\alpha\zeta^\alpha)\Omega(\zeta)(d\zeta)^\alpha, \quad (3.1)$$

where $\eta \in \mathbb{C}$, and $E_\alpha(\zeta)$ is Mittag-Leffler function $E_\alpha\Omega(\zeta) = \sum_{k=0}^{\infty}\frac{\zeta^\alpha}{\Gamma(\alpha k+1)}$.

3.1 Existence of Fractional order Laplace-Carson transform

Theorem 3.2. *If function $\Omega(\zeta)$ is non-negative piecewise continuous in interval $0 \leq \zeta \leq \xi$ and it is of exponential order α then its fractional Laplace-Carson transform $\Omega_\alpha^{'}(\eta)$ exist.*

Proof. Suppose that,

$$\begin{aligned}\eta^\alpha\int_0^\infty & E_\alpha(-\eta^\alpha\zeta^\alpha)\Omega(\zeta))(d\zeta)^\alpha \\ &= \eta^\alpha\int_0^\xi E_\alpha(-\eta^\alpha\zeta^\alpha)\Omega(\zeta))(d\zeta)^\alpha \\ &+ \eta^\alpha\int_\xi^\infty E_\alpha(-\eta^\alpha\zeta^\alpha)\Omega(\zeta))(d\zeta)^\alpha. \quad (3.2)\end{aligned}$$

Since $\Omega(\zeta)$ is pricewise continuous in interval $[0,\ \xi]$ with $0 \leq \zeta \leq \xi$ then first integral of RHS of (3.2) exist, since $\Omega(\zeta)$ is of exponential order α for $\xi < \zeta$, to check the existence we concentrate on second term of RHS of (3.2) then,

$$
\begin{aligned}
&|\eta^{\alpha}\int_{\xi}^{\infty}E_{\alpha}(-\eta^{\alpha}\zeta^{\alpha})\Omega(\zeta)(d\zeta)^{\alpha}|\\
&\leq \eta^{\alpha}\int_{\xi}^{\infty}|E_{\alpha}(-\eta^{\alpha}\zeta^{\alpha})\Omega(\zeta)|(d\zeta)^{\alpha}\\
&\leq \eta^{\alpha}\int_{\xi}^{\infty}E_{\alpha}(-\eta^{\alpha}\zeta^{\alpha})|\Omega(\zeta)|(d\zeta)^{\alpha}\\
&\leq \eta^{\alpha}\int_{\xi}^{\infty}E_{\alpha}(-\eta^{\alpha}\zeta^{\alpha})CE_{\alpha}(u^{\alpha}\zeta^{\alpha})(d\zeta)^{\alpha}\\
&=C\eta^{\alpha}\int_{\xi}^{\infty}E_{\alpha}(-(\eta-u)^{\alpha}\zeta^{\alpha})(d\zeta)^{\alpha}\\
&=C\eta^{\alpha}\lim_{n\to\infty}\int_{\xi}^{n}E_{\alpha}(-(\eta-u)^{\alpha}\zeta^{\alpha})(d\zeta)^{\alpha}\\
&=\frac{-C\eta^{\alpha}}{(\eta-u)^{\alpha}}\lim_{n\to\infty}E_{\alpha}(-(\eta-u)^{\alpha}\zeta^{\alpha})]_{\xi}^{n}\\
&=\frac{-C\eta^{\alpha}}{(\eta-u)^{\alpha}}[0-E_{\alpha}(-(\eta-u)^{\alpha}\zeta^{\alpha})].
\end{aligned}
$$

But as $\xi\to 0$ then we get the existence of Second term of RHS also,

$$
\left|\int_{\xi}^{\infty}E_{\alpha}(-\eta^{\alpha}\zeta^{\alpha})\Omega(\zeta))(d\zeta)^{\alpha}\right|\leq\frac{C}{(\eta-u)^{\alpha}}=M. \quad (3.3)
$$

This completes the proof. □

Now we prove basic properties related to fractional order Laplace-Carson transform.

