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Connected 2-domination polynomials of some graph operations

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Abstract
In this paper, we derive the connected 2-domination polynomials of some graph operations. The connected 2-domination polynomial of a graph G of order m is the polynomial $D_{c2}(G,x)=\sum_{j=\gamma_{c_2}(G)}^{m} d_{c2}(G,j)x^j$, where $d_{c2}(G,j)$ is the number of connected 2-dominating sets of G of size j and $\gamma_{c2}(G)$ is the connected 2-domination number of G.

Keywords
Corona, connected 2-dominating sets, connected 2-domination polynomials, connected 2-domination number.

AMS Subject Classification
53C05.

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Contents

1. Introduction

Let $G=(V,E)$ be a simple graph of order, $|V|=m$. For any vertex $v \in V$, the open neighbourhood of v is the set $N(v)=\{u \in V / uv \in E\}$ and the closed neighbourhood of v is the set $N[v]=N(v)\cup\{v\}$. For a set $S \subseteq V$, the open neighbourhood of S is $N(S)=\cup_{v\in s}N(v)$ and the closed neighbourhood of S is $N(S)\cup S$.

A set $D \subseteq V$ is a dominating set of G, if $N[D]=V$ or equivalently, every vertex in $V-D$ is adjacent to atleast one vertex in D.

The domination number of a graph G is defined as the cardinality of a minimum dominating set D of vertices in G and is denoted by $\gamma(G)$.

A dominating set D of G is called a connected dominating set if the induced sub-graph $<D>$ is connected.

The connected domination number of a graph G is defined as the cardinality of a minimum connected dominating set D of vertices in G and is denoted by $\gamma_c(G)$.

Definition 1.1. *A walk is called a trail if all the edges appearing in the walk are distinct. It is called a path, if all the vertices are distinct; P_m denotes a path on m vertices. A cycle is a closed trail in which the vertices are all distinct; C_m denotes a cycle on m vertices.*

Definition 1.2. *The complement $\bar{G}$ of G is the graph whose vertex set is $V(G)$ and such that for each pair u,v of vertices of G, uv is an edge of $\bar{G}$ if and only if uv is not an edge of G.*

Definition 1.3. *The complete graph on m vertices, denoted by K_m is the simple graph that contains exactly one edge between each pair of distinct vertices.*

Definition 1.4. *The corona of two graphs G_1 and G_2 is denoted by $G_1 \circ G_2$, formed from one copy of G_1 and $|V(G_1)|$ copies of G_2, where the j^{th} vertex of G_1 is adjacent to every vertex in the j^{th} copy of G_2. The corona, $G\circ K_1$, in particular, is the graph constructed from a copy of G, where for each vertex $v \in V(G)$, a new vertex v' and a pendant edge vv' are added.*

Definition 1.5. *Let G and H be two graphs. G adding H at u and v is defined as the graph with $V(G_u \oplus H_v)=V(G)\cup V(H)$ and $E(G_u \oplus H_v)$=$E(G)\cup E(H)+uv$ and is denoted by $G_u \oplus H_v$. G joining H at u and v denoted by $G_u \odot H_v$ is obtained from $G_u \oplus H_v$ by contracting the edge uv.*

Definition 1.6. *Let G_1 and G_2 be two graphs. The composition of $G_1[G_2]$ is a graph with the vertex set $V_1 \times V_2$ and* *two vertices $(u_1,u_2),(v_1,v_2)$ are adjacent if $u_1 = v_1$ and u_2 is adjacent to v_2 or u_1 is adjacent to v_1.*

2. Connected 2-Domination Polynomials of Some graph Operations

In this section, we state the connected 2- domination polynomial and derive the connected 2- domination polynomials of some graph operations.

