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Some properties for subclass of analytic functions with nonzero coefficients

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Abstract

In the given article, we introduced new subclass of normalized analytic function namely $R^t(1,\ A,\ B,\alpha)$. Coefficient inequality, necessary and sufficient condition for the functions in this class are given. The inclusion property and condition for univalancy with linear n^{th} derivative operator is also pointed out.

Keywords

Univalent, Analytic, Starlike.

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1. Introduction

2. Introduction

Let N denotes subclass of normalized analytical function in open unit disc $U = \{z : |z| < 1\}$ given by

$$\psi(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{2.1}$$

For the function $f \in N$, [1] has introduced the class namely $R^1(\ A,\ B,\alpha)$ as

$$\left|\frac{\psi'(z)-1}{l(A-B)-B(\psi'(z)-1)}\right| < \infty, \quad (z \in U). \tag{2.2}$$

Where A and B are complex numbers with $A \neq B, t \in \mathbb{C}\backslash\{0\}$ and $\alpha \in \mathbb{R}^+$. Dixit and Pal [5] studied the class $R^1(\ A,\ B,1)$. Moreover, the researchers in $([6,7,8])$ studied the class $R^1(\ A,\ B,\alpha)$ for different values of t, A, B and α The Gaussian hypergeometric function $F(a,b,c,z)$ is given by

$$F(a,b,c,z) = \sum_{k=2}^{\infty} \frac{(a)_n (b)_n}{(c)_n} z^k$$

$(a,b,c \in \mathbb{C}; c \neq 0,-1,-2\ldots\ldots\ldots; \quad z \in U)$, where $(v)_n$ is defined in terms of gamma function as given bellow

$$(v)_n = \frac{\tau(v+n)}{\tau(v)} = \begin{cases} 1 & ,\text{if } n=0 \text{ and } v \in \mathbb{C}\backslash\{0\} \\ v(v+1)\ldots\ldots(v+n-1) & ,\text{if } n \in \mathbb{N} \text{ and } v \in \mathbb{C}. \end{cases}$$

We note that

$$F(a,b,c,1) = \frac{\tau(c-a-b)\tau(c)}{\tau(c-a)\tau(c-b)} \quad (\Re\{c-a-b\} > 0).$$

The geometric properties of $zF(a,b,c,z)$ like univalence, star likeness and convexity have been studied by Vuorinen and Ponnusamy [3] and Ruscheweyh and singh [4]. For $\psi \in N$, [1] has defined the operator $I_{a,b,c}\psi$ by

$$I_{a,b,c}\psi(z) = zF(a,b,c,z) * \psi(z) \tag{2.3}$$

$$I_{a,b,c}\psi(z) = z + \sum_{k=2}^{\infty} \frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}} a_k z^k \tag{2.4}$$

Here * denotes the usual Hamdard product.

3. Linear n^{th} differential operator and class $R^{1}(t,A,B\alpha)$

[1] has used the linear operator as defined in (3). We introduced Linear n^{th} differential operator which is obtained by taking n^{th} derivative of (3) as given bellow.

Definition 3.1. *For $\psi \in N$, we define the operator $\left(I_{a,b,c}\psi(z)\right)^{t}$ by*

$$\left(I_{a,b,c}\psi(z)\right)^{t} = t!L_t + \sum_{k=2}^{\infty} k(k-1)\ldots\ldots(k-t+1)L_k z^{k-t}, \tag{3.1}$$

where

$$L_t = \frac{(a)_{t-1}(b)_{t-1}}{(c)_{t-1}} a_t \quad a_1 = 1, t \geq 1.$$

Definition 3.2. *A function $\psi \in N$ is said to be in class $R^{1}(t,A,B\alpha)$ if*

$$\left|\frac{\psi^{t}(z) - t!a_t}{l(A-B) - B\left(\psi^{t}(z) - t!a_t\right)}\right| < \infty, \quad (z \in U, a_1 = 1, t \geq 1), \tag{3.2}$$

where A and B are complex numbers with $A \neq B, t \in \mathbb{C}\backslash\{0\}$ and $\alpha \in \mathbb{R}^{+}$ For $t = 1$, we get the subclass defined by [1].

4. Main results

In this theorem, we found necessary condition for the class $R^{1}(t,A,B\alpha)$.

