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Group mean cordial labeling of some splitting graphs

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Abstract

Let G be a (p,q) graph and let A be a group. Let $f: V(G) \longrightarrow A$ be a map. For each edge uv assign the label $\left\lfloor \frac{o(f(u))+o(f(v))}{2} \right\rfloor$. Here $o(f(u))$ denotes the order of $f(u)$ as an element of the group A. Let $\mathbb{I}$ be the set of all integers that are labels of the edges of G. f is called a group mean cordial labeling if the following conditions hold:

(1) For $x,y \in A$, $|v_f(x) - v_f(y)| \leq 1$, where $v_f(x)$ is the number of vertices labeled with x .

(2) For $i,j \in \mathbb{I}$, $|e_f(i) - e_f(j)| \leq 1$, where $e_f(i)$ denote the number of edges labeled with i.

A graph with a group mean cordial labeling is called a group mean cordial graph. In this paper, we take A as the group of fourth roots of unity and prove that,the splitting graphs of Path (P_n), Cycle(C_n), Comb($P_n \odot K_1$) and Complete Bipartite graph ($K_{n,n}$ when n is even) are group mean cordial graphs. Also we characterized the group mean cordial labeling of the splitting graph of $K_{1,n}$.

Keywords

Cordial labeling, mean labeling, group mean cordial labeling.

AMS Subject Classification

05C78.

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Contents

1. Introduction

Graphs considered here are finite, undirected and simple.Terms not defined here are used in the sense of Harary [4] and Gallian [3]. Somasundaram and Ponraj [6] introduced the concept of mean labeling of graphs.

Definition 1.1. [6] A graph G with p vertices and q edges is a mean graph if there is an injective function f from the vertices of G to $0,1,2,\ldots,q$ such that when each edge uv is labeled with $\frac{f(u)+f(v)}{2}$ if $f(u)+f(v)$ is even and $\frac{f(u)+f(v)+1}{2}$ if $f(u)+f(v)$ is odd then the resulting edge labels are distinct.

Cahit [2] introduced the concept of cordial labeling.

Definition 1.2. [2] Let $f: V(G) \to \{0,1\}$ be any function. For each edge xy assign the label $|f(x)-f(y)|$. f is called a cordial labeling if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1. Also the number of edges labeled 0 and the number of edges labeled 1 differ by at most 1.

Ponraj et al. [5] introduced mean cordial labeling of graphs.

Definition 1.3. [5] Let f be a function from the vertex set $V(G)$ to $\{0,1,2\}$. For each edge uv assign the label $\left\lceil \frac{f(u)+f(v)}{2} \right\rceil$. f is called a *mean cordial labeling* if $\left|v_f(i) - v_f(j)\right| \leq 1$ and $\left|e_f(i) - e_f(j)\right| \leq 1$, $i,j \in \{0,1,2\}$, where $v_f(x)$ and $e_f(x)$ respectively denote the number of vertices and edges labeled with x $(x = 0,1,2)$. A graph with a mean cordial labeling is called a mean cordial graph.

Athisayanathan et al. [1] introduced the concept of group A cordial labeling.

Definition 1.4. [1] Let A be a group. We denote the order of an element $a \in A$ by $o(a)$. Let $f : V(G) \to A$ be a function. For each edge uv assign the label 1 if $(o(f(u)), o(f(v))) =$ 1or 0 otherwise. f is called a group A Cordial labeling if $|v_f(a) - v_f(b)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$, where $v_f(x)$ and $e_f(n)$ respectively denote the number of vertices labelled with an element x and number of edges labelled with $n(n = 0, 1)$. A graph which admits a group A Cordial labeling is called a group A Cordial graph.

Motivated by these , we define group mean cordial labeling of graphs.

For any real number x, we denoted by $\lfloor x \rfloor$, the greatest integer smaller than or equal to x and by $\lceil x \rceil$, we mean the smallest integer greater than or equal to x.

Definition 1.5. *The Splitting graph of G,$S'(G)$ is obtained from G by adding for each vertex v of G a new vertex v' so that v' is adjacent of every vertex that is adjacent to v.*

2. Main Results

Definition 2.1. *Let G be a (p,q) graph and let A be a group. Let f be a map from $V(G)$ to A. For each edge uv assign the label $\left\lfloor \frac{o(f(u))+o(f(v))}{2} \right\rfloor$. Let $\mathbb{I}$ be the set of all integers that are labels of the edges of G. f is called group mean cordial labeling if the following conditions hold:*
(1) For $x, y \in A$, $|v_f(x) - v_f(y)| \leq 1$, where $v_f(x)$ is the number of vertices labeled with x.
(2) For $i, j \in \mathbb{I}$, $|e_f(i) - e_f(j)| \leq 1$, where $e_f(i)$ denote the number of edges labeled with i.
A graph with a group mean cordial labeling is called a group mean cordial graph.

