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Fixed point theorems in S- metric spaces

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Abstract

Sedghi et al. [7,8] introduced $S-$metric space and established some fixed point theorems for a selfmapping on a complete $S-$metric space. In this paper, we prove some fixed point theorems for surjection satisfying various expansive type conditions in the setting of an $S-$metric space.

Keywords

$S-$Metric spaces, surjection, expansive mapping, fixed point.

AMS Subject Classification

30L05.

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1. Introduction

Metric spaces are very important in mathematics and applied sciences. Sedghi et al. have introduced $S-$ metric space. The $S-$ metric space is a space with three dimensions. The study of expansive mappings is very interesting research area of fixed point theory. In 1984, Wang et.al introduced the concept of expanding mappings and proved some fixed point theorems in complete metric spaces. In 1992, Daffer and Kaneko defined an expanding condition for a pair of mappings and proved some common fixed point theorems for two mappings in complete metric spaces. Sedghi, N. Shobe and A. Aliouche have introduced the notion of an $S-$metric space. Also they have proved properties of $S-$ metric spaces and some fixed point theorems for a self map on an $S-$metric spaces. Ozgur and Tas discussed new contractive conditions of intergral type on complete $S-$metric spaces. In this paper, we prove some fixed point theorems for surjection satisfying various expansive type conditions in the setting of an $S-$ metric space.

2. Preliminaries

Definition 2.1. *Let X be a non empty set. A generalized metric ($S-$ Metric) on X is a function $S : X^3 \to [0,\infty)$ that satisfies the following conditions for each $x,y,z,a \in X$*

1. $S(x,y,z) \geq 0$
2. $S(x,y,z) = 0 \iff x = y = z$
3. $S(x,y,z) \leq S(x,x,a)+S(y,y,a)+S(z,z,a)$
   The pair (X,S) is called generalized metric ($S-$Metric) space.

Example 2.2. .

1. *Let $X = R^n$ and $||.||$ a norm on X. Then*
   $S(x,y,z) = ||y+z-2x||+||y-z||$ *is an $S-$Metric space.*
2. *Let X be a nonempty set and d be an ordinary metric on X. Then*
   $S(x,y,z) = d(x,z)+d(y,z)$ *is an $S-$Metric space. where d is an ordinary metric on X*

Remark 2.3. *Let (X,S) be an S-Metric space.Then we have* $S(x,x,y) = S(y,y,x)$
For
(i) $S(x,x,y) \leq S(x,x,x)+S(x,x,x)+S(y,y,x) = S(y,y,x)$
(ii) $S(y,y,x) \leq S(y,y,y)+S(y,y,y)+S(x,x,y) = S(x,x,y)$
By (i) and (ii) $S(x,x,y) = S(y,y,x)$

Definition 2.4. *A sequence $\{x_n\}$ in X converges to x if and only if $S(x_n,x_n,x) = S(x,x,x_n) \to 0$ as $n \to \infty$*

Definition 2.5. *A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\varepsilon > 0$ there exists $n_0 \in N$ such that $S(x_n,x_n,x_m) < \varepsilon$ for each $n,m \geq n_0$. The $S-$Metric space (X,S) is said to complete if every Cauchy sequence is convergent.*

Definition 2.6. *A point x in X is a fixed point of the map $T : X \to X$ if $Tx = x$.*

3. Main results

Theorem 3.1. *Let (X,S) be a complete $S-$ Metric spaces. Let T be a surjective map from X into X such that*
$S(Tx,Tx,Ty)+mS(Tx,Tx,y) \geq aS(x,x,y)+b \max\{S(Tx,Tx,x),S(Ty,Ty,x)\}$ ....(3.1.1) *for all $x,y \in X$, where $a,b,m > 0, a > m$ and $a+b-3m > 1$. Then T has a unique fixed point.*

Proof. Let x_0 be an arbitrary point in X
Define a sequence $\{x_n\}$ in X as

$$Tx_n = x_{n-1} \text{ for } n = 1,2,3\ldots.$$

Put $x = x_n$ and $y = x_{n+1}$ in (3.1.1)

$S(Tx_n,Tx_n,Tx_{n+1})+mS(Tx_n,Tx_n,x_{n+1}) \geq aS(x_n,x_n,x_{n+1})+ +b \max \{S(Tx_n,Tx_n,x_n),S(Tx_{n+1},Tx_{n+1},x_n)\}$

$S(x_{n-1},x_{n-1},x_n) + mS(x_{n-1},x_{n-1},x_{n+1}) \geq aS(x_n,x_n,x_{n+1}) + b \max \{S(x_{n-1},x_{n-1},x_n),S(x_n,x_n,x_n)\}$