Definition 2.1. *Let G be a simple graph of order m with no isolated vertices. A subset $D \subseteq V$ is a 2- dominating set of the graph G if every vertex $v \in V - D$ is adjacent to atleast two vertices in D. A 2-dominating set is called a connected 2-dominating set if the induced subgraph $< D >$ is connected.*

Definition 2.2. *Let $D_{c2}(G, j)$ be the family of connected 2-dominating sets of the graph G with cardinality j. Then the connected 2-domination number of G is defined as the minimum cardinality taken over all connected 2-dominating sets of vertices in G and is denoted by $\gamma_{c2}(G)$.*

Definition 2.3. *Let $D_{c2}(G, j)$ be the family of connected 2-dominating sets of the graph G with cardinality j and let $d_{c2}(G, j) = |D_{c2}(G, j)|$. Then the connected 2-domination polynomial $D_{c2}(G,x)$ of G is defined as $D_{c2}(G,x)=\sum_{j=\gamma_{c2(G)}}^{|V(G)|} d_{c2}(G, j)x^j$, where $\gamma_{c2}(G)$ is the connected 2-domination number of G.*

Theorem 2.4. *The connected 2-domination polynomial of $P_2[K_m]$ is $D_{c_2}(P_2[K_m],x) = (1+x)^{2m} - (1+2mx)$.*

Proof. Let P_2 be the path with order 2 and K_m be the complete graph with order m. Then, $P_2[K_m]$ has $2m$ vertices.

Let $\{v_1, v_2\}$ be the vertices of P_2 and $\{u_1,u_2,...,u_m\}$ be the vertices of K_m.

Then, $V(P_2[K_m]) = \{(v_1,u_1),(v_1,u_2),(v_1,u_3),...,(v_1,u_m),(v_2,u_1)$ $(v_2,u_2),...,(v_2,u_3),...,(v_2,u_m)\}$

The minimum cardinality of $P_2[K_m]$ is $\gamma_{c_2}(P_2[K_m]) = 2$.

There are $\binom{2m}{j}$ possibilities of connected 2-dominating sets of $P_2[K_m]$ of cardinality j.

Hence, $D_{c_2}(P_2[K_m],x) = \sum_{j=\gamma_{c_2}(P_2[K_m])}^{|V(P_2[K_m])|} d_{c_2}(P_2[K_m], j)x^j$

$= \sum_{j=2}^{2m} d_{c_2}(P_2[K_m], j)x^j$

$= \binom{2m}{2}x^2 + \binom{2m}{3}x^3 + \binom{2m}{4}x^4 + ... + \binom{2m}{2m-1}x^{2m-1} + \binom{2m}{2m}x^{2m}$

$= [\sum_{j=0}^{2m} \binom{2m}{j}x^j] - 1 - 2mx$

Hence, $D_{c_2}(P_2[K_m],x) = (1+x)^{2m} - (1+2mx)$. □

Example 2.5. *For the graphs P_2 and K_3 given in Figure 1.3, the graph $P_2[K_3]$ is given in Figure 1.4.*

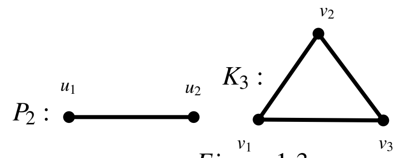


Figure 1.3

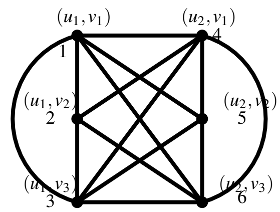


Figure 1.4 $P_2[K_3]$

The connected 2-dominating sets of $P_2[K_3]$ of cardinality 2 are $\{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{1,6\},\{2,3\},\{2,4\},\{2,5\},\{2,6\},\{3,4\},\{3,5\},\{3,6\},\{4,5\},\{4,6\},\{5,6\}\}$.

Therefore, $d_{c_2}(P_2[K_3],2) = 15$.

The connected 2-dominating sets of $P_2[K_3]$ of cardinality 3 are $\{\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,2,6\},\{1,3,4\},\{1,3,5\},\{1,3,6\},\{1,4,5\},\{1,4,6\},\{1,5,6\},\{2,3,4\},\{2,3,5\},\{2,3,6\},\{2,4,5\},\{2,4,6\},\{2,5,6\},\{3,4,5\},\{3,4,6\},\{3,5,6\},\{4,5,6\}\}$.