Theorem 3.3. *Let functions $\Omega(\zeta)\in\Delta$. Then following Fractional Laplace-Carson Transform of some standard functions hold.*

$$
\mathscr{L}\mathscr{S}_{\alpha}[1]_{\eta}=\eta^{\alpha}\Gamma(\alpha+1). \quad (3.4)
$$

Proof. By using Definition in equation (3.1) for LHS in Equation (3.4) we get,

$$
\begin{aligned}
\mathscr{L}\mathscr{S}_{\alpha}[1]_{\eta}&=\eta^{\alpha}\int_{0}^{\infty}E_{\alpha}(-\eta^{\alpha}\zeta^{\alpha})[1](d\zeta)^{\alpha}\\
&=\eta^{\alpha}\alpha!\,\Gamma_{\alpha}(1)\\
&=\eta^{\alpha}\Gamma(\alpha+1).
\end{aligned}
$$

□

Theorem 3.4 (Linearity Property)**.** *Let functions $a\Omega_1(\zeta)$, $b\Omega_2(\zeta)\in\Delta$ then $a\Omega_1(\zeta)+b\Omega_2(\zeta)\in\Delta$ where a and b are nonzero arbitrary constants and,*

$$
\mathscr{L}\mathscr{C}_{\alpha}[a\Omega_1(\zeta)+b\Omega_2(\zeta)]=a\mathscr{L}\mathscr{C}_{\alpha}[\Omega_1(\zeta)]+b\mathscr{L}\mathscr{C}_{\alpha}[\Omega_2(\zeta)]. \quad (3.5)
$$

Proof. By using Definition in equation (3.1) for LHS in Equation (3.5) we get,

$$
\begin{aligned}
&\mathscr{L}\mathscr{C}_{\alpha}[a\Omega_1(\zeta)+b\Omega_2(\zeta)]\\
&=\eta^{\alpha}\int_{0}^{\infty}E_{\alpha}(-\eta^{\alpha}\zeta^{\alpha})[a\Omega_1(\zeta)+b\Omega_2(\zeta)](d\zeta)^{\alpha}\\
&=a\eta^{\alpha}\int_{0}^{\infty}E_{\alpha}(-\eta^{\alpha}\zeta^{\alpha})\Omega_1(\zeta)(d\zeta)^{\alpha}\\
&+b\eta^{\alpha}\int_{0}^{\infty}E_{\alpha}(-\eta^{\alpha}\zeta^{\alpha})\Omega_2(\zeta))(d\zeta)^{\alpha}\\
&=a\mathscr{L}\mathscr{C}_{\alpha}[\Omega_1(\zeta)]+b\mathscr{L}\mathscr{C}_{\alpha}[\Omega_2(\zeta)].
\end{aligned}
$$

This is the complete proof. □

Theorem 3.5. *If $\Omega(\zeta)\in\Delta$ and D_{η}^{α} is the derivative of a function with respect to η of order α then*

$$
\mathscr{L}\mathscr{C}_{\alpha}[\zeta^{\alpha}\Omega(\zeta)]=D_{\eta}^{\alpha}\mathscr{L}\mathscr{C}_{\alpha}[\Omega(\zeta)]-\frac{\Gamma(\alpha+1)}{\eta^{\alpha}}\mathscr{L}\mathscr{C}_{\alpha}[\Omega(\zeta)]. \quad (3.6)
$$

Proof. By using Definition of fractional order Laplace-Carson transform in (3.1) then,

$$
\begin{aligned}
D_{\eta}^{\alpha}\mathscr{L}\mathscr{C}_{\alpha}[\Omega(\zeta)]&=D_{\eta}^{\alpha}\Omega_{\alpha}'(\eta)\\
&=D_{\eta}^{\alpha}\{\eta^{\alpha}\int_{0}^{\infty}E_{\alpha}(-\eta^{\alpha}\zeta^{\alpha})[\Omega(\zeta)](d\zeta)^{\alpha}\}
\end{aligned}
$$

