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Fixed points of generalized (φ, ψ))-Jaggi contractions in orbitally complete partially ordered metric spaces

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Abstract. In this paper, we introduce generalized (φ, ψ))-Jaggi contraction mappings and prove the existence of fixed points for such mappings in orbitally complete partially ordered metric spaces. We provide examples in support of our results. Our results generalize the results of Harjani, Lopez and Sadarangani [3].

AMS Subject Classifications: 47H10, 54H25.

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Contents

1. Introduction and Background

A number of generalizations of the Banach contraction theorem were obtained in various directions by different authors. Generalization of contraction conditions and proving the existence of fixed points is an interesting aspect. In 1977, Jaggi [4] introduced a new concept namely 'rational type contraction mappings' and proved the existence of fixed points of such mappings.

Theorem 1.1. *[4] Let f be a continuous selfmap defined on a complete metric space (X, d). Suppose that f satisfies the following condition: there exist $\alpha,\ \gamma \in [0, 1)$ with $\alpha + \gamma < 1$ such that*

$$d(fx, fy) \leq \alpha \frac{d(x, fx)d(y, fy)}{d(x, y)} + \gamma d(x, y) \ for\ all\ x, y \in X,\ x \neq y. \tag{1.1}$$

Then f has a fixed point in X.

Here we note that a mapping $f : X \to X, X$ a metric space that satisfies (1.1) is called a Jaggi contraction map on X.

Harjani, Lopez and Sadarangani [3] extended Theorem 1.1 to the context of partially ordered complete metric spaces.

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Theorem 1.2. *[3] Let $(X, \preceq)$ be a partially ordered set and suppose that there is a metric d on X such that (X, d) is a complete metric space. Let $f : X \to X$ be a non-decreasing mapping such that*

$$d(fx, fy) \leq \alpha \frac{d(x, fx)d(y, fy)}{d(x, y)} + \gamma d(x, y) \tag{1.2}$$

for all $x, y \in X$ with $x \succeq y$, $x \neq y$ where $0 \leq \alpha, \gamma < 1$ with $\alpha + \gamma < 1$.
Also, assume either

(i) *f is continuous; (or)*

(ii) *if a non-decreasing sequence $\{x_n\}$ in X is such that $x_n \to x$ as $n \to \infty$ then $x = sup\{x_n\}$.*

If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point.

In 2013, Samet, Vetro and Vetro[7] introduced a new type of contraction condition and proved fixed point theorems in complete metric spaces that generalize Banach contraction principle and Kannan fixed point results. For more works on the existence of fixed points in complete metric spaces, we refer [7].

Recently, Babu, Sailaja and Kidane[2] proved some new fixed point theorems in orbitally complete partially ordered metric spaces that generalize the fixed point theorems of Samet, Vetro and Vetro [7] and Ran and Reurings[6]. We denote
$\Psi_1 = \{\psi : [0, \infty) \to [0, \infty)/\psi$ is non-decreasing, continuous and $\psi(t) = 0 \Leftrightarrow t = 0\}$.

An element ψ in Ψ_1 is called an 'altering distance function', [5].

Theorem 1.3. (Babu, Sailaja and Kidane [2]) *Let $(X, \preceq)$ be a partially ordered set and d a metric on X. Suppose that $f : X \to X$ is a non-decreasing map and $x_0 \in X$ such that $x_0 \preceq fx_0$. Suppose that there exist a lower semi continuous function $\varphi : X \to [0, \infty)$ and $\psi \in \Psi$ such that the following condition holds.*
"For each $0 \leq a < b < \infty$, there exists $\gamma(a, b) \in [0, 1)$ such that
$a \leq \psi(d(x, y)) + \varphi(x) + \varphi(y) \leq b$ implies
$\psi(d(fx, fy)) + \varphi(fx) + \varphi(fy) \leq \gamma(a, b)M(x, y)$, where
$M(x, y) = \max\{\psi(d(x, y)) + \varphi(x) + \varphi(y), \psi(d(x, fx)) + \varphi(x) + \varphi(fx),$
$\psi(d(y, fy)) + \varphi(y) + \varphi(fy)\}$
for each $x, y \in \overline{O(x_0)}$ with $x \preceq y$."

Assume that X is f-orbitally complete. Then, the sequence $\{x_n\}$ defined by $x_{n+1} = fx_n,\ n = 0, 1, 2, ...,$ is Cauchy in X. Let $\lim_{n\to\infty} x_n = z,\ z \in X$.
Suppose that either

(i) *f is orbitally continuous at z; (or)*

(ii) *if $\{x_n\}$ is a non-decreasing sequence converging to $x \in X$, then $x_n \preceq x$, for all n.*

Then, z is a fixed point of f and $\varphi(z) = 0$.

Definition 1.4. *Let $(X, \preceq)$ be a partially ordered set. A map $f : X \to X$ is said to be non-decreasing if, for any $x, y \in X$ with $x \preceq y$ then $fx \preceq fy$.*

Definition 1.5. *Let X be a nonempty set and f be a selfmap of X. Let $x \in X$, we define the orbit of x w. r. t. f by $O(x) = \{f^n x/n = 0, 1, 2, ...\}$. Here $f^0 = I$, I is the identity map of X.*

Definition 1.6. *Let (X, d) be a metric space. Let $f : X \to X$ be a selfmap of X. A metric space X is said to be f-orbitally complete if every Cauchy sequence which is contained in $O(x)$ for all $x \in X$ converges to a point of X.*

Note: Every complete metric space is f-orbitally complete for any f; but every f-orbitally complete metric space need not be a complete metric space [9].