$= aS(x_n,x_n,x_{n+1})+bS(x_{n-1},x_{n-1},x_n)$

$S(x_{n-1},x_{n-1},x_n)(1-b) \geq aS(x_n,x_n,x_{n+1})-mS(x_{n-1},x_{n-1},x_{n+1})$

$\geq aS(x_n,x_n,x_{n+1})-m[2S(x_{n-1},x_{n-1},x_n) + S(x_{n+1},x_{n+1},x_n)]$

$= aS(x_n,x_n,x_{n+1})-m[2S(x_{n-1},x_{n-1},x_n) + S(x_n,x_n,x_{n+1})]$

$= aS(x_n,x_n,x_{n+1})-2mS(x_{n-1},x_{n-1},x_n) - mS(x_n,x_n,x_{n+1})$

$S(x_{n-1},x_{n-1},x_n)(1-b+2m) \geq (a-m)S(x_n,x_n,x_{n+1})$

$S(x_n,x_n,x_{n+1}) \leq \frac{(1-b+2m)}{(a-m)}S(x_{n-1},x_{n-1},x_n)$

$= kS(x_{n-1},x_{n-1},x_n)$ where $k = \frac{(1-b+2m)}{(a-m)} < 1$ as $a+b-3m > 1$

$\leq k^2S(x_{n-2},x_{n-2},x_{n-1})$

$\vdots$

$\leq k^nS(x_0,x_0,x_1)$

$\to 0$ as $n \to \infty [\because 0 < k < 1]$

Now we shall prove that $\{x_n\}$ is a Cauchy sequence in X.
Let $m > n > n_0$ for some $n_0 \in N$
Now

$S(x_n,x_n,x_m) \leq 2S(x_n,x_n,x_{n+1})+S(x_m,x_m,x_{n+1})$

$= 2S(x_n,x_n,x_{n+1})+S(x_{n+1},x_{n+1},x_m)$

$\leq 2S(x_n,x_n,x_{n+1})+\cdots+2S(x_{m-2},x_{m-2},x_{m-1})+S(x_{m-1},x_{m-1},x_m)$

$\to 0$ as $m,n \to \infty$

Therefore $S(x_n,x_n,x_m) \to 0$ as $m,n \to \infty$
Therefore $\{x_n\}$ is a Cauchy sequence in X
Since (X,S) is complete, $\{x_n\}$ converges to a point x in X
Since T is surjective there exist a point y in X such that $Ty = x$
Now

$S(x_n,x_n,x) = S(Tx_{n+1},Tx_{n+1},Ty)$

$\geq -mS(Tx_{n+1},Tx_{n+1},y)+aS(x_{n+1},x_{n+1},y)+ +b \max\{S(Tx_{n+1},Tx_{n+1},x_{n+1}),S(Ty,Ty,x_{n+1})\}$

$= -mS(x_n,x_n,y)+aS(x_{n+1},x_{n+1},y)+b \max \{S(x_n,x_n,x_{n+1}),S(x,x,x_{n+1})\}$

Letting $n \to \infty$

$S(x,x,x) \geq -mS(x,x,y)+aS(x,x,y)+b \max\{S(x,x,x),S(x,x,x)\}$

$0 \geq (a-m)S(x,x,y)$

$S(x,x,y) = 0$ as $a > m$

Therefore $x = y$
Hence x is a fixed point of T

Uniqueness :

Let $z \neq x$ be another common fixed point of T Then

$S(x,x,z) = S(Tx,Tx,Tz)$

$\geq -mS(Tx,Tx,z)+aS(x,x,z)+b \max\{S(Tx,Tx,x), S(Tz,Tz,x)\}$

$= -mS(x,x,z)+aS(x,x,z)+b \max\{S(x,x,x),S(z,z,x)\}$

$= -mS(x,x,z)+aS(x,x,z)+b \max\{S(x,x,x),S(x,x,z)\}$

$= -mS(x,x,z)+aS(x,x,z)+bS(x,x,z)$

$= (a+b-m)S(x,x,z)$

$S(x,x,z) > S(x,x,z)$

which is a contradiction.
Thus $x = z$
Hence x is a unique fixed point of T. □

Theorem 3.2. *Let (X,S) be a complete $S-$ metric spaces. Let T be a surjective map from X into X such that*
$S(Tx,Tx,Ty)+mS(Tx,Tx,y) \geq aS(x,x,y)+bS(Tx,Tx,x)+ cS(Ty,Ty,y)$ .... *(3.2.1)*
for all $x,y \in X$, where $a,b,c > 0, a-m > 1$ and $a+b+c-3m > 1$. Then T has a unique fixed point.