But

$$
\begin{aligned}
D_{\eta}^{\alpha}\{\eta^{\alpha}E_{\alpha}(-\eta^{\alpha}\zeta^{\alpha})\}&=\Gamma(\alpha+1)E_{\alpha}(-\eta^{\alpha}\zeta^{\alpha})\\
&+\eta^{\alpha}\zeta^{\alpha}E_{\alpha}(-\eta^{\alpha}\zeta^{\alpha}).
\end{aligned}
$$

Taking integration, we get

$$
\begin{aligned}
&\int_{0}^{\infty}D_{\eta}^{\alpha}\{\eta^{\alpha}E_{\alpha}(-\eta^{\alpha}\zeta^{\alpha})\}[\Omega(\zeta)](d\zeta)^{\alpha}\\
&=\int_{0}^{\infty}\{\Gamma(\alpha+1)E_{\alpha}(-\eta^{\alpha}\zeta^{\alpha})\\
&+\eta^{\alpha}\zeta^{\alpha}E_{\alpha}(-\eta^{\alpha}\zeta^{\alpha})\}[\Omega(\zeta)](d\zeta)^{\alpha}\\
&=\int_{0}^{\infty}\Gamma(\alpha+1)E_{\alpha}(-\eta^{\alpha}\zeta^{\alpha})[\Omega(\zeta)](d\zeta)^{\alpha}\\
&+\int_{0}^{\infty}\eta^{\alpha}\zeta^{\alpha}E_{\alpha}(-\eta^{\alpha}\zeta^{\alpha})[\Omega(\zeta)](d\zeta)^{\alpha}\\
&=\Gamma(\alpha+1)\int_{0}^{\infty}E_{\alpha}(-\eta^{\alpha}\zeta^{\alpha})[\Omega(\zeta)](d\zeta)^{\alpha}\\
&+\eta^{\alpha}\int_{0}^{\infty}E_{\alpha}(-\eta^{\alpha}\zeta^{\alpha})[\zeta^{\alpha}\Omega(\zeta)](d\zeta)^{\alpha}.
\end{aligned}
$$

Now,

$$
\begin{aligned}
&D_{\eta}^{\alpha}\{\eta^{\alpha}\int_{0}^{\infty}E_{\alpha}(-\eta^{\alpha}\zeta^{\alpha})[\Omega(\zeta)](d\zeta)^{\alpha}\}\\
&=\frac{\Gamma(\alpha+1)}{\eta^{\alpha}}\mathscr{L}\mathscr{C}_{\alpha}[\Omega(\zeta)]+\mathscr{L}\mathscr{C}_{\alpha}[\zeta^{\alpha}\Omega(\zeta)]
\end{aligned}
$$

$$\mathscr{LC}_\alpha[\zeta^\alpha \Omega(\zeta)] = D_\eta^\alpha\{\eta^\alpha \int_0^\infty E_\alpha(-\eta^\alpha\zeta^\alpha)[\Omega(\zeta)](d\zeta)^\alpha\} - \frac{\Gamma(\alpha+1)}{\eta^\alpha}\mathscr{LC}_\alpha[\Omega(\zeta)]$$

$$\mathscr{LC}_\alpha[\zeta^\alpha \Omega(\zeta)] = D_\eta^\alpha \mathscr{LC}_\alpha[\Omega(\zeta)] - \frac{\Gamma(\alpha+1)}{\eta^\alpha}\mathscr{LC}_\alpha[\Omega(\zeta)].$$

This is the final proof of above equation. □

Theorem 3.6 (Change of scale property of fractional order Laplace Carson transform)**.** *Let $\Omega(a\zeta) \in \Delta$, where a be any constant then,*

$$\mathscr{LC}_\alpha[\Omega(a\zeta)]_\eta = (\frac{1}{a})^\alpha \mathscr{LC}_\alpha[\Omega(\zeta)]_{\eta/a}. \tag{3.7}$$