In this paper, we take the group A as the group $\{1, -1, i, -i\}$ which is the group of fourth roots of unity, that is cyclic with generators i and $-i$.

Example 2.2. *The following is a simple example of a group mean cordial graph.*

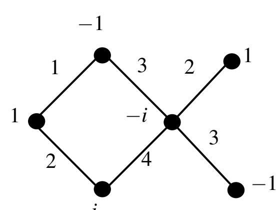

Figure 1

Theorem 2.3. *The splitting graph of the path, $S'(P_n)$ is a group mean cordial graph for every n.*

Proof. Let $P_n : u_1u_2...u_n$ be a path. Let $v_1, v_2, ..., v_n$ be the newly added vertices. Then $E(S'(P_n)) = E(P_n) \cup \{u_jv_{j+1} : 1 \leq j \leq n\} \cup \{u_jv_{j-1} : 2 \leq j \leq n\}$. Note that $S'(P_n)$ has $2n$ vertices and $3n - 3$ edges.
Define $f : V(S'(P_n)) \longrightarrow \{1, -1, i, -i\}$ as follows:

$$f(u_j) = 1 \; ; \; f(v_j) = i \quad for\ j \equiv 1\,(mod 4)$$

$$f(u_j) = -1; f(v_j) = -i \; for\ j \equiv 2\,(mod 4)$$

$$f(u_j) = i \; ; \; f(v_j) = 1 \quad for\ j \equiv 3\,(mod 4)$$

$$f(u_j) = -i; f(v_j) = -1 \; for\ j \equiv 0\,(mod 4)$$

The following tables 1 & 2 prove that the function f is a group mean cordial labeling.

Nature of n	$v_f(1)$	$v_f(-1)$	$v_f(i)$	$v_f(-i)$
n is odd	$\frac{n+1}{2}$	$\frac{n-1}{2}$	$\frac{n+1}{2}$	$\frac{n-1}{2}$
n is even	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$

Table 1

Nature of n	$e_f(1)$	$e_f(2)$	$e_f(3)$	$e_f(4)$
$n \equiv 0\,(mod\,4)$	$\frac{3n}{4} - 1$	$\frac{3n}{4} - 1$	$\frac{3n}{4}$	$\frac{3n}{4} - 1$
$n \equiv 1\,(mod\,4)$	$\frac{3n-3}{4}$	$\frac{3n-3}{4}$	$\frac{3n-3}{4}$	$\frac{3n-3}{4}$
$n \equiv 2\,(mod\,4)$	$\frac{3n-2}{4}$	$\frac{3n-2}{4}$	$\frac{3n-2}{4}$	$\frac{3n-6}{4}$
$n \equiv 3\,(mod\,4)$	$\frac{3n-1}{4}$	$\frac{3n-5}{4}$	$\frac{3n-1}{4}$	$\frac{3n-5}{4}$

Table 2

□

Example 2.4. *Group mean cordial labeling of $S'(P_7)$ is given in Figure 2*

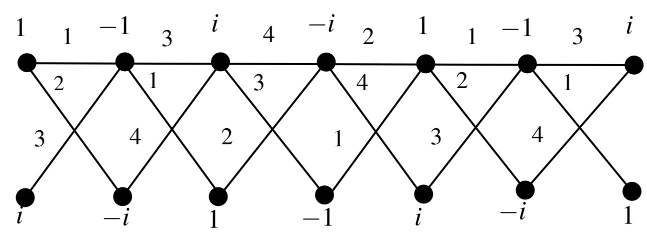

Figure 2

Theorem 2.5. *The splitting graph of cycle, $S'(C_n)$ is a group mean cordial graph for every n.*

Proof. Let $C_n : u_1u_2...u_nu_1$ be a cycle. Let $v_1, v_2, ..., v_n$ be the newly added vertices. $E(S'(C_n)) = E(C_n) \cup \{u_{j-1}v_j, u_jv_{j-1} : 2 \leq j \leq n\} \cup \{u_1v_n, u_nv_1\}$. The number of vertices and edges in $S'(C_n)$ are $2n$ and $3n$ respectively. Define $f : V(S'(C_n)) \longrightarrow \{1, -1, i, -i\}$ as follows.
Case 1: $n \equiv 0, 3\,(mod\,4)$
Label the vertices of $S'(C_n)$ as in Theorem 2.3.
Case 2: $n \equiv 1\,(mod\,4)$
Assign the labels to the vertices $u_j, v_j (1 \leq j \leq n-1)$ as in

Theorem 2.3. Then assign the labels $i, 1$ to the vertices u_n, v_n respectively.