Definition 1.7. *A selfmap f of X is said to be $orbitally\ continuous$ at a point $z \in X$ with respect to $x\ inX$, if for any sequence $\{x_n\} \subset O(x)$ with $x_n \to z$ as $n \to \infty$ implies $fx_n \to fz$ as $n \to \infty$.*

Clearly, any continuous mapping of a metric space is orbitally continuous, but its converse need not be true [9].

We use the following lemma in our main result.

Lemma 1.8. *[1] Suppose that (X,d) is a metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_n,x_{n+1}) \to 0\ as\ n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence then there exist an $\epsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ such that $d(x_{m(k)},x_{n(k)}) \geq \epsilon,\ d(x_{m(k)-1},x_{n(k)}) < \epsilon$ and*
(i) $\lim\limits_{k\to\infty} d(x_{n(k)-1},x_{m(k)+1}) = \epsilon$, (ii) $\lim\limits_{k\to\infty} d(x_{n(k)},x_{m(k)}) = \epsilon$,
(iii) $\lim\limits_{k\to\infty} d(x_{n(k)-1},x_{m(k)}) = \epsilon$, (iv) $\lim\limits_{k\to\infty} d(x_{n(k)},x_{m(k)+1}) = \epsilon$,
(v) $\lim\limits_{k\to\infty} d(x_{m(k)},x_{n(k)+1}) = \epsilon$, and (vi) $\lim\limits_{k\to\infty} d(x_{n(k)+1},x_{m(k)+1}) = \epsilon$.

Motivated by Theorem 1.3, we define generalized (φ,ψ)-Jaggi contraction maps which contain rational expressions, in orbitally partially ordered metric spaces and prove the existence of fixed points.

In the following, Ψ_2 denotes the family of non-decreasing functions $\psi : [0,+\infty) \to [0,+\infty)$ such that ψ is continuous on $[0,\infty)$ and $\Sigma_{n=1}^{\infty}\psi^n(t) < +\infty$ for each t > 0, where ψ^n is the n^{th} iterate of ψ.

Remark 1.9. *Any function $\psi \in \Psi_2$ satisfies $\lim\limits_{n\to\infty} \psi^n(t) = 0$ and $\psi(t) < t$ for any $t > 0$.*

In the following, we observe that the classes of maps Ψ_1 and Ψ_2 are different.

Example 1.10. *We define $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\psi(t) = \lambda t$, where $\lambda \geq 1$.*
Then $\psi \in \Psi_1$ but $\psi \notin \Psi_2$.

Example 1.11. *We define $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ by $\psi(t) = \begin{cases} 0 & if\ t \leq 1 \\ \frac{t-1}{2} & if\ t > 1. \end{cases}$*
Then $\psi \in \Psi_2$ but $\psi \notin \Psi_1$.

We now introduce generalized (φ,ψ)-Jaggi contraction in partially ordered metric spaces.

Definition 1.12. *Let $(X,\preceq)$ be a partially ordered metric space and suppose that $f : X \to X$ be a mapping. If there exist two functions $\varphi : X \to [0,\infty)$ lower semi continuous, $\psi \in \Psi_2$ and a point $x_0 \in X$ such that*

$$d(fx,fy) + \varphi(fx) + \varphi(fy) \leq \psi(M(x,y)), \tag{1.3}$$

where $M(x,y) = \max\{d(x,y) + \varphi(x) + \varphi(y), \frac{(d(x,fx)+\varphi(x)+\varphi(fx))(d(y,fy)+\varphi(y)+\varphi(fy))}{d(x,y)+\varphi(x)+\varphi(y)}\}$
for all $x,y \in \overline{O(x_0)}$ with $x \preceq y$ and $x \neq y$,
then we say that f is a $generalized\ (\varphi,\psi) - Jaggi\ contraction$.

Remark 1.13. *If $\varphi = 0$ in the inequality (1.3), then we say that f is a generalized ψ-Jaggi contraction.*

Note: In the context of partially ordered metric spaces, if f satisfies (1.2) with $\alpha + \gamma < 1$ then f is a generalized (φ,ψ)-Jaggi contraction with $\varphi = 0$ and $\psi(t) = (\alpha+\gamma)t$, $t \geq 0$ so that every Jaggi contraction is a generalized (φ,ψ)-Jaggi contraction. But, the following example suggests that its converse need not be true.

