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Common fixed point theorems in $\mathscr{L}$-fuzzy metric space

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Abstract

In this paper, we prove some common fixed point theorems for self mappings in complete $\mathscr{L}$ - fuzzy metric space which is introduced by Saadati, Razani and Adibi.

Keywords

Common fixed point, Complete $\mathscr{L}$ - fuzzy metric space.

AMS Subject Classification

54H25, 47H10.

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Contents

1. Introduction

In 1965, the concept of fuzzy set was introduced by Zadeh [15]. Kramosil and Michalek [10] introduced the concept of fuzzy metric spaces in terms of t- norm. George and Veeramani [6] modified the concept of fuzzy metric spaces introduced by Kramosil and Michalek and defined the Hausdorff topology of fuzzy metric spaces. Using the idea of $\mathscr{L}$ - fuzzy set, Saadati et al [13] introduced the notion of $\mathscr{L}$ - fuzzy metric spaces with the help of continuous t- norm as a generalization of fuzzy metric space due to George and Veeramani. In this paper, we prove some common fixed point theorems for self mappings in complete $\mathscr{L}$ - fuzzy metric space.

2. Preliminaries

Definition 2.1. *Let $\mathscr{L} = (L, \leq_L)$ be a complete lattice and U be a non empty set called universe. An $\mathscr{L}$ - fuzzy set $\mathscr{A}$ on U is defined as a mapping $\mathscr{A} : U \to L$. For each u in U, $\mathscr{A}(u)$ represents the degree (in L) to which u satisfies $\mathscr{A}$.*

Definition 2.2. *A triangular norm(t- norm) on $\mathscr{L}$ is a mapping $\mathscr{T} : L^2 \to L$ satisfying the following conditions:*
(i) $\mathscr{T}(x, 1_{\mathscr{L}}) = x$, for all $x \in L$ (boundary condition)
(ii) $\mathscr{T}(x,y) = \mathscr{T}(y,x)$, for all $x,y \in L$ (commutativity)
(iii) $\mathscr{T}(x, \mathscr{T}(y,z)) = \mathscr{T}(\mathscr{T}(x,y),z)$, for all $x,y,z \in L$ (associativity)
(iv) $x \leq_L x'$ and $y \leq_L y'$ implies $\mathscr{T}(x,y) \leq_L \mathscr{T}(x',y')$, for all $x,x',y,y' \in L$ (monotonicity)

Definition 2.3. *A t-norm $\mathscr{T}$ on $\mathscr{L}$ is said to be continuous if for any $x,y \in L$ and any sequences $\{x_n\}$ and $\{y_n\}$ in L which converge to x and y respectively, then we have $lim_{n\to\infty}\mathscr{T}(x_n,y_n) = \mathscr{T}(x,y)$.*

Example 2.4. *$\mathscr{T}(x,y) = min(x,y)$ and $\mathscr{T}(x,y) = xy$ are two continuous t-norm on $[0,1]$.*

Definition 2.5. *A negation on $\mathscr{L}$ is any decreasing mapping $\mathscr{N} : L \to L$ satisfying $\mathscr{N}(0_{\mathscr{L}}) = 1_{\mathscr{L}}$ and $\mathscr{N}(1_{\mathscr{L}}) = 0_{\mathscr{L}}$. If $\mathscr{N}(\mathscr{N}(x)) = x$, for all $x \in L$, then $\mathscr{N}$ is called an involutive negation. The negation $\mathscr{N}_S$ on $([0,1], \leq)$ defined as $\mathscr{N}_S(x) = 1 - x$, for all $x \in [0,1]$ is called the standard negation on $([0,1], \leq)$.*

