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On the radio antipodal geometric mean number of ladder related graphs

M. Giridaran*[1], T. Arputha Jose[2] and E. Anto Jeony[3]

[1] *Department of Mathematics, DMI-St. Eugene University, Lusaka, Zambia.*
[2] *Department of Mathematics, Sri Sivasubramaniya Nadar College of Engineering, Kalavakkam-603110, India.*
[3] *Department of Mathematics, Holy Cross College, Nagercoil, India.*

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Abstract. Let $G(V, E)$ be a graph with vertex set V and edge set E. A radio geometric mean labeling of a connected graph G is a one to one map from the vertex set $V(G)$ to the set of natural numbers N such that for two distinct vertices u and v of G, $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq 1 + diam(G)$, where $d(u,v)$ represents the shortest distance between the vertices u and v and $diam(G)$ represents the diameter of G . Based on the concept of radio geometric mean labeling, a new graph labeling called *radio antipodal geometric mean labeling* is being introduced in this paper. A radio antipodal geometric mean labeling of a graph G is a mapping from the vertex set $V(G)$ to the set of natural numbers N such that for two distinct vertices u and v of G, $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq diam(G)$. If $d(u,v) = diam(G)$, then the vertices u and v can be given the same label and if $d(u,v) \neq diam(G)$ then the vertices u and v should be assigned different labels. The radio antipodal geometric mean number of f, $r_{agmn}(f)$ is the maximum number assigned to any vertex of G. The radio antipodal geometric mean number of G, $r_{agmn}(G)$ is the minimum value taken over all radio antipodal geometric mean labeling f of G. In this paper, the radio antipodal geometric mean number of certain ladder related graphs have been investigated.

AMS Subject Classifications: 05C12, 05C15, 05C78.

Keywords: Radio labeling, Ladder graph, Triangular ladder graph, Circular ladder graph, Pagoda graph.

Contents

1. Introduction

In this paper, the graphs considered are simple, finite and undirected graphs. For definitions not given here, one can refer to [6]. In communication engineering, one of the major problem is *channel* or *frequency assignment problem* where we have to assign frequencies(channels) to different radio transmitters in such a way that the interference between any two radio transmitter is avoided. That is if the radio transmitters are close to each other, then the difference between the channel assigned should be large enough [1]. This problem was converted into a graph coloring problem by William Hale in 1980 [20]. Later graph labeling techniques were also developed to solve this problem. The process of assigning integers to the vertices, edges or to both based on certain condition is known as *graph labeling* [14]. The first paper on graph labeling was presented by A Rosa in 1966 [18] and up

Corresponding author.* Email address: **iamgiridaran@gmail.com (M. Giridaran)

to till date, there are lot of researches going on graph labeling. The main reason is that it has many applications. To list a few among them, graph labeling techniques are useful in coding theory, astronomy, circuit design, communication network addressing, secret sharing [7, 12].

In order to solve the channel assignment problem, the first graph labeling technique was introduced by Jerrold R. Griggs and Roger K. Yeh [11] in the year 1992, known as $L(2,1)$ labeling or distance two labeling. The $L(2,1)$ labeling was defined as follows. Given a real number $d > 0$, an $L_d(2,1)$ - labeling of G is a non-negative real-valued function $f : V(G) \to [0,\infty)$ such that, whenever x and y are two adjacent vertices in V, then $\mid f(x) - f(y) \mid \geq 2d$, and whenever the distance between x and y is 2, then $\mid f(x) - f(y) \mid \geq d$. In the year 2001, Gary Chartrand et al. [4] modified the definition of $L(2,1)$ labeling and introduced a new graph labeling technique called Radio Labeling which was just an extension of the existing $L(2,1)$ labeling. A *radio labeling* of a graph G is a function $f : V(G) \to N$(set of natural numbers) such that, $d(u,v) + |f(u) - f(v)| \geq diam(G) + 1$. It has been proved that finding the radio number of an arbitrary graph is an *NP-complete problem* [13]. Gary Chartrand et al. [5] have also introduced the concept of radio antipodal labeling in the year 2002. A *radio antipodal labeling* of a graph G is a function $f : V(G) \to N$(set of natural numbers) such that, $d(u,v) + |f(u) - f(v)| \geq diam(G)$. The difference between radio labeling and radio antipodal labeling is that the former one is an one to one function whereas the latter one is not since the vertices which are at diametric distance can receive the same label in the latter. From this there are few new graph labeling techniques which were defined by modifying the definition of existing radio and radio antipodal labeling. One can refer to [2, 3, 8, 15, 17, 19] for different types of labeling techniques which were originated from radio labeling and radio antipodal labeling.

