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2-Dominating sets and 2-domination polynomials of pan graph $P_{m,2}$

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Abstract

Let G be a simple graph of order m. Let $D_2(G,x)$ be the family of 2-dominating sets in G with size 1. The polynomial $D_2(G,x) = \sum_{i=\gamma_{2(G)}}^{m} d_2(G,i)x^i$ is called the 2-domination polynomial of G. Let $D_2(P_{m,2},i)$ be the family of 2-dominating sets of the pan graph $P_{m,2}$ with cardinality i and let $d_2(P_{m,2},i) = |D_2(P_{m,2},i)|$. Then, the 2-domination polynomial $D_2(P_{m,2},x)$ of $P_{m,2}$ is defined as, $D_2(P_{m,2},x) = \sum_{i=\gamma_2(P_{m,2})}^{m+2} d_2(P_{m,2},i)x^i$, where $\gamma_2(P_{m,2})$ is the 2 - domination number of $P_{m,2}$. In this paper we obtain a recursive formula for $d_2(P_{m,2},i)$. Using this recursive formula we construct the 2-domination polynomial, $D_2(P_{m,2},x) = \sum_{i=\left\lceil \frac{m+2}{2} \right\rceil}^{m+2} d_2(P_{m,2},i)x^i$, where $d_2(P_{m,2},i)$ is the number of 2-dominating sets of $P_{m,2}$ of cardinality i and some properties of this polynomial have been studied.

Keywords

Pan, 2-dominating set, 2-domination number, 2-domination polynomial.

AMS Subject Classification

97K30, 05C69, 05C31.

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Contents

1. Introduction

A Subset D of V is a dominating set of G if $N[D] = V$ or equivalently, every vertex in $V - D$ is adjacent to at least one vertex in D. The domination number of the graph G is defined as the minimum size taken over all dominating sets D of vertices in G and is denoted by (G). Let $P_{m,2}$ be a pan graph with m + 2 vertices. Let $D_2(P_{m,2},i)$ be the family of 2-dominating sets of a pan graph $P_{m,2}$ with cardinality i, let $d_2(P_{m,2},i) = |D_2(P_{m,2},i)|$. We call the polynomial $D_2(P_{m,2},x) = \sum_{i=\left\lceil \frac{m+2}{2} \right\rceil}^{m+2} d_2(P_{m,2},i)x^i$, as a 2-domination polynomial of a pan graph $P_{m,2}$. We use $\lceil m \rceil$, for the smallest integer greater than or equal to m. In this paper $[m]$ denotes the set $\{1,2,...,m\}$

2. 2-DOMINATING SETS OF PAN GRAPH $P_{m,2}$

Definition 2.1. *A set $D \subseteq V$ is a 2-dominating set if every vertex in $V - D$ is adjacent to at least two vertices in D. The 2-domination number of a graph G is defined as the minimum size taken over all 2-dominating sets of vertices in G and is denoted by $\gamma_2(G)$.*

Definition 2.2. *The pan graph is the graph obtained by joining a cycle graph to a Singleton graph with a bridge. The pan graph is therefore isomorphic with the tadpole graph.*

Definition 2.3. *Let $P_{m,2}$ be the pan graph with $m+2$ vertices. Label the vertices of $P_{m,2}$ as $v_1, v_2, v_3, \ldots, v_m, v_{m+1}, v_{m+2}$ where v_m is the vertex of degree 3, v_{m+1} is the vertex of degree 2 and v_{m+2} is the vertex of degree 1.*

Lemma 2.4. *Let $P_{m,2}$ be the pan graph with $m+2$ vertices, then its 2-domination number is $\gamma_2(P_{m,2}) = \left\lceil \frac{m+2}{2} \right\rceil$.*

Lemma 2.5. *Let $P_{m,2}, m \geq 5$ be the pan graph with $|V(P_{m,2})| = m+2$. Then, $d_2(P_{m,2}, i) = 0$ if $i < \left\lceil \frac{m+2}{2} \right\rceil$ or $i > m+2$ and $d_2(P_{m,2}, i) > 0$ if $\left\lceil \frac{m+2}{2} \right\rceil \leq i \leq m+2$.*

