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Analytical solution of linear Volterra integral equations of first and second kind by using Elzaki transform

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Abstract
In recent decades, the use of integral transforms becomes a powerful tool to solve problems in science and technology. Integral transforms have been used to find the solution to problems governed by ordinary and partial differential equations and special types of integral equations. The integral transforms aim to transform a given problem into a simpler form that can be solved easily. The integral transforms are used to find the solution to initial value problems. The integral transforms are also useful in the evaluation of certain integrals and the solution of certain differential equations, partial differential equations, and integral equations. In this paper, we have applied a new integral transform, i.e., Elzaki Transform for the solution of linear Volterra integral equation of the first and second kind and some numerical examples discussed for the validity of results.

Keywords
Elzaki transform, Inverse Elzaki transform, Convolution theorem, Volterra integral equation, Linear Volterra integral equation.

AMS Subject Classification
44A10, 44A35, 45D05.

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Contents

1. Introduction

The linear Volterra integral equation of the first kind is an integral equation in which unknown function occurs only inside the integral sign and is given by

$$f(x)=\int_0^x K(x,t)u(t)dt, \tag{1.1}$$

where the unknown function $u(x)$ is to be determined, the kernel $K(x,t)$ and the function $f(x)$ are given real-valued functions.

A linear integral Equation of the second kind is given by

$$u(x)=f(x)+\int_0^x K(x,t)u(t)dt, \tag{1.2}$$

where the unknown function $u(x)$ is to be determined, the kernel $K(x,t)$ and the function $f(x)$ are given real-valued functions.

Definition 1.1. *The Elzaki transform of the function $F(t), t\geq 0$ is defined as*

$$E\{F(t)\}=v\int_0^\infty F(t)e^{(-v/t)}dt=H(v),$$

where E is the Elzaki transform operator.

The Elzaki transform of function $F(t), t\geq 0$ exist if $F(t)$ is piecewise continuous and is of exponential order and is the sufficient conditions for the existence of the Elzaki transform of the function $F(t)$.

Integral transforms have a variety applications in the disciplines of engineering and science such as solving the problems of heat transfer, mass springs, mixing problems, electrical networks, physics, aerodynamics, regular variations in thermodynamics, neural networks, biophysics, blood flow phenomena, electrical circuits, biology, and many more. In fact, the tools Elzaki transform are used in mathematical modeling of many real world problems. Also, some of researchers have applied integral transform in solving the advanced problems of science, engineering and real-life by using new integral transform.

Elzaki et al. [1] defined basic properties of Elzaki transform and there applications. HwaJoon Kim [2] describe the time shifting theorem and convolution for Elzaki transform. Elzaki and Elzaki [3] discussed the connections between Laplace & Elzaki transforms. Elzaki and Elzaki [4] used Elzaki transform is used for solving ordinary differential equation with variable coefficients. The partial differential equations are solved by using Elzaki transform was given by Elzaki and Ezaki [5], Shendkar and Jadhav [6] used Elzaki transform for the solving differential equations, Agarwal [7] studied the Elzaki transform of Bessels functions. Agarwal et al. [8] applied Mahgoub transform studied for solution of linear Volterra integral equations of first kind, Agarwal et.al [9] had studied application of Mahgoub transform for solving linear volterra integral equations of first kind, Agarwal et al. [10] have studied the application of Elzaki transform for solving population growth and decay problems. Agarwal et.al. [11] used Kamal transform for solving linear Volterra integral equations of first kind. Applications of Mohand transform for solving linear Volterra integral equations of first kind studied by Kumar et al. [12], R.B. Thete et.al. [15] have temperature distribution of an inverse steady state thermo-elastic problem of thin rectangular plate by numerical techniques, Tarik M. Elzaki et. al. [16] have investigated solution of Volterra integro differential equation by triple Laplace transform, R.B. Thete et.al. [17] have studied estimation of temperature distribution and thermal stress analysis of composite circular rod by finite element method.

The aim of this work is to establish exact solutions for linear Volterra integral equation of of first and second kind by using Elzaki transform without large computational work.