The concept of *radio geometric mean labeling* of graphs was first introduced by Hemalatha V et al. [8] in the year 2017. The radio geometric mean labeling of a graph G is a mapping from the vertex set $V(G)$ to the set of natural numbers N such that for two distinct vertices u and v of G, $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq 1 + diam(G)$. The radio geometric mean number of f, $r_{gmn}(f)$ is the maximum number assigned to any vertex of G. The radio geometric mean number of G, $r_{gmn}(G)$ is the minimum value taken over all radio geometric mean labeling f of G. In that work, the authors have studied the radio geometric mean number of some star like graphs [8]. They have also investigated the radio geometric mean number of splitting of star and bistar [9]. The radio geometric mean number of some subdivision graphs have been obtained by Hemalatha V and Mohanaselvi V [10]. Based on the concept of radio geometric mean labeling, a new graph labeling called *radio antipodal geometric mean labeling* have been introduced in this paper by modifying the existing radio geometric mean labeling condition. A radio antipodal geometric mean labeling of a graph G is a mapping from the vertex set $V(G)$ to the set of natural numbers N such that for two distinct vertices u and v of G, $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq diam(G)$. If $d(u,v) = diam(G)$, then the vertices u and v can be given the same label and if $d(u,v) \neq diam(G)$ then the vertices u and v should be assigned different labels. The radio antipodal geometric mean number of f, $r_{agmn}(f)$ is the maximum number assigned to any vertex of G. The radio antipodal geometric mean number of G, $r_{agmn}(G)$ is the minimum value taken over all radio geometric mean labeling f of G, which will be denoted as $ragmn(G)$.

We were motivated to study the radio antipodal geometric mean number of ladder related graphs, since ladder and ladder related graphs have wide range of applications in various fields. To name a few, ladder networks have been useful in electronics, electrical and wireless communication networks [16].

In this paper, the upper bounds of radio antipodal geometric mean number of ladder related graphs have been investigated.

2. Radio antipodal geometric mean number of ladder and triangular ladder graphs

In this section, the radio antipodal geometric mean number of ladder and triangular ladder graph have been obtained.

Definition 2.1. *[14] The Ladder graph denoted by $LG(n)$, is a graph obtained by the Cartesian product of two path graphs P_2 and P_n, $n \geq 2$. The n^{th} dimension of a ladder graph has $2n$ vertices and $3n-2$ edges. The* *diameter of $LG(n)$ is n. See Figure 1.*

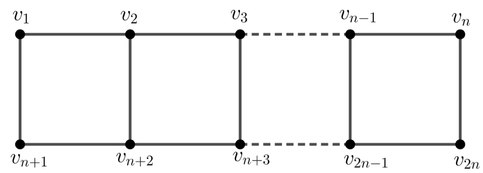

Figure 1: $LG(n)$

Definition 2.2. *[14] A Triangular ladder graph, denoted by $TLG(n)$, is a ladder graph obtained by adding the edges $(v_i, v_{n+i-1}), i = 2, 3, ..., n$. $TLG(n)$ has $2n$ vertices and its diameter is n. See Figure 2.*

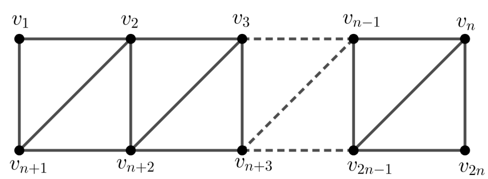

Figure 2: $TLG(n)$

Remark 2.3. *For our convenience, the vertex set of $LG(n)$ and $TLG(n)$ is partitioned into two disjoint sets V_1 and V_2, where $V_1 = \{v_i : 1 \leq i \leq n\}$ and $V_2 = \{v_i : n+1 \leq i \leq 2n\}$.*

Theorem 2.4. *The radio antipodal geometric mean number of ladder graph, $ragmn(LG(n)) \leq 3n - 6, n \geq 4$.*

Proof. Let $\{v_1, v_2, ..., v_n, v_{n+1}, ..., v_{2n}\}$ be the vertices of $LG(n)$.
In this vertex set, the vertices v_1 and v_{2n} are at diametric distance and hence they receive the same labeling. Therefore, $f(v_1) = f(v_{2n})$.
Similarly, the vertices v_n and v_{n+1} are at diametric distance and hence can be given same label, so that $f(v_n) = f(v_{n+1})$.
The remaining $2n - 2$ vertices of $LG(n)$ are labeled by the mapping,

$$f(v_i) = \begin{cases} n + i - 3, 1 \leq i \leq n - 2 \\ n - 3, i = n - 1 \\ 2n - 4, i = n \\ n + i - 5, n + 1 < i < 2n \end{cases} \tag{2.1}$$

Claim. The mapping (2.1) is a valid radio antipodal geometric mean labeling.
Let u, v be any two distinct vertices of $LG(n)$.
Case 1. Let $u, v \in V_1$.