Definition 2.6. *The 3-tuple $(X, \mathscr{M}, \mathscr{T})$ is said to be an $\mathscr{L}$-fuzzy metric space if X is an arbitrary non empty set, $\mathscr{T}$ is a continuous t-norm on $\mathscr{L}$ and $\mathscr{M}$ is an $\mathscr{L}$-fuzzy set on $X^2 \times (0, +\infty)$ satisfying the following conditions for every x, y, z in X and t, s in $(0, \infty)$:*
(i) $\mathscr{M}(x,y,t) >_L 0_{\mathscr{L}}$,

(ii) $\mathscr{M}(x,y,t) = 1_{\mathscr{L}}$, *for all* $t > 0$ *if and only if x=y,*
(iii)$\mathscr{M}(x,y,t) = \mathscr{M}(y,x,t)$,
(iv)$\mathscr{T}(\mathscr{M}(x,y,t),\mathscr{M}(y,z,s)) \leq_L \mathscr{M}(x,z,t+s)$,
(v)$\mathscr{M}(x,y,.) : (0,\infty) \to L$ *is continuous.*

Example 2.7. *Let* (X,d) *be a metric space. Define* $\mathscr{T}(a,b) = ab$, *for all* $a,b \in L'$ *and let* $\mathscr{M}$ *be an* $\mathscr{L}$ *- fuzzy set defined as*

$$\mathscr{M}(x,y,t) = \frac{ht^n}{ht^n + md(x,y)}$$

for all $t,h,m,n \in \mathbb{R}^+$. *Then* $(X,\mathscr{M},\mathscr{T})$ *is an* $\mathscr{L}$ *- fuzzy metric space. If h = m = n = 1, then the above equation gives*

$$\mathscr{M}(x,y,t) = \frac{t}{t + d(x,y)}$$

In this case $(X,\mathscr{M},\mathscr{T})$ *is called the standard* $\mathscr{L}$ *- fuzzy metric space. Hence every metric induces an* $\mathscr{L}$ *- fuzzy metric.*

Definition 2.8. *Let* $(X,\mathscr{M},\mathscr{T})$ *be an* $\mathscr{L}$*-fuzzy metric space.For* $t \in (0,\infty)$, *the open ball B(x,r,t) with center* $x \in X$ *and radius* $r \in L \setminus \{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ *is defined by*

$$B(x,r,t) = \{y \in X : \mathscr{M}(x,y,t) >_L \mathscr{N}(r)\}$$

A subset $A \subseteq X$ *is called open if for each* $x \in A$, *there exist* $t > 0$ *and* $r \in L \setminus \{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ *such that* $B(x,r,t) \subseteq A$.

Definition 2.9. *Let* $(X,\mathscr{M},\mathscr{T})$ *be an* $\mathscr{L}$*-fuzzy metric space and* $\{x_n\}$ *be a sequence in X. Then*
(i) $\{x_n\}$ *is called a Cauchy sequence if for each* $\varepsilon \in L \setminus \{0_{\mathscr{L}}\}$ *and* $t > 0$, *there exists* $n_0 \in \mathbb{N}$ *such that for all* $m \geq n \geq n_0$ $(n \geq m \geq n_0)$, $\mathscr{M}(x_m,x_n,t) >_L \mathscr{N}(\varepsilon)$.
(ii) $\{x_n\}$ *is said to be convergent to a point* $x \in X$ *if* $\mathscr{M}(x_n,x,t) = \mathscr{M}(x,x_n,t) \to 1_{\mathscr{L}}$ *as* $n \to \infty$ *for every* $t > 0$.
(iii) A $\mathscr{L}$*-fuzzy metric space is said to be complete if every Cauchy sequence is convergent.*

Lemma 2.10. *Let* $(X,\mathscr{M},\mathscr{T})$ *be an* $\mathscr{L}$*-fuzzy metric space. Then* $\mathscr{M}(x,y,t)$ *is nondecreasing with respect to t, for all x, y in X.*