Proof. By using Definition of fractional Laplace-Carson transform,

$$\mathscr{LC}_\alpha[\Omega(a\zeta)]_\eta = \eta^\alpha \int_0^\infty E_\alpha(-\eta^\alpha\zeta^\alpha)[\Omega(a\zeta)](d\zeta)^\alpha.$$

Put, $a\zeta = \lambda$, then $\zeta = \frac{\lambda}{a}$ then,

$$\begin{aligned}\mathscr{LC}_\alpha[\Omega(a\zeta)]_\eta &= \eta^\alpha \int_0^\infty E_\alpha(-\eta^\alpha(\frac{\lambda}{a})^\alpha)[\Omega(\lambda)]\frac{(d\lambda)^\alpha}{a^\alpha} \\ &= (\frac{1}{a})^\alpha \eta^\alpha \int_0^\infty E_\alpha(\frac{(-\eta\lambda)^\alpha}{a^\alpha})[\Omega(\lambda)](d\lambda)^\alpha \\ &= (\frac{1}{a})^\alpha \eta^\alpha \int_0^\infty E_\alpha(\frac{(-\eta\zeta)^\alpha}{a^\alpha})[\Omega(\zeta)](d\zeta)^\alpha \\ &= (\frac{1}{a})^\alpha \mathscr{LC}_\alpha[\Omega(\zeta)]_{\eta/a}.\end{aligned}$$

□

Theorem 3.7 (Shifting property)**.** *Let $\Omega(\zeta) \in \Delta$ then for $\Omega(\zeta - b) \in \Delta$ where b is constant, following holds,*

$$\mathscr{LC}_\alpha[\Omega(\zeta - b)] = E_\alpha(-\eta^\alpha b^\alpha)\mathscr{LC}_\alpha[\Omega(\zeta)]. \tag{3.8}$$

Proof. By using Definition of fractional Laplace-Carson transform,
$L.H.S. = \mathscr{LC}_\alpha[\Omega(\zeta - b)]$
$= \eta^\alpha \int_0^\infty E_\alpha(-\eta^\alpha\zeta^\alpha)\Omega(\zeta - b)(d\zeta)^\alpha$.
Put $\zeta - b = \lambda$ then $\zeta = \lambda + b$ then,

$$\begin{aligned}&\mathscr{LC}_\alpha[\Omega(\zeta - b)] = \eta^\alpha \int_0^\infty E_\alpha(-\eta^\alpha(\lambda+b)^\alpha)\Omega(\lambda)(d\lambda)^\alpha \\ &= \eta^\alpha \int_0^\infty E_\alpha(-\eta^\alpha\lambda^\alpha)E_\alpha(-\eta^\alpha b^\alpha)\Omega(\lambda)(d\lambda)^\alpha \\ &= \eta^\alpha E_\alpha(-\eta^\alpha b^\alpha)\int_0^\infty E_\alpha(-\eta^\alpha\lambda^\alpha)\Omega(\lambda)(d\lambda)^\alpha \\ &= E_\alpha(-\eta^\alpha b^\alpha)\eta^\alpha \int_0^\infty E_\alpha(-\eta^\alpha\zeta^\alpha)\Omega(\zeta)(d\zeta)^\alpha \\ &= E_\alpha(-\eta^\alpha b^\alpha)\mathscr{LC}_\alpha[\Omega(\zeta)] = R.H.S.\end{aligned}$$

□

Theorem 3.8. *Let $E_\alpha(\zeta)$ be the Mittag-Leffler function and $\Omega(\zeta) \in \Delta$ then,*

$$\mathscr{LC}_\alpha[E_\alpha(-c^\alpha\zeta^\alpha)\Omega(\zeta)]_\eta = \mathscr{LC}_\alpha[\Omega(\zeta)]_{\eta+c}. \tag{3.9}$$