Theorem 4.1. *Let function of the form (1) be in class $R^{1}(t,A,B\alpha)$ with*

$$a_n = |a_n| e^{i\frac{i(3n+t)}{2}\pi},$$

then

$$\sum_{k=t+1}^{\infty} k(k-1)\ldots\ldots(k-t+1)(1-\alpha|B|)\,|a_k| \leq \alpha|t||A-B|. \tag{4.1}$$

Proof. Suppose $f \in R^{1}(t,A,B\alpha)$

$$\therefore \left|\psi^{t}(z) - t!a_t\right| < \alpha\left|l(A-B) - B\left(\psi^{t}(z) - t!a_t\right)\right|$$

$$\psi^{t}(z) = a_t t! + \sum_{k=t+1}^{\infty} k(k-1)\ldots\ldots(k-t+1)a_k z^{k-t} \tag{4.2}$$

$$\left|\sum_{k=t+1}^{\infty} k(k-1)\ldots\ldots(k-t+1)a_k z^{k-t}\right|$$
$$< \alpha\left|l(A-B) - B\left(\psi^{t}(z) - t!a_t\right)\right|$$
$$= \alpha \mid l(A-B) - B\sum_{k=t+1}^{\infty} k(k-1)\ldots\ldots(k-t+1)a_k z^{k-t}$$

Put $z = r.e^{i\frac{\pi}{2}}$.

$$\begin{aligned} a_k z^{k-t} &= |a_k|\, e^{i\frac{(3k+t)}{2}\pi} r^{k-t} e^{i\frac{(k-t)}{2}\pi} \\ &= |a_k|\, r^{k-t} e^{2k\pi i} \\ &= |a_k|\, r^{k-t}. \end{aligned}$$

$$\left|\sum_{k=t+1}^{\infty} k(k-1)\ldots\ldots(k-t+1)a_k z^{k-t}\right|$$
$$< \alpha \mid l(A-B) - B\sum_{k=t+1}^{\infty} k(k-1)\ldots\ldots(k-t+1)a_k z^{k-t} \mid$$
$$\sum_{k=t+1}^{\infty} k(k-1)\ldots\ldots(k-t+1)\,|a_k|\, r^{k-t}$$
$$< \alpha|l(A-B)| + \alpha|B|\sum_{k=t+1}^{\infty} k(k-1)\ldots\ldots(k-t+1)\,|a_k|\, r^{k-t}.$$

Now,

$$\sum_{k=t+1}^{\infty} k(k-1)\ldots\ldots(k-t+1)(1-\alpha|B|)\,|a_k|\, r^{k-t} < \alpha|l||A-B|.$$

Letting limit $r \to 1^{-}$, we have

$$\sum_{k=t+1}^{\infty} k(k-1)\ldots\ldots(k-t+1)(1-\alpha|B|)\,|a_k| < \alpha|l||A-B|.$$

Hence proved the theorem. □

The following result gives us the coefficient estimates for the functions in the class $R^{1}(t,A,B\alpha)$.

Corollary 4.2. *The function of the form (1) be in class $R^{1}(t,A,B\alpha)$ with*

$$a_n = |a_n|\, e^{i\frac{(3n+t)}{2}\pi}.$$

Then

$$|a_k| \leq \frac{\alpha|||A-B\,|}{k(k-1)\ldots\ldots(k-t+1)(1-\alpha|B|)} \quad k \geq 2. \tag{4.3}$$

Proof. For the function $\psi \in R^{l}(t,A,B,\alpha)$,

$$\psi(z) = z + \sum_{k=2}^{\infty} a_k.$$

□

Theorem 1 gives us that,

$$\sum_{k=t+1}^{\infty} k(k-1)\ldots\ldots(k-t+1)(1-\alpha|B|)\,|a_k| < \alpha|l||A-B|.$$

Hence,

$$|a_k| \leq \frac{\alpha|l|A-B\,|}{k(k-1)\ldots\ldots(k-t+1)(1-\alpha|B|)}, \quad k \geq 2.$$

In the next theorem we found sufficient condition for $R^{1}(t,A,B\alpha)$.

Theorem 4.3. *Let a function of the form* (1) *be in class N. If*

$$\sum_{k=t+1}^{\infty} k(k-1)\ldots\ldots(k-t+1)(1+\alpha|B|)\,|a_k| \le \alpha|l||A-B|. \tag{4.4}$$

Then, $\psi \in R^1(t,A,B\alpha)$.