Lemma 2.11. *Let* $(X,\mathscr{M},\mathscr{T})$ *be an* $\mathscr{L}$*-fuzzy metric space. If we define* $E_{\lambda,\mathscr{M}} : X^2 \to \mathbb{R}^+ \cup \{0\}$ *by*

$$E_{\lambda,\mathscr{M}}(x,y) = inf\{t > 0 : \mathscr{M}(x,y,t) >_L \mathscr{N}(\lambda)\}$$

for each $\lambda \in L \setminus \{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ *and* $x,y \in X$. *Then we have*
(i) For any $\mu \in L \setminus \{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ *there exists* $\lambda \in L \setminus \{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ *such that* $E_{\mu,\mathscr{M}}(x_1,x_n) \leq E_{\mu,\mathscr{M}}(x_1,x_2) + E_{\mu,\mathscr{M}}(x_2,x_3) + ... + E_{\mu,\mathscr{M}}(x_{n-1},x_n)$ *for any* $x_1,x_2,...,x_n \in X$.
(ii) The sequence $\{x_n\}$ *is convergent to x with respect to* $\mathscr{L}$*-fuzzy metric* $\mathscr{M}$ *if and only if* $E_{\lambda,\mathscr{M}}(x_n,x) \to 0$. *Also the sequence* $\{x_n\}$ *is Cauchy with respect to* $\mathscr{L}$*-fuzzy metric* $\mathscr{M}$ *if and only if it is Cauchy with* $E_{\lambda,\mathscr{M}}$.

Definition 2.12. *An* $\mathscr{L}$*-fuzzy metric space* $(X,\mathscr{M},\mathscr{T})$ *has the property(C) if it satisfies condition* $\mathscr{M}(x,y,t) = C$, *for all* $t > 0$ *implies* $C = 1_{\mathscr{L}}$.

Lemma 2.13. *Let* $(X,\mathscr{M},\mathscr{T})$ *be an* $\mathscr{L}$*-fuzzy metric space, which has the property (C). If for all* $x,y \in X, t > 0$ *and for a number* $k \in (0,1)$, $\mathscr{M}(x,y,kt) \geq_L \mathscr{M}(x,y,t)$, *then x = y.*

3. Main Results

Theorem 3.1. *Let* $(X,\mathscr{M},\mathscr{T})$ *be a complete* $\mathscr{L}$ *- fuzzy metric spacespace with property (C). Let A and B be two self mappings of X satisfying*

$$\mathscr{M}^2(Ax,By,kt) \geq_L \mathscr{M}^2(x,y,t) + \mathscr{M}(x,Ax,t)..\mathscr{M}(y,By,t)$$

for all $x,y \in X$, $t > 0$ *and* $k \in (0,1)$.*Then A and B have a unique common fixed point.*

Proof. Let $x_0 \in X$ be an arbitrary point.
Define a sequence $\{x_n\}$ in X by $Ax_{2n} = x_{2n+1}$ and $Bx_{2n+1} = x_{2n+2}$ for $n = 0,1,2,...$
Now we prove $\{x_n\}$ is a cauchy sequence in X.
For $n > 0$,we have

$$\begin{aligned}\mathscr{M}^2(x_{2n+1},x_{2n+2},kt) &= \mathscr{M}^2(Ax_{2n},Bx_{2n+1},kt)\\ &\geq_L \mathscr{M}^2(x_{2n},x_{2n+1},t) + \mathscr{M}(x_{2n},Ax_{2n},t)..\mathscr{M}(x_{2n+1},Bx_{2n+1},t)\\ &= \mathscr{M}^2(x_{2n},x_{2n+1},t) + \mathscr{M}(x_{2n},x_{2n+1},t).\mathscr{M}(x_{2n+1},x_{2n+2},t)\end{aligned}$$