Case 3: $n \equiv 2\,(mod\,4)$

Here also assign the labels to the vertices $u_j, v_j (1 \leq j \leq n-2)$ as in Theorem 2.3. Then assign the labels $i, 1$ respectively to the vertices u_{n-1}, u_n and $-i, -1$ to the vertices v_{n-1}, v_n respectively.

Table 1 in Theorem 2.3 and Table 3 estatablish that f is a group mean cordial labeling.

Nature of n	$e_f(1)$	$e_f(2)$	$e_f(3)$	$e_f(4)$
$n \equiv 0\,(mod\,4)$	$\frac{3n}{4}$	$\frac{3n}{4}$	$\frac{3n}{4}$	$\frac{3n}{4}$
$n \equiv 1\,(mod\,4)$	$\frac{3n-3}{4}$	$\frac{3n+1}{4}$	$\frac{3n+1}{4}$	$\frac{3n+1}{4}$
$n \equiv 2\,(mod\,4)$	$\frac{3n-2}{4}$	$\frac{3n+2}{4}$	$\frac{3n+2}{4}$	$\frac{3n-2}{4}$
$n \equiv 3\,(mod\,4)$	$\frac{3n+3}{4}$	$\frac{3n-1}{4}$	$\frac{3n-1}{4}$	$\frac{3n-1}{4}$

Table 3

□

Theorem 2.6. *The splitting graph of star, $S'(K_{1,n})$ is a group mean cordial graph iff $n \leq 4$ and $n = 6$.*

Proof. Let $V(K_{1,n}) = \{u, u_j : 1 \leq j \leq n\}$. Let $v, v_j (1 \leq j \leq n)$ be the newly added vertices. Then $E(S'(K_{1,n})) = \{uu_j, uv_j, vu_j : 1 \leq j \leq n\}$. Clearly this graph has $2n+2$ vertices and 3n edges.

For $n \leq 4$, assign the labels $i, 1$ to the vertices u, v respectively. Next assign $-1, 1, -i, 1$ to the vertices u_1, u_2, u_3, u_4 respectively. Then assign $-i, -1, i, -1$ to the vertices v_1, v_2, v_3, v_4 respectively. By this labeling, we get $S'(K_{1,n})$ is a group mean corial graph when $n \leq 4$.

The group mean cordial labeling of $S'(K_{1,6})$ is given in Figure 3

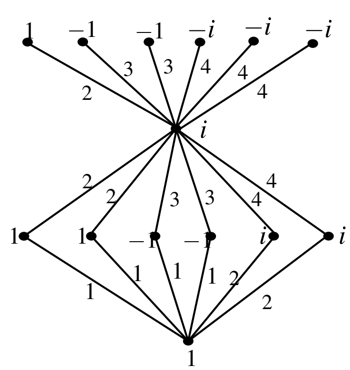

Figure 3

Now, assume $n \geq 5$ and $n \neq 6$.

Let f be a group mean cordial labeling.

First we consider the following cases.

(a) If $f(u) = f(v) = 1$ or -1. Then $e_f(4) = 0$.

(b) If $f(u) = f(v) = i$ or $-i$. Then $e_f(1) = 0$.

It is clear that u and v doesn't get the same labels.

Without loss of generality, the following cases may arise.

(1) $f(u) = 1$ and $f(v) = i$.

(2) $f(u) = -1$ and $f(v) = i$.

(3) $f(u) = i$ and $f(v) = 1$ or -1.

Case 1: $n \equiv 0\,(mod\,4)$

Let $n = 4s, s > 1$. Here the splitting graph of $K_{1,n}$ has $8s+2$ vertices and $12s$ edges.

Clearly, $v_f(x) = 2s$ or $2s+1$, $x \in \{1, -1, i, -i\}$ and $e_f(j) = 3s$, $j \in \{1, 2, 3, 4\}$

Subcase 1.1: $f(u) = 1$ and $f(v) = i$.