Example 1.14. *Let $X = [0,1)$ with the usual metric. We define partial order $\preceq$ on X as follows:*
$\preceq := \{(x,y) \in X \times X : x = y\} \cup \{(x,y) \in X \times X / x \preceq y \Leftrightarrow x \leq y$, *where* $\leq$ *is the usual order*$\}$.
We define $f : X \to X$ *by* $fx = \begin{cases} 0 & if\ x = 0 \\ \frac{x+1}{2} & if\ x \in (0, \frac{2}{5}) \\ \frac{3}{4} & if\ x \in [\frac{2}{5}, 1). \end{cases}$
We define $\varphi : X \to [0,\infty)$ *by* $\varphi(x) = \begin{cases} \frac{x}{2} & if\ x \in [0, \frac{3}{4}) \\ x - \frac{3}{4} & if\ x \in [\frac{3}{4}, 1) \end{cases}$ *and*
$\psi : [0,\infty) \to [0,\infty)$ *by* $\psi(t) = \frac{4}{5}t$ *for all* $t \geq 0$.
Let $x_0 = \frac{1}{8},\ fx_0 = \frac{9}{16}$ *then* $x_0 \preceq fx_0$. *Here* $O(x_0) = \{\frac{1}{8}, \frac{9}{16}, \frac{3}{4}, \frac{3}{4}, ...\}$ *and*
$\overline{O(x_0)} = \{\frac{1}{8}, \frac{9}{16}, \frac{3}{4}\} = O(x_0)$. *Let* $x, y \in O(x_0)$.
The following three cases arise to verify the inequality (1.3).
Case (i): $x = \frac{1}{8}$ *and* $y = \frac{9}{16}$.
In this case, $d(fx,fy) + \varphi(fx) + \varphi(fy) = \frac{15}{32}$ *and* $M(x,y) = \frac{25}{32}$.
$d(fx,fy) + \varphi(fx) + \varphi(fy) = \frac{15}{32} \leq \psi(\frac{25}{32}) = \psi(M(x,y))$.
Case (ii): $x = \frac{9}{16}$ *and* $y = \frac{3}{4}$.
In this case, the inequality (1.3) *holds trivially.*
Case (iii): $x = \frac{1}{8}$ *and* $y = \frac{3}{4}$.
In this case, $d(fx,fy) + \varphi(fx) + \varphi(fy) = \frac{15}{32}$ *and* $M(x,y) = \frac{11}{16}$.
$d(fx,fy) + \varphi(fx) + \varphi(fy) = \frac{15}{32} \leq \psi(\frac{11}{16}) = \psi(M(x,y))$.
Hence f *is a generalized* (φ,ψ)*-Jaggi contraction.*
Also we observe that the inequality (1.2) fails to hold.
For, by choosing $x = 0$ *and* $y = \frac{3}{4}$ *we have*
$d(f0, f(\frac{3}{4})) = \frac{3}{4} \nleq \alpha(0) + \gamma(\frac{3}{4}) < \frac{3}{4} = \alpha\frac{d(0,f0)d(\frac{3}{4},f\frac{3}{4})}{d(0,\frac{3}{4})} + \gamma d(0, \frac{3}{4})$.
$i.e.,$ f *is not a* $Jaggi\ contraction\ map$.

Thus we conclude that the class of generalized (φ,ψ)-Jaggi contractions is more general than the class of Jaggi contraction maps.

In Section 2, we prove the existence of fixed points of generalized (φ,ψ)-Jaggi contraction mappings in orbitally complete partially ordered metric spaces. In Section 3, we deduce some corollaries to the main results and provide examples in support of our results.

2. Main Results

Theorem 2.1. *Let $(X,d,\preceq)$ be a partially ordered metric space. Suppose that $f : X \to X$ is a non-decreasing map and $x_0 \in X$ such that $x_0 \preceq fx_0$. Suppose that f is a generalized (φ,ψ)-Jaggi contraction and X is f-orbitally complete. Then, the sequence $\{x_n\}$ defined by $x_{n+1} = fx_n,\ n = 0,1,2,...,$ is Cauchy in X. Let $\lim\limits_{n\to\infty} x_n = z,\ z \in X$. Assume that f is orbitally continuous at z. Then z is a fixed point of f and $\varphi(z) = 0$.*

Proof. Let $x_0 \in X$ be such that $x_0 \preceq fx_0$. We write $x_1 \in X$ so that $x_1 = fx_0$ then $x_0 \preceq x_1$. Since f is non-decreasing $x_1 = fx_0 \preceq fx_1$. Now, we write $x_2 \in X$ so that $x_2 = fx_1$ then $x_1 \preceq x_2$. On continuing this process, we get a sequence $\{x_n\} \subseteq \overline{O(x_0)}$ such that

$$x_{n+1} = fx_n\ for\ n = 0,1,2,... \tag{2.1}$$

satisfying $x_0 \preceq x_1 \preceq x_2 \preceq ... \preceq x_n \preceq x_{n+1} \preceq ...$.
If $x_n = x_{n+1}$ for some n, then the conclusion of the theorem trivially holds. Hence, without loss of generality, we assume that $x_n \neq x_{n+1}$ for all n. We denote

$$r_n = d(x_{n-1},x_n) + \varphi(x_{n-1}) + \varphi(x_n)\ forn = 1,2,... . \tag{2.2}$$