Therefore, $\mathscr{M}^2(x_{2n+1},x_{2n+2},kt) \geq_L \mathscr{M}^2(x_{2n},x_{2n+1},t) + \mathscr{M}(x_{2n},x_{2n+1},t)..\mathscr{M}(x_{2n+1},x_{2n+2},t)$
Dividing both sides by $\mathscr{M}^2(x_{2n},x_{2n+1},t)$ and putting $r = \frac{\mathscr{M}(x_{2n+1},x_{2n+2},kt)}{\mathscr{M}(x_{2n},x_{2n+1},t)}$ we get

$$\begin{aligned}\frac{\mathscr{M}^2(x_{2n+1},x_{2n+2},kt)}{\mathscr{M}^2(x_{2n},x_{2n+1},t)} &\geq_L 1_{\mathscr{L}} + \frac{\mathscr{M}(x_{2n+1},x_{2n+2},t)}{\mathscr{M}(x_{2n},x_{2n+1},t)}\\ &\geq_L 1_{\mathscr{L}} + \frac{\mathscr{M}(x_{2n+1},x_{2n+2},kt)}{\mathscr{M}(x_{2n},x_{2n+1},t)}\end{aligned}$$

Therefore, $r^2 \geq_L 1_{\mathscr{L}} + r$
That is, $r^2 - r - 1_{\mathscr{L}} \geq_L 0_{\mathscr{L}}$
Suppose $r <_L 1_{\mathscr{L}}$
Thus $r^2 - r - 1_{\mathscr{L}} <_L 0_{\mathscr{L}}$ (since $r >_L 0_{\mathscr{L}}$)
which is contradiction to $r^2 - r - 1_{\mathscr{L}} \geq_L 0_{\mathscr{L}}$
Thus $r \geq_L 1_{\mathscr{L}}$
Therefore, $\mathscr{M}(x_{2n+1},x_{2n+2},kt) \geq_L \mathscr{M}(x_{2n},x_{2n+1},t)$
Similarly, $\mathscr{M}(x_{2n+2},x_{2n+3},kt) \geq_L \mathscr{M}(x_{2n+1},x_{2n+2},t)$
Hence $\mathscr{M}(x_{n+1},x_{n+2},kt) \geq_L \mathscr{M}(x_n,x_{n+1},t)$ for all n
By induction we have,
$\mathscr{M}(x_n,x_{n+1},t) \geq_L \mathscr{M}(x_{n-1},x_n,\frac{t}{k}) \geq_L \mathscr{M}(x_{n-2},x_{n-1},\frac{t}{k^2}) \geq_L ... \geq_L \mathscr{M}(x_0,x_1,\frac{t}{k^n})$
For every $\lambda \in L \setminus \{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ we have,

$$\begin{aligned}E_{\lambda,\mathscr{M}}(x_n,x_{n+1}) &= inf\{t > 0 : \mathscr{M}(x_n,x_{n+1},t) \geq_L \mathscr{N}(\lambda)\}\\ &\leq inf\{t > 0 : \mathscr{M}(x_0,x_1,\tfrac{t}{k^n}) \geq_L \mathscr{N}(\lambda)\}\\ &= k^n inf\{t > 0 : \mathscr{M}(x_0,x_1,t) \geq_L \mathscr{N}(\lambda)\}\\ &= k^n E_{\lambda,\mathscr{M}}(x_0,x_1)\end{aligned}$$

Therefore, for every $\mu \in L \setminus \{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ there exists $\gamma \in L \setminus \{0_{\mathscr{L}}, 1_{\mathscr{L}}\}$ such that $E_{\mu,\mathscr{M}}(x_n,x_m) \leq E_{\gamma,\mathscr{M}}(x_n,x_{n+1}) + E_{\gamma,\mathscr{M}}(x_{n+1},x_{n+2}) + ... + E_{\gamma,\mathscr{M}}(x_{m-1},x_m)$
$\leq k^n E_{\gamma,\mathscr{M}}(x_0,x_1) + k^{n+1} E_{\gamma,\mathscr{M}}(x_0,x_1) + ... + k^{m-1} E_{\gamma,\mathscr{M}}(x_0,x_1)$
$\leq E_{\gamma,\mathscr{M}}(x_0,x_1) \sum_{j=n}^{m-1} k^j \to 0$ as $m,n \to \infty$
Therefore by Lemma 2.11(ii), $\{x_n\}$ is a Cauchy sequence in $\mathscr{L}$ - fuzzy metric space.
Since X is complete, $\{x_n\}$ converges to a point $x \in X$.
Now we prove x is a common fixed point of A and B.
Now consider