Case 1.1. Let $u = v_i$ and $v = v_j, 1 \leq i, j \leq n-2$.
In this case, $d(u,v) \geq 1$.
By mapping (2.1), we have $f(v_i) = n+i-3$ and $f(v_j) = n+j-3$.
Therefore, $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq$
$1 + \lceil \sqrt{(n+i-3)(n+j-3)} \rceil \geq n$.
Case 1.2. Let $u = v_i, 1 \leq i \leq n-2, v = v_{n-1}$.
By (2.1), we have $f(v_i) = n+i-3$ and $f(v_{n-1}) = n-3$.
Also, $d(u,v) \geq 1$.
This makes, $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq$
$1 + \lceil \sqrt{(n+i-3)(n-3)} \rceil \geq n$.
Case 1.3. Suppose $u = v_i, 1 \leq i \leq n-2$ and $v = v_n$.
Here, $f(v_i) = n+i-3$ and $f(v_n) = 2n-4$. Also $d(u,v) \geq 2$.
Hence, $2 + \lceil \sqrt{(n+i-3)(2n-4)} \rceil \geq n$.
Case 1.4. If $u = v_{n-1}$ and $v = v_n$.
In this case, the distance between the vertices u and v will be 1.
Also, $f(v_{n-1}) = n-3$ and $f(v_n) = 2n-4$.
Hence, $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq n$.
Case 2. Let $u, v \in V_2$.
Case 2.1. Suppose $u = v_i$ and $v = v_j, n+2 \leq i, j \leq 2n-1$.
In this case, $d(u,v) \geq 1$.
By mapping (2.1), we have $f(v_i) = n+i-5$ and $f(v_j) = n+j-5$.
Consequently, $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq$
$1 + \lceil \sqrt{(n+i-5)(n+j-5)} \rceil \geq n$.
Case 2.2. Let $u = v_{n+1}$ and $v = v_{2n}$.
Here, the distance between the vertices u and v will be $n-1$.
By mapping (2.1), we have $f(v_{n+1}) = 2n-4$ and $f(v_{2n}) = n-2$.
Therefore, $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil > n$.
Case 3. Let $u \in V_1$ and $v \in V_2$.
Case 3.1. If $u = v_i, 1 \leq i \leq n-2$ and $v = v_j, n+1 < j < 2n$.
In this case, $d(u,v) \geq 1$.
Here by (2.1), we have $f(v_i) = n+i-3$ and $f(v_j) = n+j-5$.
Therefore, $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq$
$1 + \lceil \sqrt{(n+i-3)(n+j-5)} \rceil \geq n$.
Case 3.2. If $u = v_n$ and $v = v_{n+1}$.
Here $d(u,v) = n$. Also, $f(u) = f(v) = 2n-4$.
Hence, $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq n + \lceil \sqrt{(2n-4)^2} \rceil > n$.
Case 3.3. Suppose $u = v_1$ and $v = v_{2n}$.
In this case, the distance between the vertices u and v will be n. As these two vertices are at diametric distance, $f(u) = f(v) = n-2$.
Therefore, $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq n + \lceil \sqrt{(n-2)^2} \rceil > n$.
Hence, in all the cases it can be seen that the mapping (2.1) satisfies the radio antipodal geometric mean labeling condition, $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq n$.
Therefore, (2.1) is a valid radio antipodal geometric mean labeling.
By the mapping (2.1) the vertex v_{2n-1} receives the maximum label which is given by,
$f(v_{2n-1}) = 3n-6$.
Hence, $ragmn(LG(n)) \leq 3n-6, n \geq 4$ ■

Theorem 2.5. *The radio antipodal geometric mean number of triangular ladder graph,* $ragmn(TLG(n)) \leq 3n-5, n \geq 5$.

Proof. Let $\{v_1, v_2, ..., v_n, v_{n+1}, ..., v_{2n}\}$ be the vertices of $TLG(n)$.
In these vertices v_1 and v_{2n} are at diametric distance and hence they receive the same labeling. Therefore, $f(v_1) = f(v_{2n})$.
The remaining $2n - 1$ vertices of $TLG(n)$ are labeled by the mapping,

$$f(v_i) = \begin{cases} n + i - 3, 1 \leq i \leq n - 2 \\ n - 3, i = n - 1 \\ 2n - 4, i = n \\ n + i - 4, n + 1 \leq i < 2n \end{cases} \tag{2.2}$$