Therefore, $d_{c_2}(P_2[K_3],3) = 20$.

The connected 2-dominating sets of $P_2[K_3]$ of cardinality 4 are $\{\{1,2,3,4\},\{1,2,3,5\},\{1,2,3,6\},\{1,2,4,5\},\{1,2,4,6\},\{1,2,5,6\},\{1,3,4,5\},\{1,3,4,6\},\{1,3,5,6\},\{1,4,5,6\},\{2,3,4,5\},\{2,3,4,6\},\{2,3,5,6\},\{2,4,5,6\},\{3,4,5,6\}\}$.

Therefore, $d_{c_2}(P_2[K_3],4) = 15$.

The connected 2-dominating sets of $P_2[K_3]$ of cardinality 5 are $\{\{1,2,3,4,5\},\{1,2,3,4,6\},\{1,2,3,5,6\},\{1,2,4,5,6\},\{1,3,4,5,6\},\{2,3,4,5,6\}\}$.

Therefore, $d_{c_2}(P_2[K_3],5) = 6$.

The connected 2-dominating set of $P_2[K_3]$of cardinality 6 is $\{1,2,3,4,5,6\}$.

Therefore, $d_{c_2}(P_2[K_3],6) = 1$.

Since, the minimum cardinality is 2,$\gamma(c_2)(P_2[K_3]) = 2$.

Therefore,$D_{c_2}(P_2[K_3],x) = \sum_{j=\gamma_{c_2}(P_2[K_3])}^{|V(P_2[K_3])|} d_{c_2}(P_2[K_3], j)x^j$

$= \sum_{j=2}^{6} d_{c_2}(P_2[K_3], j)x^j$

$= 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$.

Hence, $D_{c_2}(P_2[K_3],x) = (1+x)^6 - (1+6x)$.

Theorem 2.6. *The connected 2-domination polynomial of $C_m \odot C_n$ is*

$D_{c_2}(C_m \odot C_n,x) = 2(m+n-4)x^{m+n-3} + (m+n-2)x^{m+n-2} + x^{m+n-1}$.

Proof. Let $\{u_1,u_2,...,u_{m-1},u\}$ be the vertex set of C_m and let $\{v_1,v_2,...,v_{n-1},v\}$ be the vertex set of C_n. Therefore, $\{u_1,u_2,...,u_{m-1},u = v,v_1,v_2,...,v_{n-1}\}$ be the vertex set of $C_m \odot C_n$.

Hence, $C_m \odot C_n$ has $m+n-1$ vertices.

There is no connected 2-dominating sets of cardinality less than $m+n-3$.

There are $2(m+n-4)$ connected 2-dominating sets of cardinality $m+n-3$.
Therefore, $d_{c_2}(C_m \odot C_n, m+n-3) = 2(m+n-4)$.
There are $m+n-2$ connected 2-dominating sets of cardinality $m+n-2$.
Therefore, $d_{c_2}(C_m \odot C_n, m+n-2) = m+n-2$.
There is only one connected 2-dominating sets of cardinality $m+n-1$.
Therefore, $d_{c_2}(C_m \odot C_n, m+n-1) = 1$.
Since, the minimum cardinality is $m+n-3, \gamma_{c_2}(C_m \odot C_n) = m+n-3$.
Therefore, $D_{c_2}(C_m \odot C_n, x) = \sum_{j=m+n-3}^{m+n-1} d_{c_2}(C_m \odot C_n, j)x^j$

$$= d_{c_2}(C_m \odot C_n, m+n-3)x^{m+n-3} + d_{c_2}(C_m \odot C_n, m+n-2)x^{m+n-2} + d_{c_2}(C_m \odot C_n, m+n-1)x^{m+n-1}$$

Hence, $D_{c_2}(C_m \odot C_n, x) = 2(m+n-4)x^{m+n-3} + (m+n-2)x^{m+n-2} + x^{m+n-1}$.