We consider $r_{n+1} = d(x_n, x_{n+1}) + \varphi(x_n) + \varphi(x_{n+1}) = d(fx_{n-1}, fx_n) + \varphi(fx_{n-1}) + \varphi(fx_n)$

$$\leq \psi(M(x,y)), \tag{2.3}$$

where
$M(x,y) = \max\{d(x_{n-1}, x_n)+\varphi(x_{n-1})+\varphi(x_n), \frac{(d(x_{n-1},fx_{n-1})+\varphi(x_{n-1})+\varphi(fx_{n-1}))(d(x_n,fx_n)+\varphi(x_n)+\varphi(fx_n))}{d(x_{n-1},x_n)+\varphi(x_{n-1})+\varphi(x_n)}\}$
$= \max\{r_n, \frac{r_n.r_{n+1}}{r_n}\} = \max\{r_n, r_{n+1}\}$.
If $\max\{r_n, r_{n+1}\} = r_{n+1}$ then from (2.3) we have
$r_{n+1} \leq \psi(r_{n+1}) < r_{n+1}$,
a contradiction.
Hence $\max\{r_n, r_{n+1}\} = r_n$ then from (2.3) we have

$$r_{n+1} \leq \psi(r_n) < r_n. \tag{2.4}$$

Thus it follows that $\{r_n\}$ is strictly decreasing sequence of non-negative real numbers and hence $\lim_{n\to\infty} r_n$ exists and it is r(say). $i.e.$, $\lim_{n\to\infty} r_n = r \geq 0$.
We now show that $r = 0$.
Suppose that $r > 0$. Then from (2.4), we have
$r_{n+1} \leq \psi(r_n)$.
On letting $n \to \infty$, we have
$r \leq \lim_{n\to\infty} \psi(r_n) = \psi(\lim_{n\to\infty} r_n) = \psi(r) < r$,
a contradiction.
Hence $\lim_{n\to\infty} d(x_{n+1}, x_n) + \varphi(x_{n+1}) + \varphi(x_n) = 0$, which implies
$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$ and $\lim_{n\to\infty} \varphi(x_n) = 0$.
We now show that $\{x_n\}$ is a Cauchy sequence in X.
Suppose that $\{x_n\}$ is not a Cauchy sequence. Then, there exist $\epsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ such that

$$d(x_{m(k)}, x_{n(k)}) \geq \epsilon \tag{2.5}$$

We choose $m(k)$, the least positive integer satisfying (2.5). Then, we have
$m(k) > n(k) > k$ with
$d(x_{m(k)}, x_{n(k)}) \geq \epsilon, \qquad d(x_{m(k)-1}, x_{n(k)}) < \epsilon.$
Now by Lemma 1.8, it follows that $\lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) = \epsilon$.
Now from (1.3), we have
$d(x_{m(k)+1}, x_{n(k)+1}) + \varphi(x_{m(k)+1}) + \varphi(x_{n(k)+1}) = d(fx_{m(k)}, fx_{n(k)}) + \varphi(fx_{m(k)}) + \varphi(fx_{n(k)})$

$$\leq \psi(M(x,y)), \tag{2.6}$$

where $M(x,y) = \max\{d(x_{m(k)}, x_{n(k)}) + \varphi(x_{m(k)}) + \varphi(x_{n(k)}),$
$\frac{(d(x_{m(k)},x_{m(k)+1})+\varphi(x_{m(k)})+\varphi(x_{m(k)+1}))(d(x_{n(k)},x_{n(k)+1})+\varphi(x_{n(k)})+\varphi(x_{n(k)+1}))}{d(x_{m(k)},x_{n(k)})+\varphi(x_{m(k)})+\varphi(x_{n(k)})}\}$.
Now, on letting $k \to \infty$ in (2.6) we have
$\epsilon \leq \psi(\epsilon) < \epsilon$,
a contradiction.
Therefore $\{x_n\} \subset O(x_0)$ is a Cauchy sequence in (X, d). Since X is f-orbitally complete, there exists $z \in X$ such that

$$\lim_{n\to\infty} x_n = z. \tag{2.7}$$

Since φ is lower semi continuous, we have
$\varphi(z) \leq \liminf_{n\to\infty} \varphi(x_n) = 0$.
Hence $\varphi(z) = 0$.
Since f is orbitally continuous at z w.r.t. x_0, from (2.1), we have
$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} fx_n = fz$.
This completes the proof of the theorem. ■

Theorem 2.2. *Let $(X, d, \preceq)$ be a partially ordered metric space. Suppose that $f : X \to X$ is a non-decreasing map and $x_0 \in X$ such that $x_0 \preceq fx_0,\ \varphi : X \to \mathbb{R}^+$ lower semi continuous and $\psi \in \Psi_2$ such that*

$$d(fx, fy) + \varphi(fx) + \varphi(fy) \le \psi(M(x, y)) \tag{2.8}$$

$M(x, y) = \max\{d(x, y) + \varphi(x) + \varphi(y), \frac{(d(x,fx)+\varphi(x)+\varphi(fx))(d(y,fy)+\varphi(y)+\varphi(fy))}{d(x,y)+\varphi(x)+\varphi(y)}\}$
for all $x, y \in \cup_{x_0 \preceq fx_0,\ x_0 \in X} \overline{O(x_0)}$ with $x \preceq y$ and $x \neq y$.
Assume the following:

(i) *if $\{x_n\}$ is a non-decreasing sequence converging to $z \in X$, then $x_n \preceq z$, for all n; and*

(ii) *if $\{x_n\}$ and $\{y_n\}$ are sequences in X with $x_n \preceq y_n$, for all n and $\lim_{n\to\infty} x_n = x,\ \lim_{n\to\infty} y_n = y,\ x,\ y \in X$ then $x \preceq y$.*

Assume that X is f-orbitally complete. Then, the sequence $\{x_n\}$ defined by $x_{n+1} = fx_n,\ n = 0, 1, 2, ...,$ is Cauchy in X. Let $\lim_{n\to\infty} x_n = z,\ z \in X$. Then z is a fixed point of f and $\varphi(z) = 0$. Further, f is orbitally continuous at z.