Claim. The mapping (2.2) is a valid radio antipodal geometric mean labeling.
Let u, v be any two vertices of $TLG(n)$.
Case 1. Let $u, v \in V_1$.
Case 1.1. Suppose $u = v_i$ and $v = v_j, 1 \leq i, j \leq n - 2$.
In this case, by (2.2), we have $f(v_i) = n + i - 3$ and $f(v_j) = n + j - 3$.
Also, $d(u, v) \geq 1$.
This assures, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq$
$1 + \lceil \sqrt{(n + i - 3)(n + j - 3)} \rceil \geq n$.
Case 1.2. Let $u = v_i, 1 \leq i \leq n - 2, v = v_{n-1}$.
In this case, $d(u, v) \geq 1$.
Also by (2.2), we have $f(v_i) = n + i - 3$ and $f(v_{n-1}) = n - 3$.
Consequently, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq$
$1 + \lceil \sqrt{(n + i - 3)(n - 3)} \rceil \geq n$.
Case 1.3. If $u = v_i, 1 \leq i \leq n - 2$ and $v = v_n$.
Here, $f(v_i) = n + i - 3$ and $f(v_n) = 2n - 4$. Also $d(u, v) \geq 2$.
Therefore, $2 + \lceil \sqrt{(n + i - 3)(2n - 4)} \rceil \geq n$.
Case 1.4. Let $u = v_{n-1}$ and $v = v_n$.
In this case, $d(u, v) = 1$.
Also, $f(v_{n-1}) = n - 3$ and $f(v_n) = 2n - 4$.
Hence, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq n$.
Case 2. Let $u, v \in V_2$.
Case 2.1. Suppose $u = v_i$ and $v = v_j, n + 1 \leq i, j \leq 2n - 1$.
Here, $d(u, v) \geq 1$.
By mapping (2.2), we have $f(v_i) = n + i - 4$ and $f(v_j) = n + j - 4$.
This gives, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq$
$1 + \lceil \sqrt{(n + i - 4)(n + j - 4)} \rceil \geq n$.
Case 2.2. Let $u = v_i, n + 1 \leq i, j \leq 2n - 1$ and $v = v_{2n}$.
By (2.2), we have $f(v_i) = n + i - 4$ and $f(v_{2n}) = n - 2$.
Also $d(u, v) \geq 1$.
Therefore, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq$
$1 + \lceil \sqrt{(n + i - 4)(n - 2)} \rceil \geq n$.
Case 3. Let $u \in V_1$ and $v \in V_2$.
Case 3.1. If $u = v_i, 1 \leq i \leq n - 2$ and $v = v_j, n + 1 \leq j < 2n$.
In this case, the distance between the vertices u and v will be at least 1.
Here by (2.2), we have $f(v_i) = n + i - 3$ and $f(v_j) = n + j - 4$.
This assures, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq$
$1 + \lceil \sqrt{(n + i - 3)(n + j - 4)} \rceil \geq n$.
Case 3.2. If $u = v_n$ and $v = v_{n+1}$.
Here $d(u, v) = n - 1$. Also, $f(u) = 2n - 4$ and $f(v) = n + i - 4$.

Hence, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq$
$(n-1) + \lceil \sqrt{(n+i-4)(2n-4)} \rceil > n.$
Case 3.3. Suppose $u = v_1$ and $v = v_{2n}$.
As these two vertices are at diametric distance,
$f(u) = f(v) = n - 2.$
Here the distance between the vertices u and v will be n.
Therefore, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq n + \lceil \sqrt{(n-2)^2} \rceil > n.$
Hence, in all the cases it can be seen that the mapping (2.2) satisfies the radio antipodal geometric mean labeling condition.
Therefore, (2.2) is a valid radio antipodal geometric mean labeling.
By the mapping (2.2), the vertex v_{2n-1} receive the maximum label and it is given by, $3n - 5$.
Hence, $ragmn(TLG(n)) \leq 3n - 5, n \geq 5$ ■

3. Radio antipodal geometric mean number of circular ladder and pagoda graphs

In this section, the radio antipodal geometric mean number of circular ladder and pagoda graphs have been investigated.

Definition 3.1. *[21] The circular ladder graph is a graph obtained from the Cartesian product $C_n \times K_2$, where K_2 is the complete graph on two vertices and C_n represents the cycle on n vertices. It is denoted by $CLG(n)$. The n^{th} dimension of $CLG(n)$ is shown in Figure 3.*

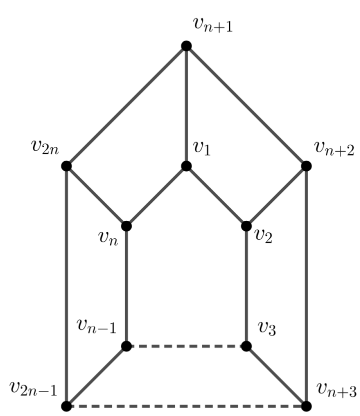

Figure 3: $CLG(n)$

Definition 3.2. *[14] A pagoda graph is a ladder graph formed by adding a vertex v_a in such a way that it is adjacent to the vertices v_1 and v_2. $PG(n)$ has $2n + 1$ vertices and it's diameter is n. See Figure 4.*