Proof. By using Definition of fractional Laplace-Carson transform,

$$\begin{aligned}L.H.S. &= \mathscr{LC}_\alpha[E_\alpha(-c^\alpha\zeta^\alpha)\Omega(\zeta)] \\ &= \eta^\alpha \int_0^\infty E_\alpha(-\eta^\alpha\zeta^\alpha)E_\alpha(-c^\alpha\zeta^\alpha)\Omega(\zeta)](d\zeta)^\alpha \\ &= \eta^\alpha \int_0^\infty E_\alpha(-[\eta^\alpha\zeta^\alpha + c^\alpha\zeta^\alpha]\Omega(\zeta)](d\zeta)^\alpha \\ &= \eta^\alpha \int_0^\infty E_\alpha(-[(\eta+c)^\alpha\zeta^\alpha]\Omega(\zeta)](d\zeta)^\alpha \\ &= \mathscr{LC}_\alpha[\Omega(\zeta)]_{\eta+c} = R.H.S\end{aligned}$$

□

Theorem 3.9. *Let $\Omega(\zeta) \in \Delta$ then,*

$$\mathscr{LC}_\alpha[D_\zeta^\alpha \Omega(\zeta)] = \eta^{2\alpha}\mathscr{LC}_\alpha[\Omega(\zeta)] - \eta^\alpha\Gamma(1+\alpha)\Omega(0). \tag{3.10}$$

Proof. By using Definition of fractional Laplace-Carson transform,

$$\mathscr{LC}_\alpha[D_\zeta^\alpha \Omega(\zeta)] = \eta^\alpha \int_0^\infty E_\alpha(-\eta^\alpha\zeta^\alpha)D_\zeta^\alpha\Omega(\zeta)(d\zeta)^\alpha.$$

By using the definition of fractional integration by part formula we get,

$$\begin{aligned}&L.H.S. = \mathscr{LC}_\alpha[D_\zeta^\alpha \Omega(\zeta)] \\ &= \eta^\alpha\Gamma(1+\alpha)\Omega(\zeta)E_\alpha(-\eta^\alpha\zeta^\alpha)|_0^\infty \\ &- \int_0^\infty D_\zeta^\alpha E_\alpha(-\eta^\alpha\zeta^\alpha)\Omega(\zeta)(d\zeta^\alpha \\ &= -\eta^\alpha\Gamma(1+\alpha)\Omega(0) - (-\eta^{2\alpha})\int_0^\infty E_\alpha(-\eta^\alpha\zeta^\alpha)\Omega(\zeta)(d\zeta)^\alpha \\ &= -\eta^\alpha\Gamma(1+\alpha)\Omega(0) + (\eta^{2\alpha})\int_0^\infty E_\alpha(-\eta^\alpha\zeta^\alpha)\Omega(\zeta)(d\zeta)^\alpha \\ &= -\eta^\alpha\Gamma(1+\alpha)\Omega(0) + (\eta^{2\alpha})\mathscr{LC}_\alpha[\Omega(\zeta)] \\ &= \eta^{2\alpha}\mathscr{LC}_\alpha[\Omega(\zeta)] - \Gamma(1+\alpha)\Omega(0) = R.H.S.\end{aligned}$$

□

Theorem 3.10. *Let $\Omega(\zeta) \in \Delta$ then,*

$$\mathscr{LC}_\alpha\left[\int_0^\zeta \Omega(\lambda)d\lambda^\alpha\right] = \Gamma(1+\alpha)^{-1}\mathscr{LC}_\alpha[\Omega(\zeta)]. \tag{3.11}$$

Proof. By using the definition of fractional order Laplace Transform integral transform [3],

$$\mathscr{L}_\alpha\left[\int_0^\zeta \Omega(\theta)(d\theta)^\alpha\right] = \Gamma(1+\alpha)^{-1}(\eta)^{-\alpha}\mathscr{L}_\alpha[\Omega(\theta)].$$