Proof. If $i < \left\lceil \frac{m+2}{2} \right\rceil$ or $i > m+2$, then there is no 2-dominating set of size i. Therefore, $D_2(P_{m,2}, i) = \phi$, if $i < \left\lceil \frac{m+2}{2} \right\rceil$ and $i > m+2$. By Lemma 2.4, the minimum size of the 2-dominating set of $P_{m,2}$ is $\left\lceil \frac{m+2}{2} \right\rceil$. Therefore, $d_2(P_{m,2}, i) > 0$, if $i \geq \left\lceil \frac{m+2}{2} \right\rceil$ and $i \leq m+2$. Thus, we get, $d_2(P_{m,2}, i) = 0$ if $i < \left\lceil \frac{m+2}{2} \right\rceil$ or $i > m+2$ and $d_2(P_{m,2}, i) > 0$ if $\left\lceil \frac{m+2}{2} \right\rceil \leq i \leq m+2$. □

Lemma 2.6. *If $D_2(P_{m,2}, i) \neq \phi$, then for every $m \geq 6$, we have*

1. *$D_2(P_{m-1,2}, i-1) = \phi, D_2(P_{m-2,2}, i-1) \neq \phi$, $D_2(P_{m-3,2}, i-1) \neq \phi$ iff $m = 2k$ and $i = k+1$ for some $k \geq 3$.*
2. *$D_2(P_{m-2,2}, i-1) = \phi, D_2(P_{m-3,2}, i-1) = \phi$ and $D_2(P_{m-1,2}, i-1) \neq \phi$ iff $i = m+2$.*
3. *$D_2(P_{m-1,2}, i-1) \neq \phi, D_2(P_{m-2,2}, i-1) \neq \phi$ and $D_2(P_{m-3,2}, i-1) = \phi$ iff $i = m+1$*

Proof. (i) Assume that, $D_2(P_{m-1,2}, i-1) = \phi$, $D_2(P_{m-2,2}, i-1) \neq \phi$ and
$D_2(P_{m-3,2}, i-1) \neq \phi$. Since, $D_2(P_{m-1,2}, i-1) = \phi$, by Lemma 2.5, $i-1 > m+1$ or $i-1 < \left\lceil \frac{m+1}{2} \right\rceil$. If $i-1 > m+1$, then $i > m+2$, which implies $D_2(P_{m,2}, i) = \phi$, which is a contradiction. Therefore, $i-1 < \left\lceil \frac{m+1}{2} \right\rceil$. That is, $i \leq \left\lceil \frac{m+1}{2} \right\rceil$.............(1).
Since, $D_2(P_{m-2,2}, i-1) \neq \phi$ and $D_2(P_{m-3,2}, i-1) \neq \phi$, we have, $\left\lceil \frac{m}{2} \right\rceil \leq i-1 \leq m-2$ and $\left\lceil \frac{(m-1)}{2} \right\rceil \leq i-1 \leq m-3$. Therefore, $\left\lceil \frac{m}{2} \right\rceil \leq i-1 \leq m-3$. This implies that, $\left\lceil \frac{m+2}{2} \right\rceil \leq i$................(2). From (1) and (2) we get $\left\lceil \frac{m+2}{2} \right\rceil \leq i \leq \left\lceil \frac{m+1}{2} \right\rceil$. This inequality is true only when $m = 2k$ and $i = k+1$ for some $k \geq 3$.
Conversely, assume that $m = 2k$ and $i = k+1$.
Therefore, $k = \frac{m}{2}$ and $i-1 = k$. $i-1 = k$ implies $i-1 = \frac{m}{2} < \left\lceil \frac{m+1}{2} \right\rceil$. Therefore, $D_2(P_{m-1,2}, i-1) = \phi$. Also, $D_2(P_{m-2,2}, i-1) = D_2(P_{2k-2,2}, k) \neq \phi$,
since $\left\lceil \frac{(2k-2+2)}{2} \right\rceil = \lceil 2k/2 \rceil = k$.
$D_2(P_{m-3,2}, i-1) = D_2(P_{2k-3,2}, k) \neq \phi$, Since $\lceil (2k-3+2)/2 \rceil = \left\lceil \frac{(2k-1}{2} \right\rceil = k$.

(ii) Assume that $D_2(P_{m-2,2}, i-1) = \phi, D_2(P_{m-3,2}, i-1) = \phi$ and $D_2(P_{m-1,2}, i-1) \neq \phi$. Since, $D_2(P_{m-2,2}, i-1) = \phi$ and $D_2(P_{m-3,2}, i-1) = \phi$, by Lemma 2.5, we have, $i-1 > m$ or $i-1 < \left\lceil \frac{m}{2} \right\rceil$ and $i-1 > m-1$ or $i-1 < \left\lceil \frac{(m-1)}{2} \right\rceil$. Therefore, $i-1 > m$ or $i-1 < \left\lceil \frac{(m-1)}{2} \right\rceil$. If $i-1 < \left\lceil \frac{(m-1)}{2} \right\rceil$, then $i-1 < \left\lceil \frac{m+2}{2} \right\rceil$ holds. Therefore, by Lemma 2.5, $D_2(P_{m,2}, i) \neq \phi$, which is a contradiction. So, we have, $i-1 > m$. Therefore, $i > m+1$. Therefore, $i \geq m+2$.........(1). Also, since $D_2(P_{m-1,2}, i-1) \neq \phi$, we have $\left\lceil \frac{m+1}{2} \right\rceil \leq i-1 \leq m+1$. Therefore, $i \leq m+2$..................(2). From (1) and (2) we get, $i = m+2$.