In $S'(K_{1,n})$, at least $2s$ vertices are labeled with 1 and atleast $2s$ vertices are labeled with -1. Then there is at least $4s-1$ edges get the label 1. This implies, $e_f(1) \geq 4s-1 > 3s$, for $s > 1$.

Subcase 1.2: $f(u) = -1$ and $f(v) = i$.

Here, at least $2s$ vertices are labeled with i and at least $2s$ vertices are labeled with $-i$. This implies, $e_f(3) \geq 4s-1 > 3s$, for $s > 1$.

Subcase 1.3: $f(u) = i$ and $f(v) = 1$ or -1.

Here, at least $2s$ vertices are labeled with i and at least $2s$ vertices are labeled with $-i$. This implies, $e_f(4) \geq 4s-1 > 3s$, for $s > 1$.

Case 2: $n \equiv 1\,(mod\,4)$

Let $n = 4s+1, s \geq 1$. Then the splitting graph of $K_{1,n}$ has $8s+4$ vertices and $12s+3$ edges.

Clearly, $v_f(x) = 2s+1$, for all $x \in \{1, -1, i, -i\}$ and $e_f(j) = 3s$ or $3s+1$, $j \in \{1, 2, 3, 4\}$.

Subcase 2.1: $f(u) = 1$ and $f(v) = i$.

In this subcase, $2s+1$ vertices are labeled with 1 and $2s+1$ vertices are labeled with -1. This implies, $e_f(1) = 4s+1 > 3s+1$, for $s \geq 1$.

Subcase 2.2: $f(u) = -1$ and $f(v) = i$.

Here, $2s+1$ vertices are labeled with i and $2s+1$ vertices are labeled with $-i$. This implies, $e_f(3) = 4s+1 > 3s+1$, for $s \geq 1$.

Subcase 2.3: $f(u) = i$ and $f(v) = 1$ or -1.

Here, $2s+1$ vertices are labeled with i and $2s+1$ vertices are labeled with $-i$. This implies, $e_f(4) = 4s+1 > 3s+1$, for $s \geq 1$.

Case 3: $n \equiv 2\,(mod\,4)$

Let $n = 4s+2, s > 1$. Clearly,the order and size of the splitting graph of $K_{1,n}$ are $8s+6$ and $12s+6$ respectively.

Here, $v_f(x) = 2s+1$ or $2s+2$, $x \in \{1, -1, i, -i\}$ and $e_f(j) = 3s+1$ or $3s+2$, $j \in \{1, 2, 3, 4\}$

Subcase 3.1: $f(u) = 1$ and $f(v) = i$.

In $S'(K_{1,n})$, at least $2s+1$ vertices are labeled with 1 and at least $2s+1$ vertices are labeled with -1. This implies, $e_f(1) \geq 4s+1 > 3s+2$, for $s > 1$.

Subcase 3.2: $f(u) = -1$ and $f(v) = i$.

Here, atleast $2s+1$ vertices are labeled with i and at least $2s+1$ vertices are labeled with $-i$. This implies, $e_f(3) \geq$

$4s+1 > 3s+2$, for $s > 1$.
Subcase 3.3: $f(u) = i$ and $f(v) = 1$ or -1.
Here, at least $2s+1$ vertices are labeled with i and at least $2s+1$ vertices are labeled with $-i$. Then, $e_f(4) \geq 4s+1 > 3s+2$, for $s > 1$.
Case 4: $n \equiv 3\,(mod\,4)$
Let $n = 4s+3, s \geq 1$. Then the splitting graph of $K_{1,n}$ has $8s+8$ vertices and $12s+9$ edges.
Clearly, $v_f(x) = 2s+2$, for all $x \in \{1,-1,i,-i\}$ and $e_f(j) = 3s+2$ or $3s+1$, $j \in \{1,2,3,4\}$.
Subcase 4.1: $f(u) = 1$ and $f(v) = i$.
In this subcase, $2s+2$ vertices are labeled with 1 and $2s+2$ vertices are labeled with -1. This implies, $e_f(1) = 4s+3 > 3s+3$, for $s \geq 1$.
Subcase 4.2: $f(u) = -1$ and $f(v) = i$.
Here, $2s+2$ vertices are labeled with i and $2s+2$ vertices are labeled with $-i$. Then, $e_f(3) = 4s+3 > 3s+3$, for $s \geq 1$.
Subcase 4.3: $f(u) = i$ and $f(v) = 1$ or -1.
Here, $2s+2$ vertices are labeled with i and $2s+2$ vertices are labeled with $-i$. This implies, $e_f(4) = 4s+3 > 3s+3$, for $s \geq 1$.
In each case, we get a contradiction.
Thus f is not a group mean cordial labeling for $n \geq 5$ and $n \neq 6$. □

