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The total geo chromatic number of a graph

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Abstract

A total geo chromatic set of a graph G is a geo chromatic set S_c such that the subgraph induced by S_c has no isolated vertices. The minimum cardinality of a total geo chromatic set of G is the total geo chromatic number of G and is denoted by $\chi_{tg}(G)$. A total geo chromatic set of cardinality $\chi_{tg}(G)$is called a χ_{tg}-set of G. The total geo chromatic number of some standard graphs are determined and some general properties satisfied by this concept are studied.

Keywords

Geodetic number, chromatic number, geo chromatic number, total geo chromatic number, connected geo chromatic number.

AMS Subject Classification

05C12.

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Contents

1. Introduction

Let $G = (V,E)$ be a finite undirected connected graph without multiple edges or loops. The order and size of G are denoted by k and l respectively. For basic graph theoretic terminology we refer to Harary [6]. For vertices p and q in a connected graph G, the distance $d(p,q)$ is the length of a shortest $p-q$ path in G. An $p-q$ path of length $d(p,q)$ is called an $p-q$ geodesic. A vertex x is said to lie on an $p-q$ geodesic P' if x is a vertex of P' including the vertices of p and q. The neighborhood of a vertex x is the set $N(x)$ consisting of all vertices y which are adjacent with x. A vertex x is an extreme vertex of G if the subgraph induced by its neighbors is complete.

The closed interval $I[p,q]$ consists of all vertices lying on some $p-q$ geodesic of G, while for $S \subseteq V, I[S] = \cup_{p,q\in S} I[p,q]$. If $I[S] = V$, then a set S of vertices is a geodetic set and the minimum cardinality of a geodetic set is the geodetic number $g(G)$. A geodetic number of a graph was introduced in [3,4] and further studied in [5,7].

The concept of geo chromatic number was introduced by S. B. Samli and S. R. Chellathurai in [1] and further studied in [2,9]. A geodetic set S is said to be a geo chromatic set S_c of G, if S is both a geodetic set and a chromatic set of G. The minimum cardinality of a geo chromatic set of G is the geo chromatic number of G and is denoted by $\chi_{gc}(G)$. A geo chromatic set S_c is said to be a connected geo chromatic set if the subgraph $< S_c >$ induced by S_c is connected. The minimum cardinality among all connected geo chromatic set of G is the connected geo chromatic number and is denoted by $\chi_{cg}(G)$.

The concept of geo chromatic set of G has motivated us to introduce the new geo chromatic set conception of totoal geo chromatic set. We call the minimum cardinality of a totoal geo chromatic set of G, the totoa geo chromatic number of G.

In this paper we introduce the new concept as total geo chromatic number of a graph. In section 2, we introduce the definition of total geo chromatic number, we determine the total geo chromatic number of some standard graphs In section In section 3, we detetmine general results. In section 3, we illustrate realization of the total geo chromatic number of G. The following theorems used in sequel.

Theorem 1.1. *[7] For any tree T with p end vertices, $g(T) =$*

p.

Theorem 1.2. *[4] Every extreme vertex of a connected graph G belongs to every connected geodetic set of G.*

2. Total Geo Chromatic Number (TGCN)

Definition 2.1. *A total geo chromatic set of a graph G is a geo chromatic set S_c such that the subgraph induced by S_c has no isolated vertices. The minimum cardinality of a total geo chromatic set of G is the total geo chromatic number of G and is denoted by $\chi_{tg}(G)$. A total geo chromatic set of cardinality $\chi_{tg}(G)$ is called a χ_{tg}-set of G.*

Example 2.2. *For the graph G in Figure 2.1, the set of vertices $\{b,d,e\}$ is a minimum geodetic set S' of G and so $g(G) = 3$. Define a proper coloring of G such that the vertices a,d receive same color, say color c_1, the vertices b,e receive same color, say color c_2, and the vertices c,f receive distinct colors, say color c_3, color c_4. Let the vertices which receive color c_1, color c_2, color c_3 and color c_4 belong to the color classes namely A,B,C and D. It is clear that S' is not a chromatic set of G and $\chi_{gc}(G) > 3$. If $c,f \in S'$, then the set becomes $S' \cup \{c,f\}$, which is a chromatic set of G. Therefore $S_c = S' \cup \{c,f\}$ is a minimum geo chromatic set of G and so $\chi_{gc}(G) = 5$. It is clear that $< S_c >$ has no isolated vertices and so $\chi_{tg}(G) = 5$.*