Proof. Given that

$$\sum_{k=t+1}^{\infty} k(k-1)\ldots\ldots(k-t+1)(1+\alpha|B|)\,|a_k| \le \alpha|l||A-B|$$

$$r<1 \Rightarrow r^{k-t}<1.$$

$$\sum_{k=t+1}^{\infty} k(k-1)\ldots\ldots(k-t+1)(1+\alpha|B|)\,|a_k|\,r^{k-t}$$

$$\le \alpha|l||A-B|\sum_{k=t+1}^{\infty} k(k-1)\ldots\ldots(k-t+1)\,|a_k|\,r^{k-t}$$

$$\le \alpha|l||A-B|-|B|\sum_{k=t+1}^{\infty} k(k-1)\ldots\ldots(k-t+1)\,|a_k|\,r^{k-t}.$$

Hence,

$$\therefore \left|\psi^t(z)-t!a_t\right| \le \alpha\left|l(A-B)-B\left(\psi^t(z)-t!a_t\right)\right|$$

$$\therefore \frac{|\psi^t(z)-t!a_t|}{|l(A-B)-B(\psi'(z)-1)|}<\alpha$$

Therefore, $\psi \in R^1(t,A,B\alpha)$. Now with operator $\left(I_{a,b,c}\psi(z)\right)^t$ we obtained inclusion relationship for the class $R^1(t,A,B,\alpha)$. □

Theorem 4.4. *Let a,* $b \in \mathbb{C}\backslash\{0\}$ *and* $c>|a|+|b|$. *If* $\psi \in R^1(t,A,B,\alpha)$ *with*

$$a_n=|a_n|\,e^{i\frac{i3n+t)}{2}\pi},\ 0<|B|<1$$

$$H_t=\sum_{k=0}^{t-1}\frac{(|a|)_k(|b|)_k}{(|c|)_k}$$

and

$$\frac{\tau(c-|a|-|b|)\tau(c)}{\tau(c-|a|)\tau(c-|b|)} \le \frac{1-\alpha|B|}{1+\alpha|B|}+H_t.$$

Then, $I_{a,b,c}\psi\left(R^1(t,A,B,\alpha)\right)\subseteq R^1(t,A,B,\alpha)$, *where the operator* $I_{a,b,c}\psi$ *is defined in (3).*

Proof. Given that $\psi \in R^1(t,A,B,\alpha)$. From theorem 3, we have

$$\sum_{k=t+1}^{\infty} k(k-1)\ldots\ldots(k-t+1)(1+\alpha|B|)\,|L_k| \le \alpha|l||A-B|,$$

where,

$$L_k=\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}}a_k.$$

From corollary 2, we have

$$|a_k| \le \frac{\alpha|l||A-B\,|}{k(k-1)\ldots\ldots(k-t+1)(1-\alpha|B|)},\quad k\ge 2.$$

$$\sum_{k=t+1}^{\infty} k(k-1)\ldots\ldots(k-t+1)(1+\alpha|B|)\left|\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}}\right||a_k|$$

$$=\sum_{k=t+1}^{\infty} k(k-1)\ldots\ldots(k-t+1)(1+\alpha|B|)$$

$$\left|\frac{(a)_{k-1}(b)_{k-1}}{(c)_{k-1}}\right|\,|\,\frac{\alpha|l||A-B|}{k(k-1)\ldots\ldots(k-t+1)(1-\alpha|B|)}$$

$$=\frac{\alpha|l||A-B\,|\,(1+\alpha|B|)}{(1-\alpha|B|)}\sum_{k=t+1}^{\infty}\left|\frac{(|a|)_{k-1}(|b|)_{k-1}}{(|c|)_{k-1}}\right|$$

$$=\frac{\alpha|l||A-B\,|\,(1+\alpha|B|)}{(1-\alpha|B|)}\sum_{k=0}^{\infty}\frac{(|a|)_{k-1}(|b|)_{k-1}}{(|c|)_{k-1}}-H_t$$

$$=\frac{\alpha|l||A-B\,|\,(1+\alpha|B|)}{(1-\alpha|B|)}\left(\frac{\tau(c-|a|-|b|)\tau(c)}{\tau(c-|a|)\tau(c-|b|)}-H_t\right)$$

$$\le\frac{\alpha|l||A-B\,|\,(1+\alpha|B|)}{(1-\alpha|B|)}\left(\frac{\tau(c-|a|-|b|)\tau(c)}{\tau(c-|a|)\tau(c-|b|)}-H_t\right)$$

$$=\frac{\alpha|l||A-B|(1+\alpha|B|)}{(1-\alpha|B|)}\frac{(1-\alpha|B|)}{(1+\alpha|B|)}$$

$$=\alpha|l||A-B|.$$

Therefore, $I_{a,b,c}\psi \in R^1(t,A,B,\alpha)$,

$$I_{a,b,c}\psi\left(R^1(t,A,B,\alpha)\right)\subseteq R^1(t,A,B,\alpha)$$

[10] has introduce lemma which is useful for the proof of next theorem. □

Lemma 4.5. *Let w be the regular in unit disc D with* $w(0)=0$ *Then* $|w(z)|$ *attains a maximum value on the circle* $|z|=r(0\le r<1)$ *at a point* Z_0 *we can write*

$$z_0w'(z_0)=kw(z_0)\quad (k\ge 1). \tag{4.5}$$

Theorem 4.6. *Let a function of the form* (1) *be in the class N. Assume*

$$\left|\left(\frac{\left(I_{a,b,c}\psi(z)\right)^t-t!L_t}{1-\alpha}\right)^{\beta}\right|\left|\frac{\left(zI_{a,b,c}\psi(z)\right)^{t+1}}{\left(I_{a,b,c}\psi(z)\right)^t-\alpha t!L_t}\right|^{\gamma} < \left(\frac{1}{1+t!\,|L_t|}\right)^{\gamma}. \tag{4.6}$$

