Malaya Journal of Matematik, Vol. 8, No. 4, 2359-2368, 2020

https://doi.org/10.26637/MJM0804/0183

2-Vertex self switching of umbrella graph

C. Jayasekaran [1]*, A. Vinoth Kumar [2] and M. Ashwin Shijo[3]

Abstract

By a graph $G=(V,E)$ we mean a finite undirected graph without loops or multiple edges. Let G be a graph and $\sigma \subseteq V$ be a non-empty subset of V. Then σ is said to be a self switching of G if and only if $G \cong G^{\sigma}$. It can also be referred to as $|\sigma|$-vertex self-switching. The set of all self switching of the graph G with cardinality k is represented by $SS_k(G)$ and its cardinality by $ss_k(G)$. A vertex v of a graph G is said to be self vertex switching if $G \cong G^v$. The set of all self vertex switchings of G is denoted by $SS_1(G)$ and its cardinality is given by $ss_1(G)$. If $|\sigma|=2$, we call it as a 2-vertex self switching. The set of all 2-vertex switchings of G is denoted by $SS_2(G)$ and its cardinality is given by $ss_2(G)$. In this paper we find the number of 2-vertex self switching vertices for the umbrella graph $U_{m,n}$.

Keywords

2-vertex switching, 2-vertex self switching, Umbrella graph.

AMS Subject Classification

05C07, 05CXX.

[1] *Department of Mathematics, Pioneer Kumaraswamy College, Nagercoil-629003, Kanyakumari District, Tamil Nadu, India.*

[2] *Research Scholar, Reg No. 20123132091003, Department of Mathematics, Pioneer Kumaraswamy College, Nagercoil-629003, Kanyakumari District, Tamil Nadu, India. Affiliated to Manonmaniam Sundaranar University, Abishekapatti-Tirunelveli-627012.*

[3] *Research Scholar, Department of Mathematics, Pioneer Kumaraswamy College, Nagercoil-629003, Kanyakumari District, Tamil Nadu, India.*

***Corresponding author**: [1] jayacpkc@gmail.com; [2] alagarrvinoth@gmail.com; [3]ashwin1992mas@gmail.com

Article History: Received **12** August **2020**; Accepted **25** November **2020**

Contents

1. Introduction

By a graph G we mean a finite undirected simple graph. The degree of a vertex v in a graph G is the number of edges incident with v. The degree of v is represented by $d_G(v)$. JJ Seidel made a brief survey of two graphs in [9]. In [8] Lint and Seidel introduces the vertex switching. For a finite undirected graph $G(V,E)$ and a subset $S \subset V$, the switching of G by S is defined as the graph $G^S(V,E')$ which is obtained from G by removing all edges between S and its complement V–S and adding as edges all non edges between S and $V-S$. For $S=\{v\}$, we write G^v instead of $G^{\{v\}}$ and the corresponding switching is called as vertex switching [3]. Hage and Harju have proved that a switching class $[G]$ contains a hamiltonian graph if and only if G is not a complete bipartite graph of odd order in [6, 7].

The concept of $2-$vertex self switchings of graphs was introduced by Jayasekaran, Christabel Sudha and Ashwin Shijo and number of 2-vertex self switching in some special graphs were studied in [1]. Selvam Avadayappan and Bhuvaneshwari studied more about self vertex switching in [10] Let G be a graph and let $\sigma \subset V$ be a non-empty subset of V. σ is said to be a self switching if $G \cong G^{\sigma}$ where G^{σ} is obtained from G by removing all edges between σ and $V-\sigma$ and adding edges between all non-adjacent vertices of σ and $V-\sigma$. We also call it as $|\sigma|$ vertex self switching. When $|\sigma|=2$, we call it as 2-vertex self switching. The set of all 2-vertex self switching sets of a graph G is denoted by $SS_2(G)$ and its cardinality by $ss_2(G)$. In [4], Sampathkumar introduced duplicate graphs. Jayasekaran and Ashwin Shijo [2] introduced the concept of anti-duplication self vertex switching. For basic definitions, we refer F Harrary [5].