S.N.	$F(t)$	$E[F(t)]=H(v)$
1.	1	v^2
2.	t	v^3
3.	t^2	$2!v^4$
4.	t^n, $n\in N$	$n!v^{n+2}$
5.	t^n $n>-1$	$\frac{\Gamma(n+1)}{v^{n+2}}$
6.	e^{at}	$\frac{v^2}{1-av}$
7.	$Sin\ at$	$\frac{av^3}{1+a^2v^2}$
8.	$Cos\ at$	$\frac{av^2}{1+a^2v^2}$
9.	$Sinh\ at$	$\frac{av^3}{1-a^2v^2}$
10.	$Cosh\ at$	$\frac{av^2}{1-a^2v^2}$

Table 1. Elzaki transform of some elementary functions

S.N.	$T(v)$	$F(t)=E^{-1}[H(v)]$
1.	v^2	1
2.	v^3	t
3.	v^4	$\frac{t^2}{2!}$
4.	v^{n+2}	$\frac{t^n}{n!}$
5.	$v^{n+2}, n\geq -1$	$\frac{t^n}{\Gamma(n+1)}$
6.	$\frac{v^2}{1-av}$	e^{at}
7.	$\frac{v^3}{1+a^2v^2}$	$\frac{sinat}{a}$
8.	$\frac{av^2}{1+a^2v^2}$	$Cos\ at$
9.	$\frac{v^3}{1-a^2v^2}$	$\frac{sinhat}{a}$
10.	$\frac{av^2}{1-a^2v^2}$	$Cosh\ at$

Table 2. Some elementary functions of Inverse Elzaki transform

2. Basic Results of Elzaki Transform

Linearity Property of Elzaki Transforms:

If $E\{F(t)\}=H(v)$ and $E\{G(t)\}=I(v)$ then, $E\{aF(t)+bG(t)\}=aE\{F(t)\}+bE\{G(t)\}=aH(v)+bI(v)$, where a,b are arbitrary constants.

Inverse Elzaki Transform

If $E[f(t)]=H(v)$ then $f(t)$ is called the inverse Elzaki transform of $H(v)$ and mathematically it can be expressed as $F(t)=E^{-1}[H(v)]$, where E^{-1} is the inverse Elzaki transform operator.

Convolution of Two Functions:

If $F(t)$ and $G(t)$ be any two function of t then Convolution of two functions $F(t)$ and $G(t)$ is denoted by $F(t)*G(t)$ and it is defined as

$$F(t)*G(t)=\int_0^t F(x)G(t-x)dx=\int_0^t F(t-x)G(x)dx.$$

Convolution Theorem for Elzaki Transforms:

If $E\{F(t)\}=H(v)$ and $E\{G(t)\}=I(v)$ then

$$E\{F(t)*G(t)\}=1/vE\{F(t)\}E\{G(t)\}=1/vH(v)I(v).$$

Elzaki transform of Bessel's function of zero order $J_0(t)$:

$$E\{J_0(t)\} = \frac{v^2}{\sqrt{1+v^2}}.$$

Elzaki transform of Bessel's function of order one $J_1(t)$:

$$\mathrm{E}\{J_1(t)\} = v - \frac{v}{\sqrt{1+v^2}}$$

3. Main Results

III-A Elzaki transform for linear Volterra integral equations of first kind:

In this work we will assume that the kernel $K(x,t)$ of equation (1.1) is a convolution type kernel that can be expressed by the difference $(x-t)$, i.e., $K(x,t) = K(x-t)$, where k is certain function of one variables the linear Volterra integral equation of first kind (1.1) can be expressed as

$$f(x) = \int_0^x K(x-t)u(t)dt. \tag{3.1}$$

Applying the Elzaki transform to both sides of (3.1) we get

$$Ef(x) = E\{\int_0^x K(x-t)u(t)dt\}. \tag{3.2}$$

Applying convolution theorem of the Elzaki transform, we have

$$\begin{aligned} &E\{f(x)\} = 1/vE\{K(x)\}E\{u(x)\} \\ &E\{u(x)\} = [(vEf(x))/EK(x)]. \end{aligned} \tag{3.3}$$

Applying Inverse Elzaki transform to both side of equation (3.3), we get

$$\{u(x)\} = E^{(-1)}\left[(vE\{f(x)\})/E\{K(x)\}\right] \tag{3.4}$$

which is the required solution of (3.1).