□

Example 2.7. *For the graphs C_3 and C_4 given in Figure 1.5, the graph $C_3 \odot C_4$ is given in Figure 1.6.*

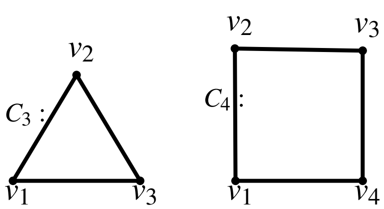


Figure 1.5

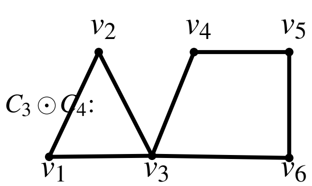


Figure 1.6 $C_3 \odot C_4$

There is no connected 2-dominating sets of $C_3 \odot C_4$ of cardinality 2 and 3.
The connected 2-dominating sets of $C_3 \odot C_4$ of cardinality 4 are $\{\{v_1,v_3,v_4,v_5\},\{v_1,v_3,v_4,v_6\},\{v_1,v_3,v_5,v_6\},\{v_2,v_3,v_4,v_5\},\{v_2,v_3,v_4,v_6\},\{v_2,v_3,v_5,v_6\}\}$.
Therefore, $d_{c_2}(C_3 \odot C_4, 4) = 6$.
The connected 2-dominating sets of $C_3 \odot C_4$ of cardinality 5 are $\{\{v_1,v_2,v_3,v_4,v_5\},\{v_1,v_2,v_3,v_4,v_6\},\{v_1,v_2,v_3,v_5,v_6\},\{v_1,v_3,v_4,v_5,v_6\},\{v_2,v_3,v_4,v_5,v_6\}\}$.
Therefore, $d_{c_2}(C_3 \odot C_4, 5) = 5$.
The connected 2-dominating set of $C_3 \odot C_4$ of cardinality 6 is $\{v_1,v_2,v_3,v_4,v_5,v_6\}$.
Therefore, $d_{c_2}(C_3 \odot C_4, 6) = 1$.
Since, the minimum cardinality is 4, $\gamma_{c_2}(C_3 \odot C_4) = 4$.
Therefore, $D_{c_2}(C_3 \odot C_4, x) = \sum_{j=\gamma_{c_2}(C_3 \odot C_4)}^{|V(C_3 \odot C_4)|} d_{c_2}(C_3 \odot C_4, j)x^j$

$$= \sum_{j=4}^{6} d_{c_2}(C_3 \odot C_4, j)x^j$$
$$= 6x^4 + 5x^5 + x^6.$$

Hence, $D_{c_2}(C_3 \odot C_4, x) = 6x^4 + 5x^5 + x^6$.

Theorem 2.8. *The connected 2-domination polynomial of $C_m \oplus C_n$ is $D_{c_2}(C_m \oplus C_n, x) = 2(m+n-4)x^{m+n-2} + (m+n-2)x^{m+n-1} + x^{m+n}$.*

Proof. Let $\{u_1,u_2,...,u_{m-1},u\}$ be the vertex set of C_m and $\{v_1,v_2,...,v_{n-1},v\}$ be the vertex set of C_n.
Therefore, $\{u_1,u_2,...,u_{m-1},u,v,v_1,v_2,...,v_{n-1}\}$ be the vertex set of $C_m \oplus C_n$.
Hence, $C_m \oplus C_n$ has $m+n$ vertices.
There is no connected 2-dominating sets of cardinality less than $m+n-2$.
There are $2(m+n-4)$ connected 2-dominating sets of cardinality $m+n-2$.
Therefore, $d_{c_2}(C_m \oplus C_n, m+n-2) = 2(m+n-4)$.
There are $m+n-2$ connected 2-dominating sets of cardinality $m+n-1$.
Therefore, $d_{c_2}(C_m \oplus C_n, m+n-1) = m+n-2$.
There is only one connected 2-dominating set of cardinality $m+n$.
Therefore, $d_{c_2}(C_m \oplus C_n, m+n) = 1$.
Since, the minimum cardinality is $m+n-2, \gamma_{c_2}(C_m \oplus C_n) = m+n-2$.
Therefore,
$D_{c_2}(C_m \oplus C_n, x) = \sum_{j=m+n-2}^{m+n} d_{c_2}(C_m \oplus C_n, j)x^j$