Theorem 2.7. *The splitting graph of Comb, $S'(P_n \odot K_1)$ is a group mean cordial graph.*

Proof. Let $V(P_n \odot K_1) = \{u_j, u'_j : 1 \leq j \leq n\}$. Then $E(P_n \odot K_1) = \{u_j u_{j+1} : 1 \leq j \leq n-1\} \cup \{u_j u'_j : 1 \leq j \leq n\}$. Let $v_1, v_2, \ldots, v_n$ and $v'_1, v'_2, \ldots, v'_n$ be the newly added vertices. Then $E(S'(P_n \odot K_1)) = E(P_n \odot K_1) \cup \{u_j v'_j, u'_j v_j : 1 \leq j \leq n\} \cup \{u_{j-1} v'_j, u'_j v_{j-1} : 2 \leq j \leq n\}$. The order and size of $S'(P_n \odot K_1)$ are $4n$ and $6n-3$ respectively. Define $f : V(S'(P_n \odot K_1)) \longrightarrow \{1,-1,i,-i\}$ as follows:

$$f(u_j) = i; f(u'_j) = -i; f(v_j) = 1; f(v'_j) = -1 \ for\, j \equiv 0,2\,(mod\,4)$$

$$f(u_j) = 1; f(u'_j) = -1; f(v_j) = i; f(v'_j) = -i \ for\, j \equiv 1\,(mod\,4)$$

$$f(u_j) = -1; f(u'_j) = 1; f(v_j) = i; f(v'_j) = -i \ for\, j \equiv 3\,(mod\,4)$$

By this labeling we get, $v_f(1) = v_f(-1) = v_f(i) = v_f(-i) = n$. Table 4 shows that $|e_f(x) - e_f(y)| \leq 1$. Hence f is a group mean cordial labeling of the splitting graph of comb.

Nature of n	$e_f(1)$	$e_f(2)$	$e_f(3)$	$e_f(4)$
$n \equiv 0\,(mod\,4)$	$\frac{6n-4}{4}$	$\frac{6n-4}{4}$	$\frac{6n}{4}$	$\frac{6n-4}{4}$
$n \equiv 1,2\,(mod\,4)$	$\frac{6n-2}{4}$	$\frac{6n-2}{4}$	$\frac{6n-2}{4}$	$\frac{6n-6}{4}$
$n \equiv 3\,(mod\,4)$	$\frac{6n-4}{4}$	$\frac{6n}{4}$	$\frac{6n-4}{4}$	$\frac{6n-4}{4}$

Table 4

□

Theorem 2.8. *The splitting graph of the complete bipartite graph, $S'(K_{n,n})$ is a group mean cordial graph when n is even.*

Proof. Let $V(K_{n,n}) = \{u_j, v_j : 1 \leq j \leq n\}$. Let $E(K_{n,n}) = \{u_i v_j : 1 \leq i, j \leq n\}$. Let u'_j, v'_j be the newly added vertices. Then $E(S'(K_{n,n})) = \{u_i v_j, u_i v'_j, u'_i v_j : 1 \leq i, j \leq n\}$. Here the order and size of the graph are $4n$ and $3n^2$ respectively.

Define $f : V(S'(K_{n,n})) \longrightarrow \{1,-1,i,-i\}$ by,

$$f(u_j) = f(u'_j) = -1$$

$$f(v_j) = f(v'_j) = 1$$

$$f(u_{\frac{n}{2}+j}) = f(u'_{\frac{n}{2}+j}) = -i$$

$$f(v_{\frac{n}{2}+j}) = f(v'_{\frac{n}{2}+j}) = i,$$

for $1 \leq j \leq \frac{n}{2}$. By this labeling, we get $v_f(1) = v_f(-1) = v_f(i) = v_f(-i) = n$ and $e_f(1) = e_f(-1) = e_f(i) = e_f(-i) = e_f(1) = e_f(2) = e_f(3) = e_f(4) = \frac{3n^2}{4}$.
Hence f is a group mean cordial labeling when n is even. □

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