Proof. Let $x_0 \in X$ be such that $x_0 \preceq fx_0$. On proceeding as in the proof of Theorem 2.1, we have $\{x_n\} \subset O(x_0)$ defined by (2.1)is a Cauchy sequence in (X, d). Since X is f-orbitally complete, there exists $z \in X$ such that

$$\lim_{n\to\infty} x_n = z \tag{2.9}$$

Since φ is lower semi continuous, we have
$\varphi(z) \le \liminf_{n\to\infty} \varphi(x_n) = 0.$
Hence $\varphi(z) = 0$.
Since $\{x_n\}$ is a non-decreasing sequence and $x_n \to z$, by (i) we have $x_n \preceq z$ for all n. Since f is non-decreasing, we have $fx_n \preceq fz$ for all n. $i.e.$, $x_{n+1} \preceq fz$ for all n. Moreover, as $x_n \preceq x_{n+1} \preceq fz$ for all n and by using (ii), we get $z \preceq fz$.

We now define a sequence $\{y_n\}$ as $y_0 = z,\ y_{n+1} = fy_n,\ n = 0, 1, 2, \ldots$. Then $y_0 \preceq fy_0$. Since f is non-decreasing, we obtain that $\{y_n\}$ is a non-decreasing sequence and $\{y_n\}$ is Cauchy (similar to the argument to show $\{x_n\}$ is Cauchy) $y_n \to y$ (say), $y \in X$. Again, by the condition (i), we have $y_n \preceq y$. Since $x_n \preceq z = y_0 \preceq fz = fy_0 \preceq y_n \preceq y$ for all n, we have $x_n \preceq y_n$ for all n, and hence $z \preceq y$.
If $x_n = y_n$ for some n, then $x_n \preceq z = y_0 \preceq fz = fy_0 \preceq y_n = x_n$ so that $fz = z$.
Hence we assume that $x_n \neq y_n$ for all n.
Suppose that $z \neq y$. Now from (2.8), we have

$$\begin{aligned} d(x_{n+1}, y_{n+1}) + \varphi(x_{n+1}) + \varphi(y_{n+1}) &= d(fx_n, fy_n) + \varphi(fx_n) + \varphi(fy_n) \\ &\le \psi(M(x, y)),\ where \end{aligned} \tag{2.10}$$

$M(x, y) = \max\{d(x_n, y_n) + \varphi(x_n) + \varphi(y_n), \frac{(d(x_n,fx_n)+\varphi(x_n)+\varphi(fx_n))(d(y_n,fy_n)+\varphi(y_n)+\varphi(fy_n))}{d(x_n,y_n)+\varphi(x_n)+\varphi(y_n)}\}$.
On letting $n \to \infty$ in (2.10), we have
$d(z, y) \le \psi(d(z, y)) < d(z, y)$,
a contradiction.
Hence $z = y$, and we have $z \preceq fz = fy_0 \preceq y_n \preceq y = z$.
Therefore z is a fixed point of f. ■

Remark: Condition (ii) of Theorem 2.2 holds trivially in $\mathbb{R}$ with the usual order. But in partially ordered metric spaces it need not hold always. For more details, we refer [8].

Now we prove the uniqueness of fixed point of f by using ‘condition (H)’ and it is the following:

Condition (H): For all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$.

Theorem 2.3. *In addition to the hypotheses of Theorem 2.1 (Theorem 2.2) if condition (H) holds, then f has a unique fixed point.*

Proof. By Theorem 2.1, we have f has a fixed point. Suppose that $x, y \in X$ are two fixed points of f. By condition (H), there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$.
Put $z = z_0,\ z_1 = fz_0$. and define a sequence $\{z_n\}$ in X by $z_{n+1} = fz_n$ for all $n \geq 0$. Then $x \preceq z_0$ and $y \preceq z_0$. By using the non-decreasing property of f, we have
$fx \preceq fz_0$ and $fy \preceq fz_0$. Hence $x \preceq z_1$ and $y \preceq z_1$.
On continuing this process, we have

$$x \preceq z_n \ \ and\ y \preceq z_n \ \ for\ \ n \geq 0. \tag{2.11}$$

In (2.11), if $x = z_n$ for some n, then $fx = fz_n$ so that $x = z_{n+1}$. In fact, we have $x = z_m$ for $m \geq n$ so that $\lim_{n\to\infty} z_n = x$.
If $x \neq z_n$ for all $n = 0, 1, 2, ...$ then by using (1.3), we have