Proof. Let x_0 be an arbitrary point in X
Define a sequence $\{x_n\}$ in X as

$$Tx_n = x_{n-1} \text{ for } n = 1,2,3,\ldots$$

Put $x = x_n$ and $y = x_{n+1}$ in (3.2.1)

$S(Tx_n,Tx_n,Tx_{n+1})+mS(Tx_n,Tx_n,x_{n+1}) \geq aS(x_n,x_n,x_{n+1})+ bS(Tx_n,Tx_n,x_n)+cS(Tx_{n+1},Tx_{n+1},x_{n+1})$

$S(x_{n-1},x_{n-1},x_n) + mS(x_{n-1},x_{n-1},x_{n+1}) \geq aS(x_n,x_n,x_{n+1}) + bS(x_{n-1},x_{n-1},x_n)+cS(x_n,x_n,x_{n+1})$

$S(x_{n-1},x_{n-1},x_n)(1-b) \geq (a+c)S(x_n,x_n,x_{n+1})-mS(x_{n-1},x_{n-1},x_{n+1})$

$\geq (a+c)S(x_n,x_n,x_{n+1})-m[2S(x_{n-1},x_{n-1},x_n)+ S(x_{n+1},x_{n+1},x_n)]$

$= (a+c)S(x_n,x_n,x_{n+1})-m[2S(x_{n-1},x_{n-1},x_n)+ S(x_n,x_n,x_{n+1})]$

$= (a+c)S(x_n,x_n,x_{n+1})-2mS(x_{n-1},x_{n-1},x_n)- mS(x_n,x_n,x_{n+1})$

$S(x_{n-1},x_{n-1},x_n)(1-b+2m) \geq (a+c-m)S(x_n,x_n,x_{n+1})$

$S(x_n,x_n,x_{n+1}) \leq \frac{(1-b+2m)}{(a+c-m)}S(x_{n-1},x_{n-1},x_n)$

$= kS(x_{n-1},x_{n-1},x_n)$ where $k = \frac{(1-b+2m)}{(a+c-m)} < 1$ as $a+b+c-3m > 1$

$\leq k^2S(x_{n-2},x_{n-2},x_{n-1})$

$\vdots$

$\leq k^nS(x_0,x_0,x_1)$

$\rightarrow 0$ as $n \rightarrow \infty$ $[\because 0 < k < 1]$

Now we shall prove that $\{x_n\}$ is a Cauchy sequence in X

Let $m > n > n_0$ for some $n_0 \in N$

Now

$$S(x_n,x_n,x_m) \leq 2S(x_n,x_n,x_{n+1}) + S(x_m,x_m,x_{n+1})$$
$$= 2S(x_n,x_n,x_{n+1}) + S(x_{n+1},x_{n+1},x_m)$$
$$\leq 2S(x_n,x_n,x_{n+1}) + \cdots + 2S(x_{m-2},x_{m-2},x_{m-1}) + S(x_{m-1},x_{m-1},x_m)$$
$$\rightarrow 0 \text{ as } m,n \rightarrow \infty$$

Therefore $S(x_n,x_n,x_m) \rightarrow 0$ as $m,n \rightarrow \infty$

Therefore $\{x_n\}$ is a Cauchy sequence in X

Since (X,S) is complete, $\{x_n\}$ converges to a point x in X

Since T is surjective there exist a point y in X such that $Ty = x$

Now

$$S(x_n,x_n,x) = S(Tx_{n+1},Tx_{n+1},Ty)$$
$$\geq -mS(Tx_{n+1},Tx_{n+1},y) + aS(x_{n+1},x_{n+1},y) + bS(Tx_{n+1},Tx_{n+1},x_{n+1}) + +cS(Ty,Ty,y)$$
$$= -mS(x_n,x_n,y) + aS(x_{n+1},x_{n+1},y) + bS(x_n,x_n,x_{n+1}) + cS(x,x,y)$$

Letting $n \rightarrow \infty$

$$S(x,x,x) \geq -mS(x,x,y) + aS(x,x,y) + bS(x,x,x) + cS(x,x,y)$$
$$0 \geq (a+c-m)S(x,x,y)$$

$S(x,x,y) = 0$ as $a+c > m$

Therefore $x = y$

Hence x is a fixed point of T

Uniqueness:

Let $z \neq x$ be another common fixed point of T

Then

$$S(x,x,z) = S(Tx,Tx,Tz)$$
$$\geq -mS(Tx,Tx,z) + aS(x,x,z) + bS(Tx,Tx,x) + cS(Tz,Tz,z)$$
$$= -mS(x,x,z) + aS(x,x,z) + bS(x,x,x) + cS(z,z,z)$$
$$= -mS(x,x,z) + aS(x,x,z)$$
$$= (a-m)S(x,x,z)$$

$S(x,x,z) > S(x,x,z)$ as $a - m > 1$

which is a contradiction

Thus $x = z$

Hence x is a unique fixed point of T

□

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