Conversely, if $i = m+2, D_2(P_{m-1,2}, i-1) = D_2(P_{m-1,2}, m+1) \neq \phi . D_2(P_{m-2,2}, i-1) = D_2(P_{m-2,2}, m+1) = \phi . D_2(P_{m-3,2}, i-1) = D_2(P_{m-3,2}, m+1) = \phi$. (iii) Assume that, $D_2(P_{m-1,2}, i-1) \neq \phi, D_2(P_{m-2,2}, i-1) \neq \phi$ and $D_2(P_{m-3,2}, i-1) = \phi$. Since, $D_2(P_{m-3,2}, i-1) = \phi$, by Lemma 2.5, we have, $i-1 > m-1$ or $i-1 < \left\lceil \frac{(m-1)}{2} \right\rceil$. If $i-1 < \left\lceil \frac{(m-1)}{2} \right\rceil$, then $i < \left\lceil \frac{m+2}{2} \right\rceil$ holds. Therefore, $D_2(P_{m,2}, i) = \phi$, which is a contradiction. Therefore, $i-1 > m-1$. ,$i > m$. Therefore, $i \geq m+1$..............(1).
Since, $D_2(P_{m-1,2}, i-1) \neq \phi$ and $D_2(P_{m-2,2}, i-1) \neq \phi$, we have $\left\lceil \frac{m+1}{2} \right\rceil \leq i-1 \leq m+1$ and $\left\lceil \frac{m}{2} \right\rceil \leq i-1 \leq m$. Therefore, $\left\lceil \frac{m+1}{2} \right\rceil \leq i-1 \leq$m. Therefore, $i \leq m+1$................(2).
From (1) and (2) we get, $i = m+1$.
Conversely, suppose $i = m+1$. Then, $D_2(P_{m-1,2}, i-1) = D_2(P_{m-1,2}, m) \neq \phi, D_2(P_{m-2,2}, i-1) = D_2(P_{m-2,2}, m) \neq \phi$ and $D_2(P_{m-3,2}, i-1) = D_2(P_{m-3,2}, m) = \phi$. □

Theorem 2.7. *For every $m \geq 2$ and $i \geq \left\lceil \frac{m+2}{2} \right\rceil$*

1. *$D_2(P_{2m,2}, m+1) = \{2, 4, 6, 8 \ldots .2m+2\}$.*
2. *If $D_2(P_{m-2,2}, i-1) = \phi, D_2(P_{m-3,2}, i-1) = \phi$ and $D_2(P_{m-1,2}, i-1) \neq \phi$, then $D_2(P_{m,2}, i) = D_2(P_{m,2}, m+2) = [m+2]$*
3. *If $D_2(P_{m-1,2}, i-1) \neq \phi, D_2(P_{m-2,2}, i-1) \neq \phi$ and $D_2(P_{m-3,2}, i-1) = \phi$, then $D_2(P_{m,2}, i) = D_2(P_{m,2}, m+1) = \{[m+2] - \{x\} / x \in [m+2]$ and $x \neq m+2\}$*
4. *If $D_2(P_{m-1,2}, i-1) = \phi, D_2(P_{m-2,2}, i-1) \neq \phi$, then $D_2(P_{m,2}, i) = \{X \cup \{m+2\}\ X \in D_2(P_{m-2,2}, i-1)\}$*

5. *If $D_2(P_{m-1,2}, i-1) \neq \phi, D_2(P_{m-2,2}, i-1) = \phi$, then $D_2(P_{m,2}, i) = \{Y \cup \{m+2\}\ Y \in D_2(P_{m-1,2}, i-1)\}$*