III-B Elzaki transform for linear Volterra integral equations of second kind:

In this work we will assume that the kernel $K(x,t)$ of equation (1.2) is a convolution type kernel that can be expressed by the difference $(x-t)$, i.e., $K(x,t) = K(x-t)$, where K is certain function of one variables The linear Volterra integral equation of Second kind(1.2) can be expressed as

$$u(x) = f(x) + \int_0^x K(x-t)u(t)dt. \tag{3.5}$$

Applying the Elzaki transform to both sides of (3.5), we get

$$E\{u(x)\} = E\{f(x)\} + E\left\{\int_0^x K(x-t)u(t)dt\right\}$$

Applying convolution theorem of the Elzaki transform, we have

$$\begin{aligned} E\{u(x)\} &= E\{f(x)\} + \frac{1}{v}E\{K(x)\}E\{u(x)\} \\ E\{u(x)\} &= \left[\frac{E\{f(x)\}}{1-\frac{1}{v}E\{K(x)\}}\right]. \end{aligned} \tag{3.6}$$

Applying Inverse Elzaki transform to both side of equation (3.6) we get

$$u(x) = E^{-1}\left[\frac{E\{f(x)\}}{1-\frac{1}{v}E\{K(x)\}}\right]$$

which is the required solution of (3.5).

4. Numerical Illustrations

In this section, some illustrative examples are given in order to demonstrate the effectiveness of Elzaki transform method for solution of linear Volterra integral equations of first kind and second kind.

Illustration-1

Let us consider linear Volterra integral equation (VIE) of first kind

$$x^2 = \int_0^x e^{(x-t)}u(t)dt \tag{4.1}$$

Applying Elzaki transform to both side of equation (4.1) we get

$$E\left\{x^2\right\} = E\left\{\int_0^x e^{(x-t)}u(t)dt\right\}. \tag{4.2}$$

Using convolution theorem of Elzaki transform on (4.2) we have

$$\begin{aligned} &\Rightarrow 2!v^4 = \frac{1}{v}E\left\{e^x\right\}E\{u(x)\} \\ &\Rightarrow E\{u(x)\} = \left[\frac{2!v^5}{\frac{v^2}{1-v}}\right] \\ &\Rightarrow E\{u(x)\} = 2!\left[v^3(1-v)\right] = 2\left[v^3 - v^4\right] \end{aligned} \tag{4.3}$$

Applying Inverse Elzaki transform to both side of equation (4.3) we get

$$\{u(x)\} = 2\left[E^{-1}\left\{v^3\right\} - E^{-1}\left\{v^4\right\}\right]$$

$$\{u(x)\} = 2\left[t - \frac{t^2}{2!}\right]$$

$$\{u(x)\} = \left[2t - t^2\right]. \tag{4.4}$$

Illustration-2

Let us consider linear Volterra integral equation (VIE) of first kind

$$\sin 2x = \int_0^x e^{(x-t)}u(t)dt. \tag{4.5}$$

Applying the Elzaki transform to both sides of (4.5), we have

$$E\{\sin 2x\} = E\left\{\int_0^x e^{(x-t)}u(t)dt\right\}. \tag{4.6}$$

Using convolution theorem of Elzaki transform on (4.6) we have

$$\Rightarrow \frac{2v^3}{1+4v^2} = \frac{1}{v}E\{e^x\}E\{u(x)\}$$

$$\Rightarrow E\{u(x)\} = \left[\frac{2\left(v^2-v^3\right)}{1+4v^2}\right]$$

$$= \left[\frac{2v^2}{1+4v^2} - \frac{2v^3}{1+4v^2}\right]. \tag{4.7}$$

Applying Inverse Elzaki transform to both side of equation (4.7), we get

$$\{u(x)\} = 2\left[E^{-1}\left\{\frac{v^2}{1+4v^2}\right\} - E^{-1}\left\{\frac{v^3}{1+4v^2}\right\}\right]$$

$$\{u(x)\} = [2\cos 2t - \sin 2t]. \tag{4.8}$$

which is exact solution of equation (4.5).

Illustration-3:

Let us consider linear Volterra integral equation (VIE) of first kind

$$\sin x = \int_0^x J_0(x-t)u(t)dt \quad \sin x = J_0(t) * u(t). \tag{4.9}$$

Applying the Elzaki transform to both sides and using convolution theorem, we have

$$E\{\sin x\} = \frac{1}{v}E\{J_0(t)\}E\{u(t)\}$$

$$\Rightarrow \frac{v^3}{1+v^2} = \frac{v^2}{\sqrt{1+v^2}}E\{u(x)\}$$

$$\Rightarrow E\{u(x)\} = \frac{v}{\sqrt{1+v^2}}. \tag{4.10}$$

Applying Inverse Elzaki transform to both side of equation (4.10), we get

$$u(x) = E^{-1}\left[\frac{v}{\sqrt{1+v^2}}\right]$$

$$u(x) = J_0(x) \tag{4.11}$$

which is exact solution of equation (4.9).