$$\begin{aligned} d(x, z_{n+1}) + \varphi(x) + \varphi(z_{n+1}) &= d(fx, fz_n) + \varphi(fx) + \varphi(fz_n) \\ &\leq \psi(\max\{d(x, z_n) + \varphi(x) + \varphi(z_n), \tfrac{(d(x,fx)+\varphi(x)+\varphi(fx))(d(z_n,fz_n)+\varphi(z_n)+\varphi(fz_n))}{d(x,z_n)+\varphi(x)+\varphi(z_n)}\} \\ &= \psi(\max\{d(x, z_n) + \varphi(z_n), 0\}) = \psi(d(x, z_n) + \varphi(z_n)) \\ d(x, z_{n+1}) + \varphi(z_{n+1}) &\leq \psi(d(x, z_n) + \varphi(z_n)) = \psi(\psi(d(x, z_{n-1}) + \varphi(z_{n-1})) \\ &\leq \psi^2(d(x, z_{n-1}) + \varphi(z_{n-1})) \\ &\leq \psi^3(d(x, z_{n-2}) + \varphi(z_{n-2})) \leq ... \leq \psi^n(d(x, z_1) + \varphi(z_1)) \to 0 \text{ as } n \to \infty. \end{aligned}$$

$$Therefore\ \lim_{n\to\infty} z_n = x. \tag{2.12}$$

Again, by applying the similar argument to $y \neq z_n$ for all $n = 0, 1, 2, ...$, it follows that

$$\lim_{n\to\infty} z_n = y. \tag{2.13}$$

From (2.12) and (2.13) we have $x = y$.
This completes the proof of the theorem. ■

3. Corollaries and examples

In the following, we deduce some corollaries to the main results of Section 2.

Corollary 3.1. *Let $(X, d, \preceq)$ be a partially ordered metric space. Suppose that $f : X \to X$ is a non-decreasing map and $x_0 \in X$ such that $x_0 \preceq fx_0$. Suppose that there exists $\psi \in \Psi_2$ such that*

$$d(fx, fy) \leq \psi[\max\{d(x, y), \frac{d(x, fx)d(y, fy)}{d(x, y)}\}] \tag{3.1}$$

for all $x, y \in \overline{O(x_0)}$ with $x \preceq y$ and $x \neq y$.
Assume that X is f-orbitally complete. Then, the sequence $\{x_n\}$ defined by
$x_{n+1} = fx_n,\ n = 0, 1, 2, ...,$ is Cauchy in X. Let $\lim_{n\to\infty} x_n = z,\ z \in X$. Suppose that f is orbitally continuous at z. Then z is a fixed point of f.

Proof. The inequality (3.1) implies the inequality (1.3) with $\varphi \equiv 0$ on X, and hence the conclusion of the corollary follows from Theorem 2.1. ■

Corollary 3.2. *Let $(X, d, \preceq)$ be a partially ordered metric space. Suppose that $f : X \to X$ is a non-decreasing map and $x_0 \in X$ such that $x_0 \preceq fx_0$. Suppose that there exist a constant $k \in (0, 1)$ such that*

$$d(fx, fy) \leq k \max\{d(x, y), \frac{d(x, fx)d(y, fy)}{d(x, y)}\} \tag{3.2}$$

for all $x, y \in \overline{O(x_0)}$ with $x \preceq y$ and $x \neq y$.
Assume that X is f-orbitally complete. Then, the sequence $\{x_n\}$ defined by $x_{n+1} = fx_n,\ n = 0, 1, 2, ...,$ is Cauchy in X. Let $\lim_{n\to\infty} x_n = z,\ z \in X$. Suppose that f is orbitally continuous at z. Then z is a fixed point of f.

Proof. By choosing $\psi(t) = kt,\ t \geq 0$ in the inequality (3.1), the conclusion of this corollary follows from Corollary 3.1. ■

Remark 3.3. *Theorem 1.2 follows as a corollary to Corollary 3.2, since the inequality (1.2) implies the inequality (3.2) with $k = \alpha + \gamma < 1$.*

Corollary 3.4. *Let $(X, d, \preceq)$ be a partially ordered metric space. Suppose that $f : X \to X$ is a non-decreasing map and $x_0 \in X$ such that $x_0 \preceq fx_0$ and $\psi \in \Psi_2$ such that*

$$d(fx, fy) \leq \psi(\max\{d(x, y), \frac{d(x, fx)d(y, fy)}{d(x, y)}\}) \tag{3.3}$$

for all $x, y \in \cup_{x_0 \preceq fx_0,\ x_0 \in X}\overline{O(x_0)}$ with $x \preceq y$ and $x \neq y$.
Assume the following:

(i) *if $\{x_n\}$ is a non-decreasing sequence converging to $z \in X$, then $x_n \preceq z$, for all n; and*

(ii) *if $\{x_n\}$ and $\{y_n\}$ are sequences in X with $x_n \preceq y_n$, for all n and $\lim_{n\to\infty} x_n = x,\ \lim_{n\to\infty} y_n = y,\ x,\ y \in X$ then $x \preceq y$.*

Assume that X is f-orbitally complete. Then, the sequence $\{x_n\}$ defined by $x_{n+1} = fx_n,\ n = 0, 1, 2, ...,$ is Cauchy in X. Let $\lim_{n\to\infty} x_n = z,\ z \in X$.
Then z is a fixed point of f.