6. *If $D_2(P_{m-1,2}, i-1) \neq \phi$ and $D_2(P_{m-2,2}, i-1) \neq \phi$, then $D_2(P_{m,2}, i) = \{X \cup \{m+2\} \cup Y \cup \{m+2\}$
   $if\ Y\ starts\ with\ 1 \cup Y - \{m+1\} \cup \{m, m+2\}$
   $if\ Y\ starts\ with\ 2\ and\ \{m-3\} \in Y \cup Y - \{m-1\} \cup \{m, m+2\}$
   $if\ Y\ starts\ with 2\ and\ \{m-2\} \in Y\}$
   where $X \in D_2(P_{m-2,2}, i-1)$
   and $Y \in D_2(P_{m-1,2}, i-1)$*

Proof. (i) For every $m \geq 2, D_2(P_{2m,2}, m+1)$ has only one 2 - dominating set as, $2, 4, 6, \ldots, 2m+2$. Therefore, $D_2(P_{2m,2}, m+1) = 2, 4, 6, \ldots 2m+2$.

(ii) Since $D_2(P_{m-2,2}, i-1) = \phi, D_2(P_{m-3,2}, i-1) = \phi$ and $D_2(P_{m-1,2}, i-1) \neq \phi$, by Lemma 2.6 (ii), $we have i = m+2$. Therefore, $D_2(P_{m,2}, i) = D_2(P_{m,2}, m+2) = [m+2]$.

(iii) Since, $D_2(P_{m-1,2}, i-1) \neq \phi, D_2(P_{m-2,2}, i-1) \neq \phi$ and $D_2(P_{m-3,2}, i-1) = \phi$, by Lemma 2.6($iii$), we have $i = m+1$. Therefore, $D_2(P_{m,2}, i) = D_2(P_{m,2}, m+1) =$
$= [m+2] - x/x \in [m+2] and x \neq m+2$.

(iv) Let X be a 2 - dominating set of P_{m-2} with size $i-1$. All the elements of $D_2(P_{m-2,2}, i-1)$ end with m. Therefore, $m \in X$, adjoin $m+2$ with X. Hence, every X of $D_2(P_{m-2,2}, i-1)$ belongs to $D_2(P_{m,2}, i)$ by adjoining $m+2$ only.

(v) Let Y be a 2 - dominating set of $P_{m-1,2}$ with size $i-1$. All the elements of $D_2(P_{m-1,2}, i-1)$ end with $m+1$. Therefore, $m+1 \in Y$, adjoin $m+2$ with Y. Hence, every Y of $D_2(P_{m-1,2}, i-1)$ belongs to $D_2(P_{m,2}, i)$ by adjoining m+2 only.

(vi) Construction of $D_2(P_{m,2}, i)$ from $D_2(P_{m-1,2}, i-1)$ and $D_2(P_{m-2,2}, i-1)$. Let X be a 2 - dominating set of P_{m-2} with size $i-1$. All the elements of $D_2(P_{m-2,2}, i-1)$ ends with m. Therefore, $m \in X$, adjoin $m+2$ with X. Hence, every X of $D_2(P_{m-2,2}, i-1)$ belongs to $D_2(P_{m,2}, i)$ by adjoining $m+2$ only. Let Y be a 2 - dominating set of $P_{m-1,2}$ with size $i-1$. All the elements of $D_2(P_{m-1,2}, i-1)$ starts with 1 or 2 and end with $m+1$. Adjoin $m+2$ with Y if Y starts with 1 and end with $m+1$. Adjoin m and $m+2$ with Y and remove $m+1$ from Y if Y starts with 2 and $\{m-3\} \in Y$. Adjoin m and $m+2$ with Y and remove $\{m-1\}$ from Y if Y starts with 2 and $\{m-2\} \in Y$. Hence, every Y of $D_2(P_{m-1,2}, i-1)$ belongs to $D_2(P_{m,2}, i)$ by adjoining $m+2$, adjoining $\{m, m+2\}$ and removing $m+1$ and adjoining $\{m, m+2\}$ and removing $\{m-1\}$. □

Theorem 2.8. *If $D_2(P_{m,2}, i)$ be the family of the 2-dominating sets of $P_{m,2}$ with size i, where $i \geq \left\lceil \frac{m+2}{2} \right\rceil$ then, $d_2(P_{m,2}, i) = d_2(P_{m-1,2}, i-1) + d_2(P_{m-2,2}, i-1)$.*

3. 2-DOMINATION POLYNOMIALS OF PAN GRAPH $P_{m,2}$.