$$\begin{aligned}\mathscr{M}^2(Ax,x,kt) &= lim_{n\to\infty}\mathscr{M}^2(Ax,x_{2n+2},kt)\\ &= lim_{n\to\infty}\mathscr{M}^2(Ax,Bx_{2n+1},kt)\end{aligned}$$

$\geq_L lim_{n\to\infty}\{\mathscr{M}^2(x,x_{2n+1},t)$
$+\mathscr{M}(x,Ax,t)..\mathscr{M}(x_{2n+1},Bx_{2n+1},t)\}$
$= lim_{n\to\infty}\{\mathscr{M}^2(x,x_{2n+1},t)$
$+\mathscr{M}(x,Ax,t)..\mathscr{M}(x_{2n+1},x_{2n+2},t)\}$
$= \mathscr{M}^2(x,x,t)+\mathscr{M}(x,Ax,t)..\mathscr{M}(x,x,t) = 1_{\mathscr{L}}+\mathscr{M}(Ax,x,t)$
Hence $\mathscr{M}(Ax,x,kt)\geq_L 1_{\mathscr{L}}$ for all $t>0$
That is, $\mathscr{M}(Ax,x,t)\geq_L 1_{\mathscr{L}}$ for all $t>0$
Therefore, $Ax = x$
Similarly, $Bx = x$
Hence x is a common fixed point of A and B.
Uniqueness: Let $y\neq x$ be another common fixed point of A and B.
$\mathscr{M}^2(x,y,kt) = \mathscr{M}^2(Ax,By,kt)$
$\geq_L \mathscr{M}^2(x,y,t)+\mathscr{M}(x,Ax,t)..\mathscr{M}(y,By,t)$
$= \mathscr{M}^2(x,y,t)+\mathscr{M}(x,x,t)..\mathscr{M}(y,y,t)$
$= \mathscr{M}^2(x,y,t)+1_{\mathscr{L}}$
$\geq_L \mathscr{M}^2(x,y,t)$
Hence $\mathscr{M}(x,y,kt)\geq_L \mathscr{M}(x,y,t)$, for all $t>0$
Therefore by Lemma 2.13, $x = y$
Hence x is a unique common fixed point of A and B. □

Theorem 3.2. *Let $(X,\mathscr{M},\mathscr{T})$ be a complete $\mathscr{L}$ - fuzzy metric space with property (C). Let A and B be two self mappings of X satisfying*

$$\mathscr{M}^2(Ax,By,kt)\geq_L \mathscr{M}^2(x,y,t)+\frac{\mathscr{M}(x,Ax,t)..\mathscr{M}(y,By,t)}{\mathscr{M}(x,y,t)}$$

for all $x,y\in X$, $t>0$ and $k\in(0,1)$. Then A and B have a unique common fixed point.