Proof. The inequality (3.3) implies the inequality (2.8) with $\varphi \equiv 0$ on X, and hence the conclusion of the corollary follows from Theorem 2.2. ■

Corollary 3.5. *Let $(X, d, \preceq)$ be a partially ordered metric space. Suppose that $f : X \to X$ is a non-decreasing map and $x_0 \in X$ such that $x_0 \preceq fx_0$ and there exist a constant $k \in (0, 1)$ such that*

$$d(fx, fy) \leq k \max\{d(x, y), \frac{d(x, fx)d(y, fy)}{d(x, y)}\} \tag{3.4}$$

for all $x, y \in \cup_{x_0 \preceq fx_0,\ x_0 \in X}\overline{O(x_0)}$ with $x \preceq y$ and $x \neq y$.
Assume the following:

(i) *if $\{x_n\}$ is a non-decreasing sequence converging to $z \in X$, then $x_n \preceq z$, for all n; and* *(ii) if $\{x_n\}$ and $\{y_n\}$ are sequences in X with $x_n \preceq y_n$, for all n and*

$\lim_{n\to\infty} x_n = x, \ \lim_{n\to\infty} y_n = y, \ x, \ y \in X$ *then* $x \preceq y$.

Assume that X is f-orbitally complete. Then, the sequence $\{x_n\}$ defined by $x_{n+1} = fx_n, \ n = 0, 1, 2, ...,$ is Cauchy in X. Let $\lim_{n\to\infty} x_n = z, \ z \in X$. Then z is a fixed point of f.

Proof. By choosing $\psi(t) = kt, \ t \geq 0$ in the inequality (3.3), the conclusion of this corollary follows from Corollary 3.4. ∎

Remark 3.6. *Theorem 1.2 follows as a corollary to Corollary 3.5, since the inequality (1.2) implies the inequality (3.4) with $k = \alpha + \gamma < 1$.*

In the following, we provide examples in support of the results that are proved in Section 2.

Example 3.7. *Let $X = [0, 2)$ with the usual metric. We define partial order $\preceq$ on X by*

$\preceq := \{(x, y) \in X \times X : x = y\} \cup \{(x, y)/x, y \in X, \ x \preceq y \Leftrightarrow x \geq y$, *where $\geq$ is the usual order*$\}$.

We define $f : X \to X$ by
$$fx = \begin{cases} 0 & if\ x = 0 \\ \frac{1}{8} & if\ x \in [\frac{1}{2}, 1) \\ \frac{x^2}{16} & if\ x \in [\frac{1}{4}, \frac{1}{2}) \\ \frac{1}{2^{n+2}} & if\ x \in [\frac{1}{2^{n+1}}, \frac{1}{2^n}), n \geq 2 \\ 2 - x & if\ x \in [1, 2). \end{cases}$$

We define $\psi : [0, \infty) \to [0, \infty)$ by $\psi(t) = \frac{2}{3}t$ for all $t \geq 0$ and
$$\varphi : X \to [0, \infty) \text{ by } \varphi(x) = \begin{cases} \frac{x}{3} & if\ x \in [0, \frac{5}{16}) \\ x - \frac{5}{16} & if\ x \in [\frac{5}{16}, 1). \end{cases}$$

Let $x_0 = \frac{3}{8}$ then $x_0 \preceq fx_0$. Here $O(x_0) = \{\frac{3}{8}, \frac{9}{2^{10}}, \frac{1}{2^8}, \frac{1}{2^9} ..., \frac{1}{2^{2n+8}}, ...\} = \{\frac{3}{8}, \frac{9}{2^{10}}\} \cup \{\frac{1}{2^n}/n \geq 8\}$ and $\overline{O(x_0)} = O(x_0) \cup \{0\}$.

We show that f is a generalized (φ, ψ)-Jaggi contraction. The following are the possible four cases.

Case (i): $x = \frac{3}{8}$ and $y = \frac{9}{2^{10}}$.

In this case, $d(fx, fy) + \varphi(fx) + \varphi(fy) = \frac{7}{3.2^8}$ and $M(x, y) = \frac{442}{2^{10}}$.

$d(fx, fy) + \varphi(fx) + \varphi(fy) = \frac{7}{3.2^8} \leq \psi(\frac{442}{2^{10}}) = \psi(M(x, y))$.

Case (ii): $x = \frac{9}{2^{10}}$ and $y = \frac{1}{2^{i+3}}, \ i \geq 2$.

In this case, $d(fx, fy) + \varphi(fx) + \varphi(fy) = \frac{2^i - 2^3}{3.2^{i+6}}$ and $M(x, y) = \frac{7}{2^3(9.2^i - 2^6)}$.

$d(fx, fy) + \varphi(fx) + \varphi(fy) = \frac{2^i - 2^3}{3.2^{i+6}} \leq \psi(\frac{7}{2^3(9.2^i - 2^6)}) = \psi(M(x, y))$.

Case (iii): $x = \frac{3}{8}$ and $y = \frac{1}{2^{i+1}}, \ i \geq 2$. In this case,
$$d(fx, fy) + \varphi(fx) + \varphi(fy) = \begin{cases} \frac{9.2^i - 2^7}{3.2^{i+8}} & if\ i \geq 2 \\ \frac{2^9 - 9.2^i}{3.2^{i+9}} & if\ i \leq 2 \end{cases} \text{ and } M(x, y) = \frac{663}{2^4(21.2^i - 16)}.$$

Sub case (a):

$d(fx, fy) + \varphi(fx) + \varphi(fy) = \frac{9.2^i - 2^7}{3.2^{i+8}} \leq \psi(\frac{663}{2^4(21.2^i - 16)}) = \psi(M(x, y))$.