Using the duality of Laplace and L-C transform [13],

$$\begin{aligned}\mathscr{LC}_\alpha\left[\int_0^\zeta \Omega(\theta)(d\theta)^\alpha\right] &= \eta^\alpha\Gamma(1+\alpha)^{-1}(\eta)^{-\alpha}\mathscr{LC}_\alpha[\Omega(\theta)] \\ &= \Gamma(1+\alpha)^{-1}\mathscr{LC}_\alpha[\Omega(\theta)].\end{aligned}$$

□

4. Convolution theorem of fractional order Laplace-Carson transform

Theorem 4.1. *If the convolution of order α of two functions $\Omega_1(\zeta)$ and $\Omega_2(\zeta)$ is define by the integral of the form,*

$$(\Omega_1(\zeta)*\Omega_2(\zeta))_\alpha = \int_0^\zeta \Omega_1(\zeta-v)\Omega_2(v)(dv)^\alpha.$$

Then we can write,

$$\mathscr{LC}_\alpha[(\Omega_1(\zeta)*\Omega_2(\zeta))_\alpha] = \frac{1}{\eta^\alpha}\mathscr{LC}_\alpha[\Omega_1(\zeta)]\mathscr{LC}_\alpha[\Omega_2(\zeta)].$$

Proof. We starts from Definition,

$$\begin{aligned}
&\mathscr{LC}_\alpha[(\Omega_1(\zeta)*\Omega_2(\zeta))_\alpha]\\
&= \eta^\alpha\int_0^\infty E_\alpha(-\eta^\alpha\zeta^\alpha)\int_0^\zeta \Omega_1(\zeta-v)\Omega_2(v)(dv)^\alpha(d\zeta)^\alpha\\
&= \eta^\alpha\int_0^\infty E_\alpha(-\eta^\alpha(\zeta-v)^\alpha)E_\alpha(-\eta^\alpha(v)^\alpha)\\
&\quad\int_0^\zeta \Omega_1(\zeta-v)\Omega_2(v)(dv)^\alpha(d\zeta)^\alpha.
\end{aligned}$$

By changing variable $\zeta - v \to \lambda$, and $V \to \xi$ taking limits from zero to infinite we get,

$$\begin{aligned}
&\mathscr{LC}_\alpha[(\Omega_1(\zeta)*\Omega_2(\zeta))_\alpha]\\
&= \eta^\alpha\int_0^\infty E_\alpha(-\eta^\alpha\lambda^\alpha)\Omega_1(\lambda)(d\lambda)^\alpha\\
&\quad\int_0^\infty E_\alpha(-\eta^\alpha\xi^\alpha)\Omega_2(\xi)(d\xi)^\alpha\\
&= [\eta^\alpha\int_0^\infty E_\alpha(-\eta^\alpha\lambda^\alpha)\Omega_1(\lambda)(d\lambda)^\alpha]\frac{1}{\eta^\alpha}\\
&\quad[\eta^\alpha\int_0^\infty E_\alpha(-\eta^\alpha\xi^\alpha)\Omega_2(\xi)(d\xi)^\alpha]\\
&= [\eta^\alpha\int_0^\infty E_\alpha(-\eta^\alpha\zeta^\alpha)\Omega_1(\zeta)(d\zeta)^\alpha]\frac{1}{\eta^\alpha}\\
&\quad[\eta^\alpha\int_0^\infty E_\alpha(-\eta^\alpha\zeta^\alpha)\Omega_2(\zeta)(d\zeta)^\alpha]\\
&= \frac{1}{\eta^\alpha}\mathscr{LC}_\alpha[\Omega_1(\zeta)]\mathscr{LC}_\alpha[\Omega_2(\zeta)].
\end{aligned}$$

□

4.1 Inversion formula for fractional Laplace-Carson Transform

Definition 4.2 ([9,12])**.** *The Dirac's distribution also known as generalized function, $\delta_\alpha(z)$ of order α, where $\alpha \in (0,1)$, is define as,*

$$\int_{\mathscr{R}} \Omega(\zeta)\delta_\alpha(\zeta-a)d\zeta^\alpha = \alpha\Omega(a). \tag{4.1}$$

In particular, $\int_{\mathscr{R}}\Omega(\zeta)\delta_\alpha(\zeta)d\zeta^\alpha = \alpha\Omega(0)$.