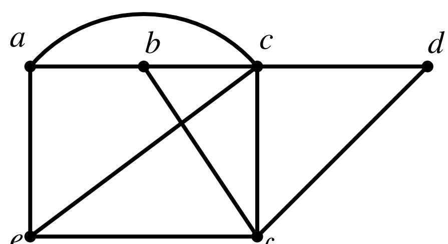

Figure 2.1 : $\chi_{tg}(G) = 5$

3. The Total Geo Chromatic Number for some connected graphs

Theorem 3.1. *For a connected graph P_k,*

$$\chi_{tg}(P_k) = \begin{cases} k \; if \; k = 2,3,4 \\ 4 \; if \; k \geqslant 5 \end{cases} .$$

Proof. Let $G = P_k$. For $k = 2, g(G) = \chi_{gc}(G)$. It is clear that $\chi_{tg}(P_k) = 2$. For $k = 3, g(G) = 2$ but $\chi_{gc}(G) = 3$. Clearly $\chi_{tg}(G) = 3$. For $k = 4, g(G) = \chi_{gc}(G) = 2$. But the subgraph induced by S_c has isolated vertices. Let $S_t = S_c \cup N(S_c)$ has no isolated vertices. Then $\chi_{tg}(G) = 4$. For $k \geqslant 5, g(G) = 2$. For $P_{2k}, g(G) = \chi_{gc}(G)$. But the induced subgraph of S_c is isolated. If $N(S_c) \in S_c$, then S_c becomes $S_t = S_c \cup N(S_c)$ is the total geo chromatic set of G. For $P_{2k+1}, \chi_{gc}(G) = 3$. The induced subgraph contains one isolated vertex. Let the isolated vertex be u. If $N(u) \in S_c$, then $S_t = S_c \cup N(u)$. It follows that $\chi_{tg}(G) = 4$. □

Theorem 3.2. *For a connected graph C_k,*

$$\chi_{tg}(C_k) = \begin{cases} k \; if \; k = 3 \\ k-1 \; if \; k = 4 \\ 4 \; if k \geqslant 5 \end{cases} .$$

Proof. Let $G = C_k$. For $k = 3, g(G) = \chi_{gc}(G) = 3$. Clearly $\chi_{tg}(P_k) = 3$. For $k = 4, g(G) = 2$ but $\chi_{gc}(G) = 3$. The induced subgraph S_c has no isolated vertices so that $g(G) = \chi_{gc}(G) = \chi_{tg}(G)$. For $k \geqslant 5$, the vertices in S_c is not adjacent to any other vertices in S_c. Clearly the induced subgraph S_c has isolated vertices so that S_c is not a total geo chromatic set of G. By choosing the $N(S_c)$, the set $S_t = S \cup N(S_c)$ is the total geo chromatic set of G. Hence $\chi_{tg}(G) = 4$. □

Theorem 3.3. *For a connected graph $K_{1,k-1}, \chi_{tg}(K_{1,k-1}) = k$.*

Proof. Since $\chi_{gc}(K_{1,k-1}) = k$, it is clear that every vertices in $K_{1,k-1}$ is the total geo chromatic set of $K_{1,k-1}$. □

Theorem 3.4. *For a connected graph $K_k, \chi_{tg}(K_k) = k$.*

Proof. Since $g(K_k) = \chi_{gc}(K_k) = k$, obviously $\chi_{tg}(K_k) = k$. □

Theorem 3.5. *For a connected graph $W_k, \chi_{tg}(W_k) = \lceil \frac{k}{2} \rceil + 1$.*

Proof. Let $G = W_k$. Since $\chi_{gc}(G) = \lceil \frac{k}{2} \rceil + 1$, the vertices in the geo chromatic set S_c is connected.so that $< S_c >$ does not have any isolated vertices. Hence it is clear that $\chi_{gc}(G) = \chi_{tg}(G) = \lceil \frac{k}{2} \rceil + 1$. □

Theorem 3.6. *For a connected graph $K_{n,k}$,*

$$\chi_{tg}(K_{n,k}) = \begin{cases} 3 \; if \; n = 2, k \geqslant 2 \\ 4 \; if \; n \geqslant 3, k \geqslant 3 \end{cases} .$$