$$= d_{c_2}(C_m \oplus C_n, m+n-2)x^{m+n-2} + d_{c_2}(C_m \oplus C_n, m+n-1)x^{m+n-1} + d_{c_2}(C_m \oplus C_n, m+n)x^{m+n}$$

Hence, $D_{c_2}(C_m \oplus C_n, x) = 2(m+n-4)x^{m+n-2} + (m+n-2)x^{m+n-1} + x^{m+n}$. □

Example 2.9. *For the graphs C_3 and C_4 given in Figure 1.7 the graph $C_3 \oplus C_4$ is given in Figure 1.8.*

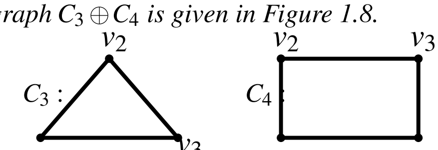


Figure 1.7

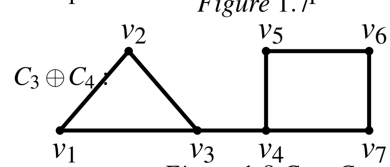


Figure 1.8 $C_3 \oplus C_4$

There is no connected 2-dominating sets of $C_3 \oplus C_4$ of cardinality 2, 3 and 4.
The connected 2-dominating sets of $C_3 \oplus C_4$ of cardinality 5 are
$\{\{v_1,v_3,v_4,v_5,v_6\},\{v_1,v_3,v_4,v_5,v_7\},\{v_1,v_3,v_4,v_6,v_7\},\{v_2,v_3,v_4,v_5,v_6\},\{v_2,v_3,v_4,v_5,v_7\},\{v_2,v_3,v_4,v_6,v_7\}\}$.
Therefore, $d_{c_2}(C_3 \oplus C_{4,5}) = 6$.
The connected 2-dominating sets of $C_3 \oplus C_4$ of cardinality 6 are $\{\{v_1,v_2,v_3,v_4,v_5,v_6\},\{v_1,v_2,v_3,v_4,v_5,v_7\},\{v_1,v_2,v_3,v_4,v_6,$

$v_7\}, \{v_1,v_3,v_4,v_5,v_6,v_7\}, \{v_2,v_3,v_4,v_5,v_6,v_7\}\}$.
Therefore, $d_{c_2}(C_3 \oplus C_{4,6}) = 5$.
The connected 2-dominating set of $C_3 \oplus C_4$ of cardinality 7 is $\{v_1,v_2,v_3,v_4,v_5,v_6,v_7\}$.
Therefore, $d_{c_2}(C_3 \oplus C_{4,7}) = 1$.
Since, the minimum cardinality is 5, $\gamma_{c_2}(C_3 \oplus C_4) = 5$.
Therefore, $D_{c_2}(C_3 \oplus C_4, x) = \sum_{j=\gamma_{c_2}(C_3 \oplus C_4)}^{|V(C_3 \oplus C_4)|} d_{c_2}(C_3 \oplus C_4, j)x^j$
$= \sum_{j=5}^{7} d_{c_2}(C_3 \oplus C_4, j)x^j \quad = 6x^5 + 5x^6 + x^7$.
Hence, $D_{c_2}(C_3 \oplus C_4, x) = 6x^5 + 5x^6 + x^7$.