Lemma 4.3 ([9,12])**.** *The equality*

$$\frac{\alpha}{(\vartheta_\alpha)^\alpha}\int_{-\infty}^{+\infty} E_\alpha(i(-\rho\zeta)^\alpha)(d\rho)^\alpha = \delta_\alpha(\zeta), \tag{4.2}$$

holds where ϑ_α is the period of complexed value Mittag-Leffler function defined by the equality, $E_\alpha(i(\vartheta_\alpha)^\alpha) = 1$.

Proof. From equation (3.9) we can write,

$$\alpha = \int_{\mathscr{R}} E_\alpha(i(\rho\zeta)^\alpha)\delta_\alpha(\zeta)d\zeta^\alpha.$$

By using the value $\delta_\alpha(\zeta)$ to LHS of the above equation we get,

$$\begin{aligned}
&= \int_{\mathscr{R}}(d\zeta)^\alpha E_\alpha(i(\rho\zeta)^\alpha)\frac{\alpha}{(\vartheta_\alpha)^\alpha}\int_{-\infty}^{+\infty}E_\alpha(i(-u\zeta)^\alpha)(du)^\alpha\\
&= \int_{\mathscr{R}}(d\zeta)^\alpha\frac{\alpha}{(\vartheta_\alpha)^\alpha}\int_{\mathscr{R}}E_\alpha(i(\rho\zeta)^\alpha)E_\alpha(i(-u\zeta)^\alpha)(du)^\alpha\\
&= \int_{\mathscr{R}}\int_{\mathscr{R}}(d\zeta)^\alpha\frac{\alpha}{(\vartheta_\alpha)^\alpha}E_\alpha(i((\rho-u)\zeta)^\alpha)(du)^\alpha\\
&= \int_{\mathscr{R}}\int_{\mathscr{R}}\frac{\alpha}{(\vartheta_\alpha)^\alpha}E_\alpha(i(-\varphi\zeta)^\alpha)(d\varphi)^\alpha(d\zeta)^\alpha\\
&= \int_{\mathscr{R}}\delta_\alpha(\zeta)(d\zeta)^\alpha.
\end{aligned}$$

□

4.2 Inversion Theorem of Fractional Order Laplace-Carson transform:

Lemma 4.4. *Fractional order Laplace-Carson transform define in Definition 4,*

$$\mathscr{LC}_\alpha[\Omega(\zeta)] = \Omega'_\alpha(\eta) = \eta^\alpha\int_0^\infty E_\alpha(-\eta^\alpha\zeta^\alpha)\Omega(\zeta)(d\zeta)^\alpha, \tag{4.3}$$

then its inversion formula is,

$$\Omega(\zeta) = \frac{1}{(\vartheta_\alpha)^\alpha}\int_{+i\infty}^{-i\infty}\eta^{-\alpha}E_\alpha(\eta^\alpha\zeta^\alpha)\Omega'_\alpha(\eta)(d\eta)^\alpha. \tag{4.4}$$