Remark 3.3. *For our convenience, the vertices in the internal cycle $\{v_1, v_2, ..., v_n\}$ of $CLG(n)$ will be denoted as C_1 and the vertices in the outer cycle $\{v_{n+1}, v_{n+2}, ..., v_{2n}\}$ as C_2.*

Remark 3.4. *The vertex set of $PG(n)$ is partitioned into two disjoint sets V_1 and V_2, where $V_1 = \{v_{2i-1} : 1 \leq i \leq n\}$ and $V_2 = \{v_{2i} : 1 \leq i \leq n\}$.*

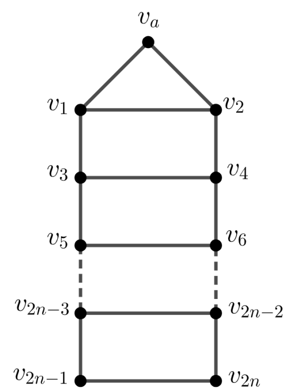

Figure 4: $PG(n)$

Theorem 3.5. *The radio antipodal geometric mean number of circular ladder graph,* $ragmn(CLG(n)) \leq 2n-3, n \equiv (1mod2), n \geq 5.$

Proof. The graph $CLG(n)$ has $2n$ vertices and $3n$ edges. In this $2n$ vertices there exists $\lceil \frac{n}{2} \rceil$ vertices which are at diametric distance and hence these vertices can receive the same label. These vertices are given by,
$f(v_i) = f(v_{n+\lceil \frac{n}{2} \rceil+i-1}), 1 \leq i \leq \lceil \frac{n}{2} \rceil$.
The vertices of $CLG(n)$ are labeled by the mapping,

$$f(v_i) = \lfloor \frac{n}{2} \rfloor + i - 2, 1 \leq i \leq n + \lfloor \frac{n}{2} \rfloor \tag{3.1}$$

Claim. The mapping (3.1) is a valid radio antipodal geometric mean labeling.
Let u, v be any two distinct vertices of $CLG(n)$.
Case 1. Let $u, v \in C_1$.
In this case, $d(u,v) \geq 1$.
By mapping (3.1), we have $f(v_i) = \lfloor \frac{n}{2} \rfloor + i - 2$ and $f(v_j) = \lfloor \frac{n}{2} \rfloor + j - 2$.
Therefore, $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq$
$1 + \lceil \sqrt{(\lfloor \frac{n}{2} \rfloor + i - 2)(\lfloor \frac{n}{2} \rfloor + j - 2)} \rceil \geq d.$
Case 2. If the vertices $u, v \in C_2$.
Case 2.1. Let $u = v_i$ and $v = v_j, n+1 \leq i, j \leq n + \lfloor \frac{n}{2} \rfloor$.
Then, $f(v_i) = \lfloor \frac{n}{2} \rfloor + i - 2$ and $f(v_j) = \lfloor \frac{n}{2} \rfloor + j - 2$ by (3.1).
Also, $d(u,v) \geq 1$.
Hence, $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq d.$
Case 2.2. Suppose $u = v_i$ and $v = v_j, n + \lfloor \frac{n}{2} \rfloor \leq i, j \leq 2n$.
This case will be similar to Case 1.
Therefore, $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq d.$
Case 2.3. If $u = v_i, n+1 \leq n + \lfloor \frac{n}{2} \rfloor$ and $v = v_j, \leq n + \lfloor \frac{n}{2} \rfloor + 1 \leq j \leq 2n$.
In this case, the distance between the vertices u and v will be at least 1.
By (3.1), we have $f(v_i) = \lfloor \frac{n}{2} \rfloor + i - 2$ and $f(v_j) = \lfloor \frac{n}{2} \rfloor + j - 2$.
This guarantees, $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq d.$
Case 3. If $u \in C_1$ and $v \in C_2$.
Case 3.1. Suppose $u = v_i, 1 \leq i \leq n$ and $v = v_j, n+1 \leq j \leq n + \lfloor \frac{n}{2} \rfloor$.
In this case, $d(u,v) \geq 1$.

Also, $f(u) = \lfloor \frac{n}{2} \rfloor + i - 2$ and $f(v_i) = \lfloor \frac{n}{2} \rfloor + j - 2$.
This assures, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq d$.
Case 3.2. Let $u = v_i, 1 \leq i \leq n$ and $v = v_i, n + \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq 2n$.
In this case, the vertices u and v will receive same labels as they are at diametric distance and hence $f(u_i) = f(v_i) = \lfloor \frac{n}{2} \rfloor + i - 2$.
This guarantees $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq d$.