In this paper we find the number of 2-vertex self switching vertices for the umbrella graph $U_{m,n}$.

2. 2-Vertex Self Switchings

Here we recall the theorems which are used in the subsequent sections.

Theorem 2.1. *[1] If $\sigma = \{u,v\} \subseteq V$ is a 2-vertex self switching of a graph G, then* $d_G(u)+d_G(v) = \begin{cases} p \text{ if } uv \in E(G) \\ p-2 \text{ if } uv \notin E(G). \end{cases}$

Theorem 2.2. *[1] Let G be a connected graph and let w be a vertex of degree 2, adjacent to u and v. Then $\sigma = \{u,v\}$ is not a 2-vertex self switching of G.*

3. 2-Vertex Self Switching of Umberlla Graph

Definition 3.1. *Consider the paths $P_m : v_1v_2...v_m$ and $P_n : u_1u_2...u_n$. For $1 \le i \le m$, join u_1 with v_i. The resultant graph is the umbrella graph $G = U(m,n)$ with vertex set $V(G) = \{v_i, u_j : 1 \le i \le m, 1 \le j \le n\}$ and edge set $E(G) = \{v_iv_{i+1}, u_ju_{j+1}, u_1v_k : 1 \le i \le m-1, 1 \le j \le n-1, 1 \le k \le m\}$. Clearly, $p = |V(G)| = m+n$ and $q = |E(G)| = 2m+n-2$.*

Example 3.2. *The umbrella graph $U(5,3)$ is given in figure 1*

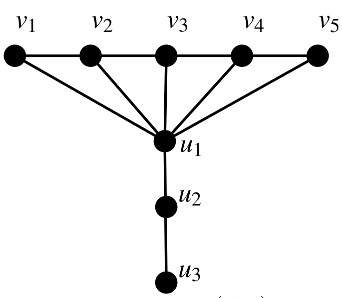


Figure 1. $U(5,3)$

Theorem 3.3. *For the umberla graph $U(m,n)$, $(m,n \ge 2)$*

$$ss_2(U(m,n)) = \begin{cases} 6 & for\ m=n=3 \\ 5 & for\ m=2, n=4 \\ 3 & for\ m=2, n=3\ and\ m=3, n=2 \\ 2 & for\ m=n=2\ and\ m=3, n=4 \\ 1 & for\ m \ge 4, 2 \le n \le 5\ and\ for\ n=5, m=2,3 \\ 0 & otherwise \end{cases}$$

Proof. Consider the paths $P_m : v_1v_2...v_m$ and $P_n : u_1u_2...u_n$. For $1 \le i \le m$, join u_1 with v_i. The resultant graph is the umbrella graph $G = U(m,n)$ with vertex set $V(G) = \{v_i, u_j : 1 \le i \le m, 1 \le j \le n\}$ and edge set $E(G) = \{v_iv_{i+1}, u_ju_{j+1}, u_1v_k : 1 \le i \le m-1, 1 \le j \le n-1, 1 \le k \le m\}$. Clearly, $p = |V(G)| = m+n$ and $q = |E(G)| = 2m+n-2$. Also $d_G(v_1) = 2 = d_G(v_n)$ and $d_G(v_i) = 3$, $2 \le i \le m-1$, $d_G(u_1) = m+1$, $d_G(u_n) = 1$ and $d_G(u_j) = 2$, $2 \le j \le n-1$. Let $\sigma = \{u,v\} \subseteq V(G)$. Then $uv \in E(G)$ or $uv \notin E(G)$. Also $d_G(u)+d_G(v) \in \{3,4,5,6,m+2,m+3,m+4\}$. Since $m,n \ge 2, p \ge 4$.

Case 1. $uv \in E(G)$

By Theorem 2.1 , if $\sigma = \{u,v\}$ is a 2-vertex self switching of G, then $d_G(u)+d_G(v) = p = m+n$. If $n > 4$, then $d_G(u)+d_G(v) = p = m+n > m+4$ which is not possible and thereby $2 \le n \le 4$.