Proof. By substituting equation (3.1) in (4.4) and using (4.1) in (4.2) we get,

$$\begin{aligned}
\Omega(\zeta) &= \frac{1}{(\vartheta_\alpha)^\alpha}\int_{+i\infty}^{-i\infty}\eta^{-\alpha}E_\alpha(\eta^\alpha\zeta^\alpha)(d\eta)^\alpha\eta^\alpha\\
&\quad\int_0^\infty E_\alpha(-\eta^\alpha\lambda^\alpha)\Omega(\lambda)(d\lambda)^\alpha\\
&= \frac{1}{(\vartheta_\alpha)^\alpha}\int_0^\infty\Omega(\lambda)(d\lambda)^\alpha\int_{+i\infty}^{-i\infty}E_\alpha(-\eta^\alpha\lambda^\alpha)\\
&\quad E_\alpha(\eta^\alpha\zeta^\alpha)(d\eta)^\alpha\\
&= \frac{1}{(\vartheta_\alpha)^\alpha}\int_0^\infty\Omega(\lambda)(d\lambda)^\alpha\int_{+i\infty}^{-i\infty}E_\alpha(-\eta^\alpha(\lambda-\zeta)^\alpha)(d\eta)^\alpha\\
&= \frac{1}{(\vartheta_\alpha)^\alpha}\int_0^\infty\frac{(\vartheta_\alpha)^\alpha}{\alpha}\Omega(\lambda)\delta_\alpha(\lambda-\zeta)(d\lambda)^\alpha\\
&= \frac{1}{\alpha}\int_{\mathscr{R}}\Omega(\lambda)\delta_\alpha(\zeta-\lambda)(d\lambda)^\alpha\\
&= \frac{1}{\alpha}\alpha\Omega(\zeta)\\
&= \Omega(\zeta).
\end{aligned}$$

□

5. Conclusion

From the above study we have developed fractional order Laplace-Carson transform. Also, we establish properties of fractional order Laplace-Carson transform. Further, some main results like convolution and inversion theorem.

References

[1] Hassan Eltayeb, Adem Kilicman, and Mohemed B. Jleli, Fractional integral transform and its applications, *Hindawi publication Corporation*, (2014).

[2] Xiao-Jun Yang, Dhumtru Beleanu, Shrivastava, H. M., Local fractional integral transform and their applications, *Ebook,* 1(2015), 0–262.

[3] Donolato, C., Analytical and numerical inversion of the Laplace-Carson transform by a differential Method, *Computer physics Communications*, 145(2002), 298–309.

[4] Toshikazu Kimura,, Applications the Laplace-Carson transform to Option Pricing: A Tutorial, *Chinees Publication*, (2016), 68–89.

[5] Makarov, A. M., Applications of the Laplace- Carson method of integral transform to the theory of unsteady visco-plastic flows, *J. Engtg. Physics Theosophy's,* 19(1970), 94–99.

[6] Tukaram G. Thange, Anantha R. Gade, Analytical solution for time fractional diffusion equation by Laplace-Carson Decomposition method, *Our heritage*, 68(6)(2020), 45–50.

[7] Guy Jumarie, (2019), Laplace transform of fractional order via the Mittag-Leffler function and modified Riemann-Liouville derivative, *Applied mathematics letters,* 22, 1659–1664.

[8] Tukaram G. Thange, Anantha R. Gade, on the Aboodh transform of fractional differential operator, *Malaya Journal of mathematic*, 8(01)(2020), 225–229.

[9] Lakshmikantam, V. and Vatsala, A. S., Basic theory of fractional differential equation, *Nonlinear Analysis*, 69(2008).

[10] Joseph M. Kimeu, *Fractional calculus definitions and application*, top scholar master's thesis and specialist project.

[11] Raman Chauhan, Kumar, N. and Aggarwal, S., Dualities between Laplace-Carson transform and Some useful integral transform, *International journal of innovative technology and exploring engineering,* 8(12)(2019), 2278–3075.

[12] Elif Demirci, Nuri O Zalp, A methods for solving differential equation of fractional order, *Journal of computational and applied mathematics*, 236(2012).

[13] Alfaqeih, S. and Ozis, T. Note on double Aboodh transform of fractional order and its properties, *online mathematics*, 01(2019), 2672–7501.

[14] Thange T. G. and Gade A. R., A new approach for solving a system of fractional partial differential Equation, *Advances in Mathematics:Sci Journal,* 09(2020), 1933–1944.

⋆⋆⋆⋆⋆⋆⋆⋆⋆
ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
⋆⋆⋆⋆⋆⋆⋆⋆⋆