Proof. Let $x_0\in X$ be an arbitrary point.
Define a sequence $\{x_n\}$ in X by $Ax_{2n} = x_{2n+1}$ and $Bx_{2n+1} = x_{2n+2}$ for $n = 0,1,2,...$
Now we prove $\{x_n\}$ is a cauchy sequence in X.
For $n>0$, we have
$\mathscr{M}^2(x_{2n+1},x_{2n+2},kt) = \mathscr{M}^2(Ax_{2n},Bx_{2n+1},kt)$
$\geq_L \mathscr{M}^2(x_{2n},x_{2n+1},t)+\frac{\mathscr{M}(x_{2n},Ax_{2n},t)..\mathscr{M}(x_{2n+1},Bx_{2n+1},t)}{\mathscr{M}(x_{2n},x_{2n+1},t)}$
$= \mathscr{M}^2(x_{2n},x_{2n+1},t)+\frac{\mathscr{M}(x_{2n},x_{2n+1},t)..\mathscr{M}(x_{2n+1},x_{2n+2},t)}{\mathscr{M}(x_{2n},x_{2n+1},t)}$
$= \mathscr{M}^2(x_{2n},x_{2n+1},t)+\mathscr{M}(x_{2n+1},x_{2n+2},t)$
Therefore, $\mathscr{M}^2(x_{2n+1},x_{2n+2},kt)\geq_L \mathscr{M}^2(x_{2n},x_{2n+1},t)$
$+\mathscr{M}(x_{2n+1},x_{2n+2},t)$
Dividing both sides by $\mathscr{M}^2(x_{2n},x_{2n+1},t)$ and putting $r = \frac{\mathscr{M}(x_{2n+1},x_{2n+2},kt)}{\mathscr{M}(x_{2n},x_{2n+1},t)}$ we get
$\frac{\mathscr{M}^2(x_{2n+1},x_{2n+2},kt)}{\mathscr{M}^2(x_{2n},x_{2n+1},t)}\geq_L 1_{\mathscr{L}}+\frac{\mathscr{M}(x_{2n+1},x_{2n+2},t)}{\mathscr{M}^2(x_{2n},x_{2n+1},t)}$
$\geq_L 1_{\mathscr{L}}+\frac{\mathscr{M}(x_{2n+1},x_{2n+2},t)}{\mathscr{M}(x_{2n},x_{2n+1},t)}$
$\geq_L 1_{\mathscr{L}}+\frac{\mathscr{M}(x_{2n+1},x_{2n+2},kt)}{\mathscr{M}(x_{2n},x_{2n+1},t)}$
Therefore, $r^2\geq_L 1_{\mathscr{L}}+r$
That is, $r^2-r-1_{\mathscr{L}}\geq_L 0_{\mathscr{L}}$
Suppose $r<_L 1_{\mathscr{L}}$
Thus $r^2-r-1_{\mathscr{L}}<_L 0_{\mathscr{L}}$ (since $r>_L 0_{\mathscr{L}}$)
which is contradiction to $r^2-r-1_{\mathscr{L}}\geq_L 0_{\mathscr{L}}$
Thus $r\geq_L 1_{\mathscr{L}}$
Therefore, $\mathscr{M}(x_{2n+1},x_{2n+2},kt)\geq_L \mathscr{M}(x_{2n},x_{2n+1},t)$
Similarly, $\mathscr{M}(x_{2n+2},x_{2n+3},kt)\geq_L \mathscr{M}(x_{2n+1},x_{2n+2},t)$
Hence $\mathscr{M}(x_{n+1},x_{n+2},kt)\geq_L \mathscr{M}(x_n,x_{n+1},t)$ for all n
By induction we have,
$\mathscr{M}(x_n,x_{n+1},t)\geq_L \mathscr{M}(x_{n-1},x_n,\frac{t}{k})\geq_L \mathscr{M}(x_{n-2},x_{n-1},\frac{t}{k^2})\geq_L ...\geq_L \mathscr{M}(x_0,x_1,\frac{t}{k^n})$
For every $\lambda\in L\setminus\{0_{\mathscr{L}},1_{\mathscr{L}}\}$ we have,
$E_{\lambda,\mathscr{M}}(x_n,x_{n+1}) = inf\{t>0:\mathscr{M}(x_n,x_{n+1},t)\geq_L \mathscr{N}(\lambda)\}$