Subcase 1.a. $n = 2$

Subcase 1.a.a. $m = 2$

The graph $G = U(2,2)$ is given in figure 2. Since $p = 4$, $d_G(u)+d_G(v) = p = 4$ implies that $uv \in \{u_1u_2, v_1v_2\}$. The graphs $G^{\{u_1,u_2\}}$ and $G^{\{v_1,v_2\}}$ are given in figures 3 and 4. Clearly, $G^{\{u_1,u_2\}} \cong G \cong G^{\{u_1,u_2\}}$ and hence $\{u_1,u_2\}$ and $\{v_1,v_2\}$ are 2-vertex self switchings of $U(2,2)$.

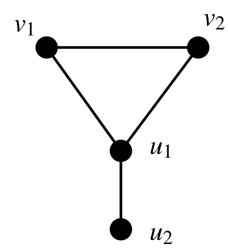


Figure 2. $G = U(2,2)$

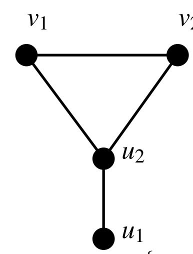


Figure 3. $G^{\{u_1,u_2\}}$

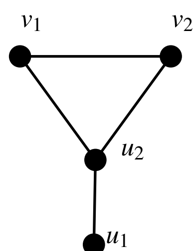
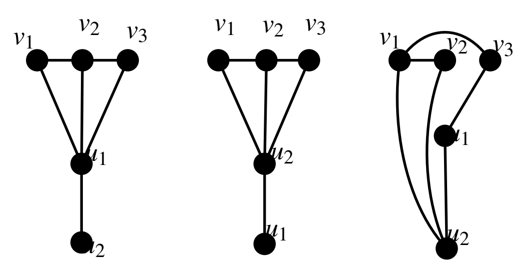


Figure 4. $G^{\{v_1,v_2\}}$

Subcase 1.a.b. $m = 3$

The graph $G = U(3,2)$ is given in figure 5 (a). Now $p = m+n = 5$ and so $d_G(u)+d_G(v) = 5$ implies that $uv \in \{u_1u_2, v_1v_2, v_2v_3\}$. Clearly $G^{\{v_1,v_2\}} \cong G^{\{v_2,v_3\}}$. The graphs $G^{\{u_1,u_2\}}$ and $G^{\{v_1,v_2\}}$ are given in figure 5 (b) and (c). Since $G^{\{v_1,v_2\}}$ has no vertex of degree 1, $G^{\{v_1,v_2\}} \ncong G$ and so $\{v_1,v_2\}$ and $\{v_2,v_3\}$ are not 2-vertex self switchings of G. Clearly, $G^{\{u_1,u_2\}} \cong G$ and so $\{u_1,u_2\}$ is a 2-vertex self switching of $U(3,2)$.



a. $G = U(3,2)$ b. $G^{\{u_1,u_2\}}$ c. $G^{\{v_1,v_2\}}$

Figure 5. $G, G^{\{u_1,u_2\}}, G^{\{v_1,v_2\}}$

Subcase 1.a.c. $m = 4$

The graph $G = U(4,2)$ is given in figure 6. Then $p = m+n = 6$ and so $d_G(u)+d_G(v) = p = 6$ implies that $uv \in \{u_1u_2, v_2v_3\}$. The graphs $G^{\{u_1,u_2\}}$ and $G^{\{v_2,v_3\}}$ are given in figure 7 and 8, repectively.

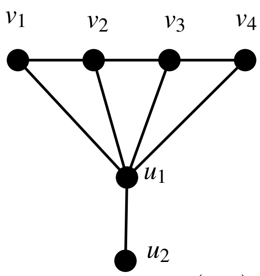


Figure 6. $G = U(4,2)$

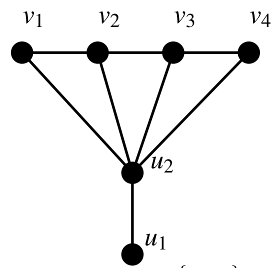


Figure 7. $G^{\{u_1,u_2\}}$

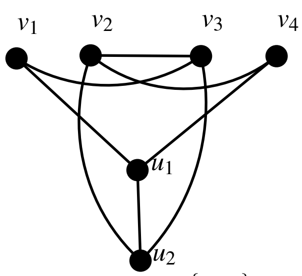


Figure 8. $G^{\{v_1,v_2\}}$

Since $G^{\{v_1,v_2\}}$ has no vertex of degree 1 and G has a vertex of degree 1, $G^{\{v_1,v_2\}} \ncong G$. Clearly, $G^{\{u_1,u_2\}} \cong G$ and thereby $\{u_1,u_2\}$ is a 2-vertex self switching of $U(4,2)$.