Proof. Let $G = K_{n,k}$. For $n = 2$ and $k \geqslant 2, g(G) = 2$. The geodetic set S receive same color, It follows that $\chi_{gc}(G) > 2$, Since the neighborhood of S receive diffiernt color other than S. Let one of the neighborhood of S be u. The set $S_c = S \cup \{u\}$ is the geo chromatic set of G, $\chi_{gc}(G) = 3$. Also $< S_c >$ has no isolated vertices. It is clear that $\chi_{gc}(G) = \chi_{tg}(G) = 3$. For $n \geqslant 3, k \geqslant 3, g(G) = \chi_{gc}(G) = 4$. The vertices in geodetic chromatic set of G has not isolated vertices. Hence $\chi_{tg}(G) = 4$. □

4. Bounds and some results on TGCN

Theorem 4.1. *For a connected graph G of order $k, 2 \leqslant g(G) \leqslant \chi_{tg}(G) \leqslant k$.*

Proof. Let G be a connected graph of order k. Since the geodetic set contains at least two vertices of G. There exists a connected graph G with induced subgraph of the geodetic set has isolated vertices. But the total geodetic chromatic set has no isolated vertices. It is clear that $g(G) \leqslant \chi_{tg}(G)$. Every vertices of G is also the total geodetic chromatic set of G, Hence $\chi_{tg}(G) \leqslant k$. □

Remark 4.2. *Let $G = P_2$, then $g(G) = 2$. So that the lower bound of the theorem 2.4.1 is sharp. For $G = K_k$, then $\chi_{tg}(G) = k$. So that the upper bound of the theorem 2.4.1 is sharp and $g(G) = \chi_{tg}(G)$. Also, all the inequalities of the theorem 2.4.1 is strict. For the graph G given in figure 2, $g(G) = 3, \chi_{tg}(G) = 6$. So that $2 < g(G) < \chi_{tg}(G) < k$.*

Theorem 4.3. *For a connected graph G of order $k, 2 \leqslant \chi_{gc}(G) \leqslant \chi_{tg}(G) \leqslant k$.*

Proof. Let G be a connected graph of order k. Since the geodetic set S contains atleast two vertices of G. So the geodetic chromatic set S_c of G contains atleast two vertices, $2 \leqslant \chi_{gc}(G)$. There exists a graph G with $< S_c >$ has isolated vertices. But the total geo chromatic set has no isolated vertices. It is clear that $\chi_{gc}(G) \leqslant \chi_{tg}(G)$. Every vertices of G is also the total geodetic chromatic set of G, So that $\chi_{tg}(G) \leqslant k$. □

Remark 4.4. *Let $G = P_{2k}$, then $\chi_{gc}(G) = 2$. Then the lower bound of the theorem 2.4.3 is sharp. For $G = K_k$, then $\chi_{tg}(G) = k$. So that the upper bound of the theorem 2.4.3 is sharp and for $G = K_{1,k-1}, \chi_{gc}(G) = \chi_{tg}(G)$. Also, all the inequalities of the theorem 2.4.3 is strict. For the graph G given in figure 2, $\chi_{gc}(G) = 4$, $\chi_{tg}(G) = 6$. Hence $2 < \chi_{gc}(G) < \chi_{tg}(G) < k$.*

Theorem 4.5. *Let G be a connected graph of order k. If $\chi_{tg}(G) = 2$ then $g(G) = 2$.*

Proof. Suppose $\chi_{tg}(G) = 2$, then the geo chromatic S_c of G has no isolated vertices. So that $\chi_{gc}(G) = 2$. Clearly the geo chromatic set S_c of G is connected. So that $g(G) = 2$. □

Corollary 4.6. *For any connected graph G of order $k, \chi_{tg}(G) = 2$ iff $G = K_2$.*

Theorem 4.7. *Let G be a connected graph with atleast 2 vertices. Then $\chi_{tg}(G) \leqslant 2g(G)$.*

Proof. Let G be a connected graph of order at least 2. We prove this theorem by two cases.
Case 1:

Suppose the pendent vertices is the geodetic set S of G, then for the total geodetic set contains all the neighborhood of S. So that $g_t(G) \leqslant 2g(G)$. There exists a graph G with either the pendent vertices are in the geodetic chromatic set or the pendent vertices and one of the neighborhood of the pendent vertices are in the geodetic chromatic set of G. If the leaves of G belongs to the geodetic chromatic set then it is clear that $g_t(G) = \chi_{tg}(G)$. So that $\chi_{tg}(G) \leqslant 2g(G)$. If one of the neighborhood of the leaves belongs to the geodetic chromatic set of Gthere is an isolated vertices in $< S_c >$. By replacing the isolated vertices into without isolated vertices we obtain $\chi_{tg}(G) \leqslant 2g(G)$.
Case 2:

Suppose there is no pendent vertices in the geodetic set $S \neq V(G)$ of G, then the induced subgraph S_c has an isolated vertices. By taking all the neighborhood of the geodetic set and replace an isolated vertices into without isolated vertices. Hence 2 times the geodetic set is the total geodetic chromatic set S_t of G. Hence $\chi_{tg}(G) \leqslant 2g(G)$. □

Theorem 4.8. *For any nontrivial tree T, the set of all end vertices and support vertices of T is the unique minimum total geodetic chromatic set of G.*

Proof. The set of all leaves is the unique minimum geodetic set S of T. But it is not true for geodetic chromatic set S_c of T. There exists a tree T with the set of all leaves does not form a geo chromatic set S_c of T. Let us consider two cases.

Case 1:

Suppose the set of leaves is the unique minimum geodetic chromatic set of T, then it is clear that the neighborhood of the leaves and the leaves of T form a total geo chromatic set of T, which is minimum.

Case 2:

Suppose the set of all leaves does not form a geo chromatic set of T, then there exists T with an isolated vertices in the induced subgraph of S_c. Choose the support vertices of $< S_c >$, then $S_t = S_c \cup N(S_c)$ is the unique total geo chromatic set of T.

□

5. Realization Results

Theorem 5.1. *For positive integers r,d and $k \geqslant 4$ with $r \geqslant d \geqslant 2r$, there exists a connected graph G with $rad(G) = r, diam(G) = d$ and $\chi_{tg}(G) = k$.*

Proof. If r=1, then d=1 or 2. For d=1, let $G = K_k$. Then $\chi_{tg}(G) = k$. For d=2, let $G = K_{1,k-1}$. It is clear that $\chi_{tg}(G) = k$. Now, let $r \geqslant 2$. We construct a graph G with the desired properties as follows:

Case 1: $r = d$.

Let $C_{2k} : u_1, u_2, \ldots u_2k$ be the cycle with 2k vertices. Take k/2 copies of P_3. Let G be the graph obtained by joining the pendent vertices of $k/2$ copies of P_3 with the vertices u_1 and u_{2k} of C_{2k}. Let the vertices which are joined with u_1 and u_{2k} of C_{2k} be $\{v_1, v_2, \ldots v_{k-3}\}$. Hence r=d. Now, $S = \{v_1, v_2, \ldots v_{k-3}, u_{\frac{k+2}{2}}\}$ is the geodetic set of G. Define the proper coloring of G such that the set of vertices $\{v_1, v_2, \ldots v_{k-3}\}$ receive the same color and the vertices u_1 and $u_{\frac{k+2}{2}}$ receive two distict colors. This is shown in figure 2.2.

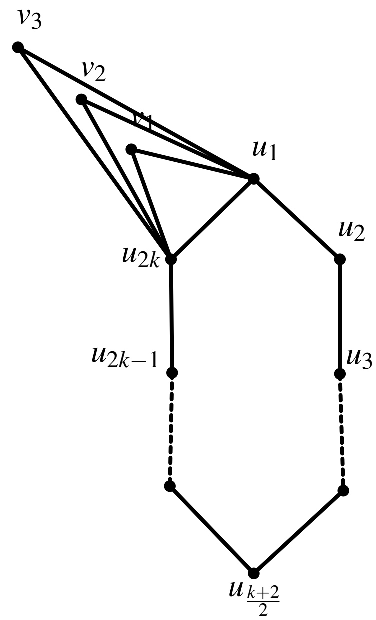

Figure 2.2

Thus $S_t = \{v_1, v_2, .., v_{k-3}, u_1, u_{\frac{k+2}{2}}\}$ is the total geodetic set as well as chromatic set of G. Hence $\chi_{tg}(G) = k$.

Case 2: $r < d$.

Consider the cycle of even order. Let the vertex set of cycle be $\{u_1, u_2, ..., u_{2k}\}$. Take a copy of P_{2k+1} and join with the vertex u_1 of C_{2k}. Let $\{v_1, v_2, ..., v_{2k+1}\}$ be the set of vertices of P_{2k+1}. The graph G obtained by joining one copy of $K_{1,r} : m_1, m_2, ..., m_r$ with the vertex v_{2k}. Hence $rad(G) = r, diam(G) = d$ and $r < d$. Now, $S = \{m_1, m_2, ..., m_r, u_{\frac{k+2}{2}}, v_{2k+1}\}$ is the geodetic set of G. Define a proper coloring of G such that the geodetic set S receive the same color. The graph G is 2-colorable. The set $S_t = \{m_1, m_2, ..., m_r, u_{\frac{k+2}{2}}, u_{\frac{k}{2}}, v_{2k}, v_{2k+1}\}$ is the total geodetic chromatic set of G. So that $\chi_{tg}(G) = k$.