For some real $\alpha(0\le\alpha<1)\quad \beta>0, r>0$. *Then,*

$$\left|\left(I_{a,b,c}\psi(z)\right)^t-t!L_t\right|<1-\alpha. \tag{4.7}$$

Where,

$$Lt=\frac{(a)_{t-1}(b)_{t-1}}{(c)_{t-1}},\ a_1=1, t\ge 1.$$

Proof. Let

$$lw(z)=\frac{\left(I_{a,b,c}\psi(z)\right)^t-t!L_t}{1-\alpha}$$

$$\frac{zw'(z)}{w(z)+t!Lt}=\frac{\frac{\left(I_{a,b,c}\psi(z)\right)^{t+1}}{1-\alpha}}{\frac{\left(I_{a,b,c}\psi(z)\right)^t-t!L_t}{1-\alpha}+t!L_t}$$

$$=\frac{\left(I_{a,b,c}\psi(z)\right)^{t+1}}{\left(I_{a,b,c}\psi(z)\right)^t-\alpha t!L_t}$$

$$\therefore |w(z)|^{\beta}\left|\frac{zw'(z)}{w(z)+t!Lt}\right|^{\gamma}=\left|\left(\frac{\left(I_{a,b,c}\psi(z)\right)^{t}-t!L_t}{1-\alpha}\right)^{\beta}\right|$$

$$\left|\frac{\left(zI_{a,b,c}\psi(z)\right)^{t+1}}{\left(I_{a,b,c}\psi(z)\right)^{t}-\alpha t!L_t}\right|^{\gamma}<\left(\frac{1}{1+t!\,|L_t|}\right)^{\gamma} \qquad (4.8)$$

Suppose there exist point $z_0 \in U$ such that

$$\text{Max}\,|w(z)|=|w(z_0)|=1, \quad |z|\le|z_0|.$$

$\therefore$ by lemma (10) we have

$$\left|\frac{z_0 w'(z_0)}{w(z_0)}\right|=k\ge 1.$$

$$\begin{aligned}|w(z_0)|^{\beta}\left|\frac{zw'(z_0)}{w(z_0)+t!Lt}\right|^{\gamma} &= \left|\frac{zw'(z_0)}{w(z_0)+t!Lt}\right|^{\gamma}\\ &= \left|\frac{z_0w'(z_0)}{w(z_0)+t!Lt}\right|^{\gamma}\left|\frac{w(z_0)}{w(z_0)+t!Lt}\right|^{\gamma}\\ &= k^{\gamma}\left|\frac{1}{w(z_0)+t!Lt}\right|^{\gamma}\\ &\ge \left|\frac{k}{1+t!|Lt|}\right|^{\gamma}\\ &\ge \left|\frac{1}{1+t!|Lt|}\right|^{\gamma}.\end{aligned}$$

Which contradicts (14). Hence , $|w(z)|=1$

$$\therefore \left|\frac{\left(I_{a,b,c}\psi(z)\right)^{t}-t!L_t}{1-\alpha}\right|<1.$$

This shows that

$$\left|\left(I_{a,b,c}\psi(z)\right)^{t}-t!L_t\right|<|1-\alpha|.$$

Hence, completed the proof of theorem 6. □

Corollary 4.7. *Let a function of the form* (1) *be in the class A satisfying inequality* (12)

$$\left(I_{a,b,c}\psi(z)\right)^{t-1}+(1-t!L_t)\,z$$

is univalent in D

Proof.

$$\left(\left(I_{a,b,c}\psi(z)\right)^{t-1}+(1-t!L_t)\,z\right)'=\left(I_{a,b,c}\psi(z)\right)^{t}-t!L_t+1.$$

From theorem (6), we have

$$\Re\left\{\left(\left(I_{a,b,c}\psi(z)\right)^{t-1}+(1-t!L_t)\,z\right)'\right\}>0, \quad (z\in U)$$

Therefore, by [2] Alexander-Noshiro-Warschawski theorem the

$$\left(\left(I_{a,b,c}\psi(z)\right)^{t-1}+(1-t!L_t)\,z\right)$$

is univalent in *D*. □

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