Illustration-4:

Let us consider linear Volterra integral equation (VIE)of second kind

$$u(x) = x^2 + \int_0^x u(x)\sin(x-t)dt. \tag{4.12}$$

Applying the Elzaki transform to both sides of (4.12), we have

$$E\{u(x)\} = E\left\{x^2\right\} + E\left\{\int_0^x u(x)\sin(x-t)dt\right\}. \tag{4.13}$$

Using convolution theorem of Elzaki transform on (4.13), we have

$$\Longrightarrow E\{u(x)\} = 2!v^4 + \frac{1}{v}E\{\sin t\}E\{u(x)\}$$

$$\Rightarrow E\{u(x)\} = 2!v^4 + \frac{1}{v}\frac{v^3}{1+v^2}E\{u(x)\}$$

$$\Rightarrow E\{u(x)\} = 2!v^4\left(1+v^2\right)$$

$$\Rightarrow E\{u(x)\} = 2\left[v^4+v^6\right]. \tag{4.14}$$

Applying inverse Elzaki transform to both side of equation (4.14), we get

$$\{u(x)\} = 2E^{-1}\left[v^4\right] + 2E^{-1}\left[v^6\right]$$

$$\{u(x)\} = \left[t^2 + \frac{t^4}{12}\right] \tag{4.15}$$

which is exact solution of equation (4.12).

Illustration-5

Let us consider linear Volterra integral equation (VIE) of second kind

$$u(x) = x + \int_0^x u(x)\sin(x-t)dt. \tag{4.16}$$

Taking the Elzaki transform to both sides of equation (4.16), we get

$$E\{u(x)\} = E\{x\} + E\left\{\int_0^x u(x)\sin(x-t)dt\right\}.$$

By using Elzaki transform of convolution theorem we get

$$\Rightarrow E\{u(x)\} = v^3 + \frac{1}{v}E\{\text{sint}\}E\{u(x)\}$$

$$\Rightarrow E\{u(x)\} = v^3 + \frac{1}{v}\left\{\frac{v^3}{1+v^2}\right\}E\{u(x)\}$$

$$\Rightarrow E\{u(x)\} = v^3\left(1+v^2\right). \tag{4.17}$$

Applying Inverse Elzaki transform to both side of equation (4.17), we get

$$\{u(x)\} = E^{-1}\left[v^3\right] + E^{-1}\left[v^5\right]$$

$$\{u(x)\} = t + \frac{t^3}{3!}$$

$$\{u(x)\} = t + \frac{t^3}{6}$$

which is exact solution of equation (4.16).

Illustration-6

Let us consider linear Volterra integral equation (VIE) of first kind

$$\sin x = \int_0^x \cosh(x-t)u(t)dt. \tag{4.18}$$

Applying the Elzaki transform to both sides of (4.18) and convolution theorem we have

$$E\{\sin x\} = E\left\{\int_0^x \cosh(x-t)u(t)dt\right\}$$

$$E\{\sin x\} = \frac{1}{v}E\{\cosh t\}E\{u(x)\}$$

$$\frac{v^3}{1+v^2} = \frac{1}{v}\frac{v^2}{1-v^2}E\{u(x)\}$$

$$\frac{v^2-v^4}{1+v^2} = E\{u(x)\}. \tag{4.19}$$

Applying Inverse Elzaki transform to both side of equation (4.19), we get

$$\{u(x)\} = E^{-1}\left[\frac{v^2-v^4}{1+v^2}\right] = E^{-1}\left[\frac{2v^2}{1+v^2}\right] - E^{-1}\left[v^2\right]$$

$$\{u(x)\} = 2\cos x - 1$$

which is exact solution of equation (4.18).

5. Conclusion

In this paper, we have successfully applied the Elzaki transform for the solution of the linear Volterra integral equation of the first kind and second kind. In the applications section, the illustrative example shows that the exact solution has been obtained using very less computational work and spending very little time. In the future, the proposed scheme can be applied for other linear Volterra integral equations and linear Fredholm integral equation and their systems.

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