□

Theorem 5.2. *If k,a,b are positive integers such that $4 \leqslant a \leqslant b \leqslant k$, then there exists a connected graph G of order $k, \chi_{tg}(G) = a$ and $\chi_{cg}(G) = b$.*

Proof. We prove this theorem by considering four cases:

Case 1: $a = b = k$.

Let $G = K_k$, then $\chi_{tg}(G) = \chi_{cg}(G)$.

Case 2: $a < b < k$.

Let H be a graph obtained by taking a copy of P_{2k} with the vertex set $\{u_1, u_2, ..., u_{2b}\}$. Take a copy of $K_{1,b-a}$. Let $\{u_{2k}, h_1, ..., h_{b-a}\}$ be the set of vertices of $K_{1,b-a}$. Consider the graph G by joining the vertices $\{v_1, v_2, ..., v_a\}$ with the vertices u_1 and u_3. This is shown in Figure 2.3. The set $S = \{u_1, h_1, ..., h_{b-a}\}$ is the geodetic set of G.

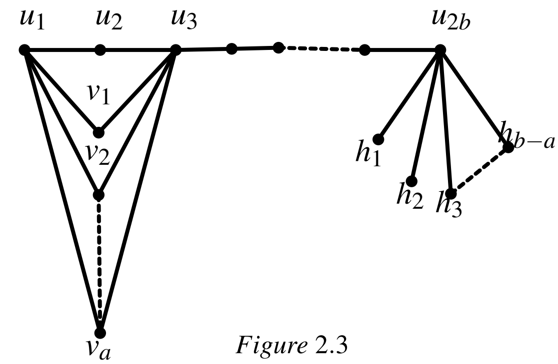

Figure 2.3

Define a proper coloring of G such that the vertices in S receive the same color. For obtaining the geo chromatic set S_c we need the vertex which receive different color. So that $S_c = S \cup \{v_1\}$ is the geo chromatic set of G. $< S_c >$ is not connected and it has an isolated vertices. The set $S_t = \{u_1, v_1, v_2, \ldots v_a, u_2k, h_1, \ldots, h_{b-a}\}$ is the total geo chromatic set of G, $\chi_{tg}(G) = a$. The induced subgraph of S_t is not connected. The set $S_{cg} = \{u_1, u_3, \ldots u_{2k}, v_1, v_2, \ldots v_a, u_{2k}, h_1, \ldots, h_{b-a}\}$ is the connected geo chromatic set of G. Hence $\chi_{cg}(G) = b$.

Case 3: $a = b < k$.

Let G be a graph obtained by taking $k-3$ copies of P_3 and join the pendent vertices of all the P_3 with the pendent vertices of one of the P_3. Let the vertex set of the $k-3$ copies of P_3 be $\{v_1, v_2, h_1, h_2, \ldots, h_{k-3}\}$. Take a copy of $K_{1,a}$ and join with v_1 of P_3. Let the set of vertices of $K_{1,a}$ be $\{v_1, s_1, \ldots, s_a\}$. The set $S = \{v_1, s_1, \ldots, s_a, h_2, \ldots, h_{k-3}, v_2\}$ is the geodetic set of G. Also the set S is the total geo chromatic set and the connected geo chromatic set of G. Hence $a = b$.

Case 4: $a < b = k$.

Consider the graph G by taking the copy of $P_2k : u_1, u_2, \ldots, u_{2k}$. Join a copy of $K_{1,a-1}$ with the vertex u_{2k}. Let the set of vertices of $K_{1,a-1}$ be $\{h_1 h_2, \ldots, h_{a-1}\}$. Now, the set $S = \{u_1, h_1 h_2, \ldots, h_{a-1}\}$ is the geodetic set of G. Define a proper coloring of G such that the geodetic set receive the same color and all the other vertices receive the another one same color. Clearly, the geodetic set is not a chromatic set. By taking one of the neighborhood of S form a geodetic chromatic set of G. The set $S_t = \{u_1, u_2, u_{2k}, h_1 h_2, \ldots, h_{a-1}\}$ is the total geo chromatic set of G, $\chi_{tg}(G) = a$. The set S_t is not the connected geo chromatic set of G. The set of vertices of G is the connected geo chromatic set of G. Hence $\chi_{cg}(G) = k = b$.

□

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