Subcase 1.a.d. $m \geq 5$

The graph $G = U(m,2)$ is given in figure 9. Here $p = m+2$ and so $d_G(u)+d_G(v) = p = m+2$ implies that uv is the edge u_1u_2. The graph $G^{\{u_1,u_2\}}$ is given in figure 10 which is isomorphic to G and so $\{u_1,u_2\}$ is a 2- vertex self switching of $G = U(m,2)$.

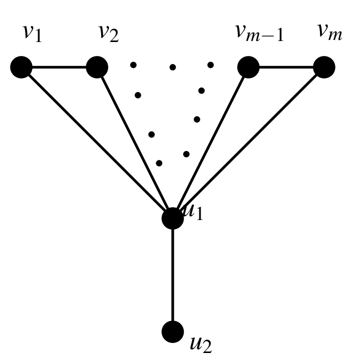


Figure 9. $G = U(m,2)(m > 4)$

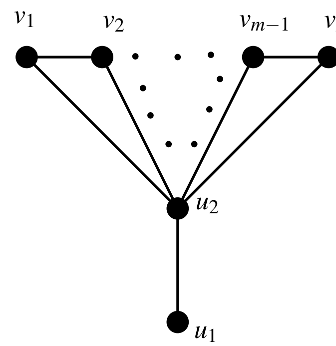


Figure 10. $G^{\{u_1,u_2\}}$

Subcase 1.b. $n = 3$

Subcase 1.b.a. $m = 2$

The graph $G = U(2,3)$ is given in figure 11. Now $p = m+n = 5$ and so $d_G(u)+d_G(v) = p = 5$ implies that $uv \in \{u_1u_2, u_1v_1, u_1v_2\}$. Clearly, $G^{\{u_1,v_1\}} \cong G^{\{u_1,v_2\}}$. The vertices u_1 and v_1 and the vertices u_1 and v_2 are adjacent to a vertex of degree 2 in G and thereby Theorem 2.2, $\{u_1,v_1\}$ and $\{u_1,v_2\}$ are not 2-vertex self switchings of G. The graph $G^{\{u_1,u_2\}}$ is given in figure 12 which is isomorphiuc to G. Hence, $\{u_1,u_2\}$ is a 2-vertex self switching of $U(2,3)$.

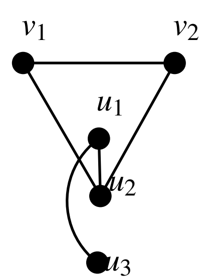


Figure 11. $G = U(2,3)$

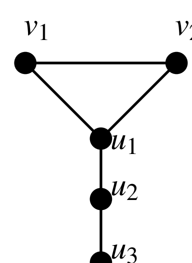


Figure 12. $G^{\{u_1,u_2\}}$

Subcase 1.b.b. $m = 3$

The graph $G = U(3,3)$ is given in figure 13. Now $p = m+n = 6$ and so $d_G(u)+d_G(v) = p = 6$ implies that $uv \in \{u_1u_2, u_1v_1, u_1v_3\}$. Clearly, $G^{\{u_1,v_1\}} \cong G^{\{u_1,v_3\}}$. The graphs $G^{\{u_1,u_2\}}$ and $G^{\{u_1,v_1\}}$ are given in figures 14 (a) and 14 (b), respectively. The graph $G^{\{u_1,v_1\}} \cong G$ and an isomorphism f between G and $G^{\{u_1,v_1\}}$ is given by $f(v_1) = u_1, f(v_2) = u_3, f(v_3) = u_2, f(u_1) = v_1, f(u_2) = v_3$ and $f(u_3) = v_2$. Hence $\{u_1,v_1\}$ and $\{u_1,v_3\}$ are 2-vertex self switchings of $U(3,3)$. Also $\{u_1,u_2\}$ is a 2-vertex self switching of $U(3,3)$ since $G^{\{u_1,u_2\}} \cong G$.