Theorem 2.10. *Let G be any connected graph with m vertices. Then, $D_{c_2}(G \circ K_1, x) = x^{2m}$.*

Proof. Since, G has m vertices, $G \circ K_1$ has $2m$ vertices.
There is no connected 2-dominating set of cardinality less than $2m$.
Clearly, $\{v_1, v_2, ..., v_{2m}\}$ is the only connected 2-dominating set of $G \circ K_1$.
Therefore, $\gamma_{c_2}(G \circ K_1) = 2m$ and $d_{c_2}(G \circ K_1, 2m) = 1$.
Hence, $D_{c_2}(G \circ K_1, x) = x^{2m}$. □

Example 2.11. *Consider the graph $C_4 \circ K_1$ given in Figure 1.9*

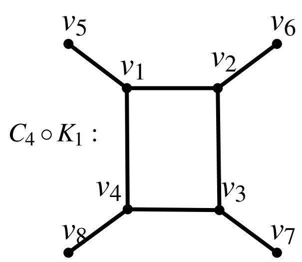


Figure 1.9 $C_4 \circ K$

There is no connected 2- dominating sets of $C_4 \circ K_1$ of cardinality 2,3,4,5,6 and 7.
The connected 2-dominating set of $C_4 \circ K_1$ with cardinality 8 is $\{v_1,v_2,v_3,v_4,v_5,v_6,v_7,v_8\}$.
Therefore, $d_{c_2}(C_4 \circ K_1, 8) = 1$.
The minimum cardinality of $C_4 \circ K_1$ is 8.
Therefore, $\gamma_{c_2}(C_4 \circ K_1, x) = 8$.
Hence, $D_{c2}(C_4 \circ K_1, x) = x^8$.

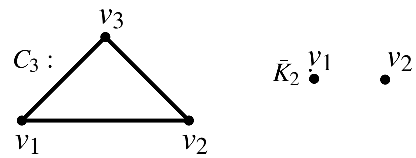
Theorem 2.12. *Let G be a simple graph of order n. Then the connected 2-domination polynomial of $G \circ \bar{K}_m$ is $D_{c_2}(G \circ \bar{K}_m) = x^{n(m+1)}$.*

Proof. G has n vertices and $\bar{K}_m$ has m vertices. $G \circ \bar{K}_m$ has $n(m+1)$ vertices.

Any set S of cardinality less than $n(m+1)$, $< s >$ is not a connected 2-dominating set. Also, the connected 2-domination number of $G \circ \bar{K}_m$ is $n(m+1)$.

Hence, $D_{c2}(G \circ \bar{K}_m, x) = x^{n(m+1)}$. □

Example 2.13. *Consider the graph $C_3 \circ \bar{K}_2$ given in Figure 1.10*



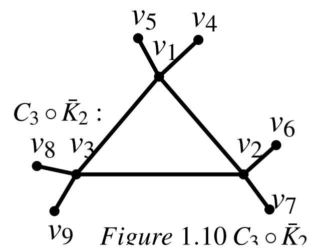


Figure 1.10 $C_3 \circ \bar{K}_2$

There is no connected 2- dominating sets of $C_3 \circ \bar{K}_2$ of cardinality 2,3,4,5,6,7 and 8.
The connected 2-dominating set of $C_3 \circ \bar{K}_2$ with cardinality 9 is $\{v_1,v_2,v_3,v_4,v_5,v_6,v_7,v_8,v_9\}$.
Therefore, $d_{c_2}(C_3 \circ \bar{K}_2, 9) = 1$.
The minimum cardinality of $C_3 \circ \bar{K}_2$ is 9.
Therefore, $\gamma_{c_2}(C_3 \circ \bar{K}_2) = 9$.
Hence, $D_{c2}(C_3 \circ \bar{K}_2, x) = x^9$.

3. Conclusion

In this paper, the connected 2-domination polynomials has been derived by identifying its connected 2-dominating sets. It also help us to characterize the connected 2-dominating sets of cardinality j. We can generalize this study to any power of graphs.

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