Accordingly, in all the cases it can be seen that the mapping (3.1) satisfies the radio antipodal geometric mean labeling condition, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq d$.
Therefore, (3.1) is a valid radio antipodal geometric mean labeling.
By the mapping (3.1) the vertex $v_{n+\lfloor \frac{n}{2} \rfloor}$ receives the maximum label which is given by,
$f(v_{n+\lfloor \frac{n}{2} \rfloor}) = 2n - 3$.
Hence, $ragmn(CLG(n)) \leq 2n - 3, n \equiv (1 mod 2), n \geq 5$ ∎

Remark 3.6. *It is easy to verify that* $ragmn(CLG(4)) = 4$ *and* $ragmn(CLG(6)) = 6$.

Theorem 3.7. *The radio antipodal geometric mean number of circular ladder graph,* $ragmn(CLG(n)) \leq 2n - 3, n \equiv (0 mod 2), n \geq 8$.

Proof. The graph $CLG(n)$ has $2n$ vertices out of which $\frac{n}{2}$ vertices are at diametric distance. Hence, these vertices can receive same label. These vertices are given as follows,
$f(v_i) = f(v_{n+\frac{n}{2}+i}), 1 \leq i \leq \frac{n}{2}$. The remaining vertices of $CLG(n)$ are labeled by the mapping:

$$f(v_i) = \begin{cases} \frac{n}{2} + i - 2, 1 \leq i \leq n - 2 \\ \frac{n}{2} - 2, i = n - 1 \\ n + \frac{n}{2} - 3, i = n \\ \frac{n}{2} + i - 3, n + 1 \leq i \leq n + \frac{n}{2} \end{cases} \tag{3.2}$$

We now claim that the mapping (3.2) is an valid radio antipodal geometric mean labeling.
Let u, v be any two distinct vertices of $CLG(n)$.
Case 1. If $u, v \in C_1$.
Case 1.1. Let $u = v_i, v = v_j, 1 \leq i, j \leq n - 2$.
In this context, by (3.2) we have $f(u) = \frac{n}{2} + i - 2$ and $f(v) = \frac{n}{2} + j - 2$. Also, $d(u, v) \geq 1$.
This makes, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq d$.
Case 1.2. If $u = v_i, 1 \leq i \leq n - 2$ and $v = v_{n-1}$.
In this instance, by mapping (3.2) $f(u) = \frac{n}{2} + i - 2$ and $f(v_{n-1}) = \frac{n}{2} - 2$.
Further, $d(u, v) \geq 1$.
As a result, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq d$.
Case 1.3. Let $u = v_{n-1}$ and $v = v_n$.
In this case, $d(u, v) = 1$.
Also by (3.2), we have $f(u) = \frac{n}{2} - 2$ and $f(v) = n + \frac{n}{2} - 3$.
Hence, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq d$.
Case 1.4. If $u = v_i, 1 \leq i \leq n - 2$ and $v = v_n$.
In the considered case, $d(u, v) \geq 2$.
By (3.2), $f(u) = \frac{n}{2} + i - 2$ and $f(v) = n + \frac{n}{2} - 3$.
As a consequence of this, we have $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq d$.
Case 2. Let $u, v \in C_2$.
Case 2.1. Suppose $u = v_i, v = v_j, n + 1 \leq i, j \leq n + \frac{n}{2}$.
By (3.2) we have $f(u) = \frac{n}{2} + i - 3$ and $f(v) = \frac{n}{2} + j - 3$.

Further more, $d(u,v) \geq 1$.
Therefore, $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq d$.
Case 2.2. If $u = v_i, v = v_j, n + \frac{n}{2} + 1 \leq i, j \leq 2n$.
This case will be similar to Case 1.1. and hence $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq d$.
Case 2.3. If $u = v_i, n + 1 \leq i \leq n + \frac{n}{2}$ and $v = v_j, n + \frac{n}{2} + 1 \leq j \leq 2n$.
In this case, the distance between the vertices u and v will be at least 1.
Also by (3.2), we have $f(u) = \frac{n}{2} + i - 3$ and $f(v) = \frac{n}{2} + j - 2$.
This assures $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq d$.
Case 3. Let $u \in C_1$ and $v \in C_2$.
Case 3.1. Suppose $u = v_i, 1 \leq i \leq n - 2$ and $v = v_j, n + 1 \leq j \leq n + \frac{n}{2}$.
In the situation under consideration, the distance between the vertices u and v will be at least 1.
By (3.2), $f(u) = \frac{n}{2} + i - 2$ and $f(v) = \frac{n}{2} + j - 3$.
Therefore, $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq d$.
Case 3.2. If $u = v_i, 1 \leq i \leq n - 2$ and $v = v_j, n + \frac{n}{2} + 1 \leq j \leq 2n$.
This case will be similar to Case 1.1.
Therefore, $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq d$.
Case 3.3. Let $u = v_{n-1}$ and $v = v_j, n + 1 \leq i \leq n + \frac{n}{2}$.
In this case, $f(v_{n-1}) = \frac{n}{2} - 2$ and $f(v) = \frac{n}{2} + i - 3$.
It can be seen that $d(u,v) \geq 3$.
Consequently, $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq d$.
Case 3.4. Let $u = v_{n-1}$ and $v = v_j, n + \frac{n}{2} + 1 \leq i \leq 2n$.
This case will be similar to Case 1.2. which guarantees $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq d$.
Case 3.5. If $u = v_n$ and $v = v_j, n + 1 \leq i \leq n + \frac{n}{2}$.
By (3.2), $f(v_n) = n + \frac{n}{2} - 3$ and $f(v) = \frac{n}{2} + i - 3$.
The distance between the vertices u and v will be at least 2.
Consequently, $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq d$.
Case 3.6. Suppose $u = v_n$ and $v = v_j, n + \frac{n}{2} + 1 \leq i \leq 2n$.
This case will be similar to Case 1.4. which assures that $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq d$.
Hence the mapping (3.2) is an valid radio antipodal geometric mean labeling.
By (3.2), the vertex $v_{n+\frac{n}{2}}$ receives the maximum label which is given by $2n - 3$.
Therefore, $ragmn(CLG(n)) \leq 2n - 3, n \equiv (0 mod 2), n \geq 8$ ■