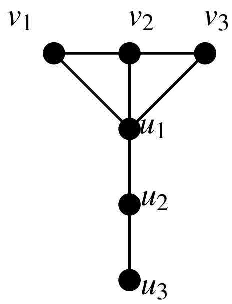
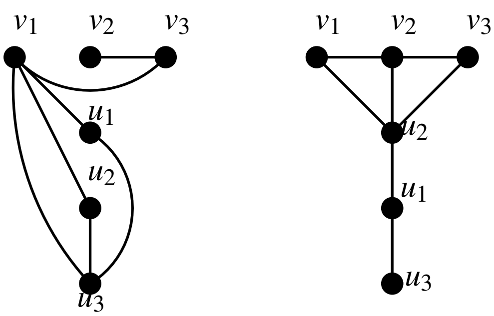


Figure 13.



a. $G^{\{u_1,v_1\}}$ *b.* $G^{\{u_1,u_2\}}$

Figure 14. $G^{\{u_1,v_1\}}, G^{\{u_1,u_2\}}$

Subcase 1.b.c. $m \geq 4$

The graph $G = U(m,3)$ is given in figure 15. Hence $p = m+3$ and so $d_G(u) + d_G(v) = p = m+3$ implies that $uv \in \{u_1u_2, u_1v_1, u_1v_m\}$. Clearly, $G^{\{u_1,v_1\}} \cong G^{\{u_1,v_m\}}$. The graphs $G^{\{u_1,v_1\}}$ and $G^{\{u_1,u_2\}}$ are given in figures 16 and 17, respectively.

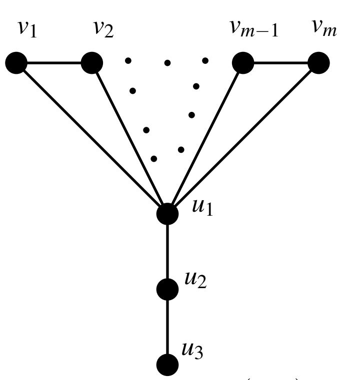


Figure 15. $G = U(m,3)$

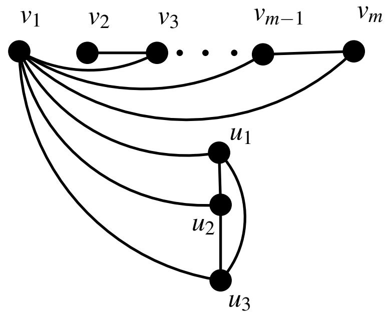


Figure 16. $G^{\{u_1,v_1\}}$

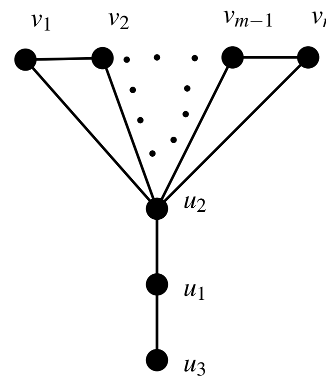


Figure 17. $G^{\{u_1,u_2\}}$

Now G has a vertex of degree 1 which is adjacent to a vertex of degree 2 whereas in $G^{\{u_1,v_1\}}$, the vertex of degree 1 is adjacent to a vertex of degree 3 and so $\{u_1, v_1\}$ and $\{u_1, v_m\}$ are not a 2-vertex self switchings of G. Clearly, $G^{\{u_1,u_2\}} \cong G$ and thereby $\{u_1, u_2\}$ is a 2- vertex self switching of $U(m,3)$.

Subcase 1.c. $n = 4$

Subcase 1.c.a. $m = 2$

The graph $G = U(2,4)$ is given in figure 18. Now $p = m+n = 6$ and so $d_G(u) + d_G(v) = p = 6$ implies that there is no edge uv exists in G since G has exactly one vertex with degree 3 and all other vertices have degree less than 3. This shows that $U(2,4)$ has no 2-vertex self switchings when uv is an edge of G.