Sub case (b): $d(fx,fy)+\varphi(fx)+\varphi(fy)=\frac{2^9-9.2^i}{3.2^{i+9}}\leq\psi(\frac{663}{2^4(21.2^i-16)})=\psi(M(x,y))$.

Case (iv): $x=\frac{1}{2^i}$ *and* $y=\frac{1}{2^j},\ i\geq 2$ *and* $j\geq i$.

In this case, $d(fx,fy)+\varphi(fx)+\varphi(fy)=\frac{2.2^j-2^i}{3.2^{i+j}}$ *and* $M(x,y)=\frac{4.2^j-2.2^i}{3.2^{i+j}}$.

$d(fx,fy)+\varphi(fx)+\varphi(fy)=\frac{2.2^j-2^i}{3.2^{i+j}}\leq\psi(\frac{4.2^j-2.2^i}{3.2^{i+j}})=\psi(M(x,y))$.

Hence, all the hypotheses of Theorem 2.1 hold and $0,\ 1$ *are two fixed points of* f *in* $\overline{O(x_0)}$. *Also* $\varphi(0)=0$.

Since, the inequality (1.3) *fails to hold at* $x=0,\ y=1$ *when* $\varphi\equiv 0,\ f$ *is not a generalized* ψ*-Jaggi contraction. Further, we observe that at* $x=0$ *and* $y=1$*, we have*

$d(f0,f1)=1\nleq\alpha.0+\gamma.1=\alpha\frac{d(0,f0)d(1,f1)}{d(0,1)}+\gamma d(0,1)$

so that the inequality (1.2) does not hold for any α *and* γ *in* $[0,1)$ *with* $\alpha+\gamma<1$. $i.e.,$ f *is not a* $Jaggi\ contraction\ map$. *Therefore Theorem 1.2 is not applicable.*

Thus, it suggests that Theorem 2.1 is a generalization of Theorem 1.2.

Remark 3.8. *For* $x=0$ *and* $y=1$*, and for any* $z\in X$ *we have either* $0\npreceq z$ *or* $1\npreceq z$. *Hence condition (H) fails to hold and* f *has more than one fixed point namely* 0 *and* 1.

Example 3.9. *Let* $X=[0,1)$ *with the usual metric. We define partial order* $\preceq$ *on* X *by*

$\preceq:=\{(x,y)\in X\times X:x=y\}\cup\{(x,y)/x,y\in X,\ x\preceq y\Leftrightarrow x\geq y,$ *where* $\geq$ *is the usual order*$\}$.

We define $f:X\to X$ *by* $fx=\begin{cases}\frac{x}{2} & if\ x\in[0,\frac{1}{2}]\\ \frac{x^2}{2} & if\ x\in(\frac{1}{2},1).\end{cases}$

We define $\psi:[0,\infty)\to[0,\infty)$ *by* $\psi(t)=\frac{5t}{6}$ *for all* $t\geq 0$ *and*

$\varphi:X\to[0,\infty)$ *by* $\varphi(x)=\begin{cases}\frac{x}{3} & if\ x\in[0,\frac{5}{16})\\ x-\frac{5}{16} & if\ x\in[\frac{5}{16},1).\end{cases}$

We choose $x_0=\frac{1}{2}$ *then* $x_0\preceq fx_0,\ O(x_0)=\{\frac{1}{2},\frac{1}{2^2},\frac{1}{2^3}...,\frac{1}{2^n},....\}=\{\frac{1}{2^n}/n\geq 1\}$ *and* $\overline{O(x_0)}=O(x_0)\cup\{0\}$.

The following two cases arise to verify the inequality (2.8).

Case (i): $x=\frac{1}{2^i}$ *and* $y=\frac{1}{2^j},\ i\geq 2$ *and* $j\geq i$.

In this case, $d(fx,fy)+\varphi(fx)+\varphi(fy)=\frac{2.2^j-2^i}{3.2^{i+j}}$ *and* $M(x,y)=\frac{4.2^j-2.2^i}{3.2^{i+j}}$.

$$d(fx,fy)+\varphi(fx)+\varphi(fy)=\frac{2.2^j-2^i}{3.2^{i+j}}$$
$$\leq\psi(\frac{4.2^j-2.2^i}{3.2^{i+j}})=\psi(M(x,y)).$$

Case (ii): $x=\frac{1}{2^i}$ *and* $y=0,\ i\geq 1$.

In this case, $d(fx,fy)+\varphi(fx)+\varphi(fy)=\frac{2}{3.2^i}$ *and* $M(x,y)=\frac{4}{3.2^i}$.

$$d(fx,fy)+\varphi(fx)+\varphi(fy)=\frac{2}{3.2^i}$$
$$\leq\psi(\frac{4}{3.2^i})=\psi(M(x,y)).$$

Hence, all the hypotheses of Theorem 2.2 hold and 0 *is a fixed point of* f *in* $\overline{O(x_0)}$. *Also* $\varphi(0)=0$.

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