Theorem 3.8. *The radio antipodal geometric mean number of pagoda graph,* $ragmn(PG(n)) \leq 3n - 3, n \geq 3$.

Proof. Let $\{v_a, v_1, v_2, ..., v_n, v_{n+1}, ..., v_{2n}\}$ be the vertices of $PG(n)$.
Let $f(v_a) = n - 2$.
In these $2n + 1$ vertices v_a and v_{2n} are at diametric distance and hence they receive the same labeling. Therefore, $f(v_a) = f(v_{2n})$.
The remaining $2n - 1$ vertices of $PG(n)$ are labeled by the mapping,

$$f(v_i) = n + i - 2, 1 \leq i < 2n. \tag{3.3}$$

Claim. The mapping (3.3) is a valid radio antipodal geometric mean labeling.
Let u, v be any two distinct vertices of $TLG(n)$.
Case 1. Let $u, v \in V_1$.
Case 1.1. If $u = v_i$ and $v = v_j, 1 \leq i, j \leq n$.
In this case, $d(u,v) \geq 1$.
By mapping (3.3), we have $f(v_i) = n + i - 2$ and $f(v_j) = n + j - 2$.
Thus, $d(u,v) + \lceil \sqrt{f(u)f(v)} \rceil \geq n$.
Case 2. Let $u, v \in V_2$.

Case 2.1. Suppose $u = v_i$ and $v = v_j, 1 \leq i, j \leq n - 1$.
In this case, $d(u, v) \geq 1$.
By mapping (3.3), we have $f(v_i) = n + i - 2$ and $f(v_j) = n + j - 2$.
Thus, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq n$.
Case 2.2. Let $u = v_i, 1 \leq i \leq n - 1$ and $v = v_{2n}$.
In this case, by (3.3), we have $f(v_i) = n + i - 2$ and $f(v_{2n}) = n - 2$.
Also, $d(u, v) \geq 1$.
Hence, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil > n$.
Case 3. Let $v_i \in V_1$ and $v_j \in V_2$.
Case 3.1. If $u = v_i, 1 \leq i \leq n$ and $v = v_j, 1 \leq j < n$.
In this case, $d(u, v) \geq 1$.
Here by (3.3), we have $f(v_i) = n + i - 2$ and $f(v_j) = n + j - 2$.
Thus, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq$
$1 + \lceil \sqrt{(n + i - 2)(n + j - 2)} \rceil \geq n$.
Case 3.2. If $u = v_a$ and $v = v_{2n}$.
Here $d(u, v) = n$.
Also, $f(u) = f(v) = n - 2$.
Hence, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq n$.
Case 4. Suppose $u = v_a$ and $v \in V_1$ or $v \in V_2$.
We will have the following two sub cases:
Case 4.1 If $u = v_a$ and $v \in V_1$
By (3.3), $f(u) = n - 2$ and $f(v) = n + i - 2$.
Also, $d(u, v) \geq 1$.
This assures $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq n$.
Case 4.2. Suppose $u = v_a$ and $v \in V_2$
Here, $d(u, v) \geq 1$.
By (3.3), $f(u) = n - 2$ and $f(v) = n + i - 2$.
Therefore, $d(u, v) + \lceil \sqrt{f(u)f(v)} \rceil \geq$
$1 + \lceil \sqrt{(n - 2)(n + i - 2)} \rceil \geq d$.
Hence, in all the cases it can be seen that the mapping (3.3) satisfies the radio antipodal geometric mean labeling condition.
Consequently, (3.3) is a valid radio antipodal geometric mean labeling.
By the mapping (3.3), the vertex v_{2n-1} receive the maximum label and the label is given by, $f(v_{2n-1}) = 3n - 3$.
Hence, $ragmn(PG(n)) \leq 3n - 3, n \geq 3$ ■

4. Conclusion

In this paper, a new graph labeling technique called radio antipodal geometric mean labeling have been introduced. By this technique the span of the given network can be minimized as the diametric opposite vertices can receive same labels. The upper bounds of ladder, triangular ladder, circular ladder and pagoda graphs have been investigated in this paper. This work can be extended further to other communication networks like honeycomb, butterfly, mesh.