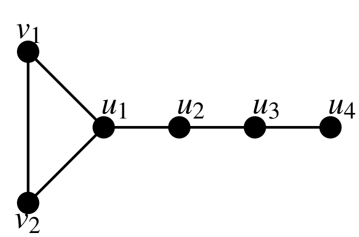


Figure 18. $G = U(2,4)$

Subcase 1.c.b. $m = 3$

The graph $G = U(3,4)$ is given in figure 19. Now $p = m+n = 7$ and so $d_G(u) + d_G(v) = p = 7$ implies that uv is u_1v_2. Since the vertices u_1 and v_2 are adjacent to a vertex of degree 2, by Theorem 2.2 , $\{u_1, v_2\}$ is not a 2-vertex self switching of G.

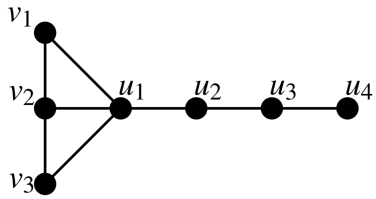


Figure 19. $G = U(3,4)$

Subcase 1.c.c. $m \geq 4$

The graph $G = U(m,4)$ is given in figure 20. Here $p = m+4$ and so $d_G(u)+d_G(v) = p = m+4$ implies that $uv \in \{u_1v_2, u_1v_3, ..., u_1v_{m-1}\}$. Clearly, $G^{\{u_1,v_2\}} \cong G^{\{u_1,v_{m-1}\}}$ and the vertices u_1 and v_2 are adjacent to a vertex of degree 2 in G. By Theorem 2.2, $\{u_1, v_2\}$ and $\{u_1, v_{m-1}\}$ are not 2-vertex self switchings of G. In G, the vertices v_{i-1} and v_{i+1} have degree 3 and adjacent to both u_1 and v_i, $3 \leq i \leq m-2$. This implies that v_{i-1} and v_{i+1} have degree 1 in $G^{\{u_1,v_i\}}$ and hence $G^{\{u_1,v_i\}} \ncong G$. This implies that $U(m,4)$ has no 2-vertex self switchings.

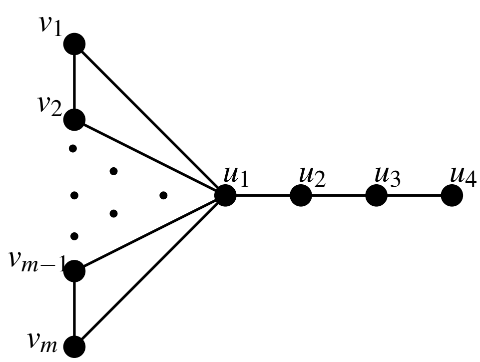


Figure 20. $G = U(m,4)(m \geq 4)$

Case 2. $uv \notin E(G)$

By Theorem 2.1, if $\sigma = \{u, v\}$ is a 2-vertex self switching of G, then $d_G(u)+d_G(v) = p-2 = m+n-2$. Since u_1 is adjacent to v_i and u_2, $1 \leq i \leq n$, $uv \in \{u_1u_j, v_iu_k, v_lv_r : 3 \leq j \leq n, 1 \leq i \leq m, 2 \leq k \leq n, 1 \leq l \neq r \leq m \text{ and } r \neq l-1, l+1\}$. Clearly, $d_G(u)+d_G(v) \in \{3,4,5,6,m+2,m+3\}$.

Subcase 2.a. $n = 2$

If $m > 6$, then $d_G(u)+d_G(v) = p-2 = m > 6$ which is not possible and so $m \leq 6$.

Subcase 2.a.a. $m = 2$

The graph $G = U(2,2)$ is given in figure 2. Since $p = 4$, $d_G(u)+d_G(v) = p-2 = 2$ which is not possible since $d_G(u)+d_G(v) \geq 3$ and so there is no 2-vertex self switchings in G when $uv \notin E(G)$.