Definition 3.1. *Let $P_{m,2}$ be the pan graph with $m+2$ vertices. Then, the 2 - domination polynomial $D_2(P_{m,2}, x)$ of $P_{m,2}$ is defined as, $D_2(P_{m,2}, x) = \sum_{i=\gamma_2(P_{m,2})}^{m+2} d_2(P_{m,2}, i) x^i$, where $\gamma_2(P_{m,2})$ is the 2 - domination number of $P_{m,2}$.*

Theorem 3.2. *For every $m \geq 5, D_2(P_{m,2}, x) = x[D_2(P_{m-1,2}, x) + D_2(P_{m-2,2}, x)]$ with initial values $D_2(P_3, 2, x) = 2x^3 + x^4 + x^5 D_2(P_4, 2, x) = x^3 + 5x^4 + 5x^5 + x^6$*

Proof. We have $d_2(P_{m,2}, i) = d_2(P_{m-1,2}, i-1) + d_2(P_{m-2,2}, i-1)$. Therefore, $d_2(P_{m,2}, i)x^i = d_2(P_{m-1,2}, i-1)x^i + d_2(P_{m-2,2}, i-1)x^i$
$\sum d_2(P_{m,2}, i)x^i = \sum d_2(P_{m-1,2}, i-1)x^i + \sum d_2(P_{m-2,2}, i-1)x^i$
$\sum d_2(P_{m,2}, i)x^i = x \sum d_2(P_{m-1,2}, i-1)xi-1 + x \sum d_2(P_{m-2,2}, i-1)x^{i-1}$
$D_2(P_{m,2}, x) = xD_2(P_{m-1,2}, x) + xD_2(P_{m-2,2}, x)$
$D_2(P_{m,2}, x) = x[D_2(P_{m-1,2}, x) + D_2(P_{m-2,2}, x)]$
with the initial values $D_2(P_3, 2, x) = 2x^3 + x^4 + x^5, D_2(P_4, 2, x) = x^3 + 5x^4 + 5x^5 + x^6$

i	3	4	5	6	7	8	9	10	11	12	13
$P_{3,2}$	2	4	1								
$P_{4,2}$	1	5	5	1							
$P_{5,2}$	0	3	9	6	1						
$P_{6,2}$	0	1	8	14	7	1					
$P_{7,2}$	0	0	4	17	20	8	1				
$P_{8,2}$	0	0	1	12	31	27	9	1			
$P_{9,2}$	0	0	0	5	29	51	35	10	1		
$P_{10,2}$	0	0	0	1	17	60	78	44	11	1	
$P_{11,2}$	0	0	0	0	6	46	111	113	54	12	1

□

Theorem 3.3. *The following properties hold for the coefficients of $D_2(P_{m,2}, x)$ for all m:*

1. $d_2(P_{m,2}, m+2) = 1$, *for every* $m \geq 3$.
2. $d_2(P_{m,2}, m+1) = m+1$, *for every* $m \geq 3$.
3. $d_2(P_{m,2}, m) = \frac{1}{2}[m^2 - m - 2]$, *for every* $m \geq 3$.
4. $d_2(P_{m,2}, m-1) = \frac{1}{6}[m^3 - 6m^2 + 5m + 18]$, *for every* $m \geq 4$.
5. $d_2(P_{m,2}, m-2 = \frac{1}{24}[m^4 - 14m^3 + 59m^2 - 22m - 240]$, *for every* $m \geq 6$.
6. $d_2(P_{2m}, 2, m+1) = 1$, *for every* $m \geq 2$.
7. $d_2(P_{2m-3}, 2, m) = m-1$, *for every* $m \geq 3$.

References

[1] Adriana Hansberg, Lutz Volkmann, On graphs with equal domination and 2- domination numbers, *Discrete Mathematics,* 308(2008), 2277 - 2281.

[2] S. Alikhani and Y.H. Peng, Dominating sets and Domination Polynomials of paths, *International Journal of Mathematics and Mathematical Science,* 1(2)(2009), 1-10.

[3] A. Vijayan and Lal Gipson, Dominating sets and Domination Polynomials of square of paths, *Open Journal of Discrete Mathematics*, 3(1)(2013), 60-69.

[4] A. Vijayan and T. Anitha Baby, Connect Total Dominating sets and connected Total Domination Polynomials of square of paths, *International Journal of Mathematics Trends and Technology*, 11 (1)(2014), 56-63.

[5] P.C. Priyanka Nair, T. Anitha Baby, V.M. Arul Flower Mary, 2-Dominating sets and 2-Domination Polynomials of Paths, *Journal of Shanghai Jiaotong University* 16(2020), 42-51.

[6] P.C. Priyanka Nair, T. Anitha Baby, 2-Dominating sets and 2-Domination Polynomials of Cycles, *Adalya Journal,* 9(11)(2020), 182-194.

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