$\leq inf\{t>0:\mathscr{M}(x_0,x_1,\frac{t}{k^n})\geq_L \mathscr{N}(\lambda)\}$
$= k^n inf\{t>0:\mathscr{M}(x_0,x_1,t)\geq_L \mathscr{N}(\lambda)\}$
$= k^n E_{\lambda,\mathscr{M}}(x_0,x_1)$
Therefore, for every $\mu\in L\setminus\{0_{\mathscr{L}},1_{\mathscr{L}}\}$ there exists $\gamma\in L\setminus\{0_{\mathscr{L}},1_{\mathscr{L}}\}$ such that $E_{\mu,\mathscr{M}}(x_n,x_m)\leq E_{\gamma,\mathscr{M}}(x_n,x_{n+1})$
$+E_{\gamma,\mathscr{M}}(x_{n+1},x_{n+2})+...+E_{\gamma,\mathscr{M}}(x_{m-1},x_m)$
$\leq k^nE_{\gamma,\mathscr{M}}(x_0,x_1)+k^{n+1}E_{\gamma,\mathscr{M}}(x_0,x_1)+...+k^{m-1}E_{\gamma,\mathscr{M}}(x_0,x_1)$
$\leq E_{\gamma,\mathscr{M}}(x_0,x_1)\sum_{j=n}^{m-1}k^j\to 0$ as $m,n\to\infty$
Therefore by Lemma 2.11(ii), $\{x_n\}$ is a Cauchy sequence in $\mathscr{L}$ - fuzzy metric space.
Since X is complete, $\{x_n\}$ converges to a point $x\in X$.
Now we prove x is a common fixed point of A and B.
Now consider
$\mathscr{M}^2(Ax,x,kt) = lim_{n\to\infty}\mathscr{M}^2(Ax,x_{2n+2},kt)$
$= lim_{n\to\infty}\mathscr{M}^2(Ax,Bx_{2n+1},kt)$
$\geq_L lim_{n\to\infty}\{\mathscr{M}^2(x,x_{2n+1},t)+\frac{\mathscr{M}(x,Ax,t)..\mathscr{M}(x_{2n+1},Bx_{2n+1},t)}{\mathscr{M}(x,x_{2n+1},t)}\}$
$= lim_{n\to\infty}\{\mathscr{M}^2(x,x_{2n+1},t)+\frac{\mathscr{M}(x,Ax,t)..\mathscr{M}(x_{2n+1},x_{2n+2},t)}{\mathscr{M}(x,x_{2n+1},t)}\}$
$= \mathscr{M}^2(x,x,t)+\frac{\mathscr{M}(x,Ax,t)..\mathscr{M}(x,x,t)}{\mathscr{M}(x,x,t)}$
$= 1_{\mathscr{L}}+\mathscr{M}(Ax,x,t)$
Hence $\mathscr{M}^2(Ax,x,kt)\geq_L 1_{\mathscr{L}}$ for all $t>0$
That is, $\mathscr{M}(Ax,x,t)\geq_L 1_{\mathscr{L}}$ for all $t>0$
Therefore, $Ax = x$
Similarly, $Bx = x$
Hence x is a common fixed point of A and B.
Uniqueness: Let $y\neq x$ be another common fixed point of A and B.
Now consider
$\mathscr{M}^2(x,y,kt) = \mathscr{M}^2(Ax,By,kt)$
$\geq_L \mathscr{M}^2(x,y,t)+\frac{\mathscr{M}(x,Ax,t)..\mathscr{M}(y,By,t)}{\mathscr{M}(x,y,t)}$
$= \mathscr{M}^2(x,y,t)+\frac{\mathscr{M}(x,x,t)..\mathscr{M}(y,y,t)}{\mathscr{M}(x,y,t)}$
$= \mathscr{M}^2(x,y,t)+\frac{1_{\mathscr{L}}}{\mathscr{M}(x,y,t)}$
$\geq_L \mathscr{M}^2(x,y,t)$
Hence $\mathscr{M}(x,y,kt)\geq_L \mathscr{M}(x,y,t)$ for all $t>0$
Therefore by Lemma 2.13, $x = y$
Hence x is a unique common fixed point of A and B. □

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