References

[1] T. Arputha Jose, Daniel Raj, P. Venugopal and M. Giridaran, Radio Antipodal Mean Number of Quadrilateral Snake Families, *Indian Journal of Science and Technology*, **14(13)**(2021), 1071–1080.

[2] T. Arputha Jose and M. Giridaran, Radial Radio Mean Labeling of Mongolian Tent and Diamond Graphs, *International Journal for Research in Applied Science and Engineering Technology*, **8(7)**(2020), 2078–2083.

[3] T. Arputha Jose, P. Venugopal and M. Giridaran, Radio antipodal mean labeling of Triangular snake families, *Advances in Mathematics: A Scientific Journal*, **9(11)**(2020), 9739–9746.

[4] Gary Chartrand, David Erwin, P. Zhang and Frank Harary, Radio labeling's of graphs, *Bulletin of the Institute of Combinatorics and its Applications*, **33**(2001), 77–85.

[5] Gary Chartrand, David Erwin, and P. Zhang, Radio antipodal colorings of graphs, *Math. Bohem.*, **127**(2002), 57–69.

[6] Douglas Brent West, *Introduction to Graph Theory*, Upper Saddle River, Prentice Hall Ltd., 2001,

[7] M. Giridaran, Application of Super Magic Labeling in Cryptography, *International Journal of Innovative Research in Science, Engineering and Technology*, **9(6)**(2020), 4816–4822.

[8] V. Hemalatha, V. Mohanaselvi and K. Amuthavalli, Radio Geometric Mean Labeling of Some Star Like Graphs, *Journal of Informatics and Mathematical Sciences*, **9(3)**(2017), 969–977.

[9] V. Hemalatha, V. Mohanaselvi, K. Amuthavalli, Radio Geometric Mean Number of Splitting of Star and Bistar, *International Journal of Research in Advent Technology,* **6(7)**(2018), 1696–1700.

[10] V. Hemalatha, V. Mohanaselvi, Radio geometric mean number of some subdivision graphs, *International Journal of Pure and Applied Mathematics*, **113(6)**(2017), 362–374.

[11] Jerrold R. Griggs, Roger K. Yeh, Labelling graphs with a condition at distance 2, *SIAM Journal on Discrete Mathematics*, **5(4)**(1992), 586–595.

[12] Joseph A. Gallian, A dynamic survey of graph labeling, *The Electronic Journal of combinatorics*, **1**(2018).

[13] M. Kchikech, R. Khennoufa and O. Togni, Linear and cyclic radio k-labelings of trees, *Discussiones Mathematicae Graph Theory*, **27(1)**(2007), 105–123.

[14] Kins Yenoke, T. Arputha Jose and P. Venugopal, On the radio antipodal mean number of certain types of ladder graphs, *International Journal of Innovative Research in Science, Engineering and Technology*, **9(6)**(2020), 4607–4614.

[15] K. N. Meera, B. Sooryanarayana, Radio Secure number of a graph, *Electronic Notes in Discrete Mathematics*, **53**(2016), 271–286.

[16] M. I. Moussa, and E. M. Badr, Ladder and subdivision of ladder graphs with pendant edges are odd graceful, *Internat. J. Appl. Graph Theory Wireless Ad hoc Networks and Sensor Networks,(GRAPH-HOC),* **8(1)**(2016), 1–8.

[17] R. Ponraj, S. Sathish Narayanan and R. Kala, Radio mean labeling of a grah, *AKCE International Journal of Graphs and Combinatorics*, **12(2 - 3)**(2015), 224-228.

[18] A. Rosa, On certain valuations of the vertices of a graph, *In Theory of Graphs (Internat. Symposium, Rome)*, (1966), 349–355.

[19] Selvam Avadayappan, M. Bhuvaneshwari and S. Vimalajenifer, The Radial Radio Number and the Clique Number of a Graph, *International Journal of Engineering and Advanced Technology (IJEAT)*, **9**(2019), 1002–1006.

[20] William K. Hale, Frequency Assignment: Theory and Applications, *Proceedings of the IEEE*, **68**(1980), 1497–1514.

[21] Yichao Chen, Jonathan L. Gross, and Toufik Mansour, Total embedding distributions of circular ladders, *Journal of Graph Theory*, **74(1)**(2013), 32–57.