Subcase 2.a.b. $m = 3$

The graph $G = U(3,2)$ is given in figure 5. Now $p = m+n = 5$ and so $d_G(u)+d_G(v) = p-2 = 3$ implies that $uv \in \{v_1u_2, v_3u_2\}$. Clearly, $G^{\{v_1,u_2\}} \cong G^{\{v_3,u_2\}}$. The graph $G^{\{v_1,u_2\}}$ is given in figure 21. Also $G^{\{v_1,u_2\}} \cong G$ and an isomorphism f between them is given by $f(v_1) = u_1, f(v_2) = v_2, f(v_3) = u_2, f(u_1) = v_3$ and $f(u_2) = v_1$. Hence $\{v_1, u_2\}$ and $\{v_3, u_2\}$ are 2-vertex self switchings of $U(3,2)$.

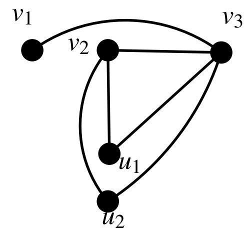


Figure 21. $G^{\{v_1,u_2\}}$

Subcase 2.a.c. $m = 4$

The graph $G = U(4,2)$ is given in figure 6.12. Here $p = m+n = 6$ and so $d_G(u)+d_G(v) = p-2 = 4$ implies that $uv \in \{v_1v_4, v_2u_2, v_3u_2\}$. Clearly, $G^{\{v_2,u_2\}} \cong G^{\{v_3,u_2\}}$. The graphs $G^{\{v_1,v_4\}}$ and $G^{\{v_2,u_2\}}$ are given in figures 22 and 23.

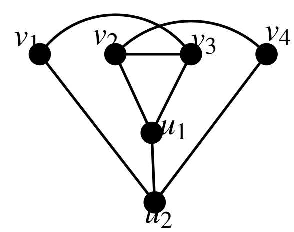


Figure 22. $G^{\{v_1,v_4\}}$

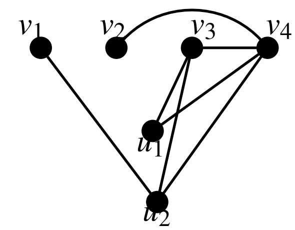


Figure 23. $G^{\{v_2,u_2\}}$

The graph G has one vertex of degree 1 and two vertices of degree 2 but the graph $G^{\{v_1,v_4\}}$ has no vertex of degree 1 and the graph $G^{\{v_2,u_2\}}$ has only one vertex of degree 2. This shows that $U(4,2)$ has no 2-vertex self switchings when $uv \notin E(U(4,2))$.

Subcase 2.a.d. $m = 5$

The graph $G = U(5,2)$ is given in figure 24. Here $p = m+n = 7$ and so $d_G(u)+d_G(v) = p-2 = 5$ implies that $uv \in \{v_1v_3, v_1u_4, v_5v_2, v_5v_3\}$. Clearly, $G^{\{v_1,v_3\}} \cong G^{\{v_3,v_5\}}$ and $G^{\{v_1,v_4\}} \cong G^{\{v_2,v_5\}}$. The graphs $G^{\{v_1,v_3\}}$ and $G^{\{v_1,v_4\}}$ are given in figures 25 and 26.

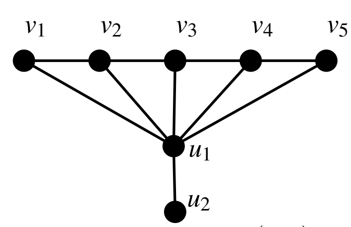


Figure 24. $G = U(5,2)$

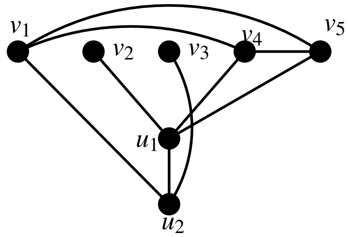


Figure 25. $G^{\{v_1,v_3\}}$

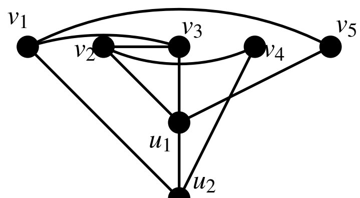


Figure 26. $G^{\{v_2,v_4\}}$

The graph G has one vertex of degree 1 and two vertices of degree 2 but $G^{\{v_1,v_3\}}$ has two vertices of degree 1 and the graph $G^{\{v_1,v_4\}}$ has no vertex of degree 1. Hence $U(5,2)$ has no 2-vertex self switchings when $uv \notin E(U(5,2))$.

Subcase 2.a.e. $m = 6$

The graph $G = U(6,2)$ is given in figure 27. Then $p = m+n = 8$ and so $d_G(u) + d_G(v) = p-2 = 6$ implies that $uv \in \{v_2v_4, v_2v_5, v_3v_5\}$. Clearly $G^{\{v_2,v_4\}} \cong G^{\{v_3,v_5\}}$. The graphs $G^{\{v_2,v_4\}}$ and $G^{\{v_2,v_5\}}$ are given in figure 28 and 29.

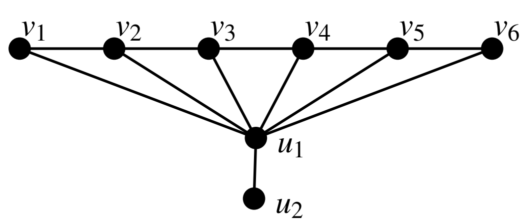


Figure 27. $G = U(6,2)$

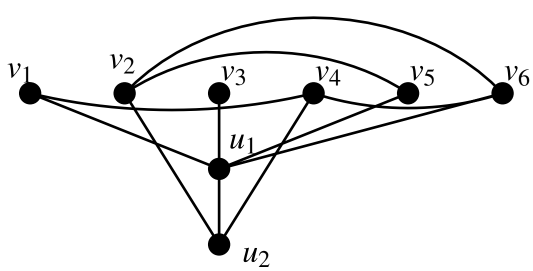


Figure 28. $G^{\{v_2,v_4\}}$

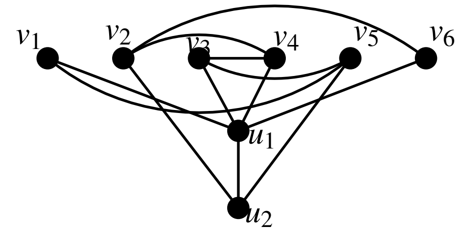


Figure 29. $G^{\{v_2,v_5\}}$

The graph $G^{\{v_2,v_4\}}$ has one vertex of degree 2 and the graph $G^{\{v_2,v_5\}}$ has no vertex of degree 1 and hence both graphs are not isomorphic to G. Thus, $U(6,2)$ has no 2-vertex self switchings when $uv \notin E(U(6,2))$.

Subcase 2.b. $n = 3$

If $m > 5$, then $d_G(u) + d_G(v) = p-2 = m+3-2 = m+1$ which is not possible and so $m \leq 5$.

Subcase 2.b.a. $m = 2$

The graph $G = U(2,3)$ is given in figure 11. Since $p = 5$, $d_G(u) + d_G(v) = p-2 = 3$ and so $uv \in \{v_1u_3, v_2u_3\}$. Clearly, $G^{\{v_1,u_3\}} \cong G^{\{v_2,u_3\}}$. The graph $G^{\{v_1,u_3\}}$ is given in figure 30. Now $G^{\{v_1,u_3\}} \cong G$ and an isomorphism f between them is given by $f(v_1) = v_2, f(v_2) = u_3, f(u_1) = u_1, f(u_2) = u_2$ and $f(u_3) = v_1$. Hence $\{v_1,u_3\}$ and $\{v_2,u_3\}$ are 2-vertex self switchings of $U(2,3)$.

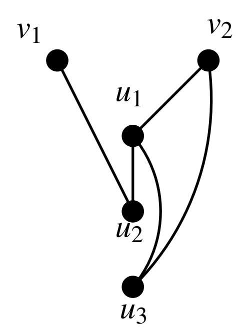
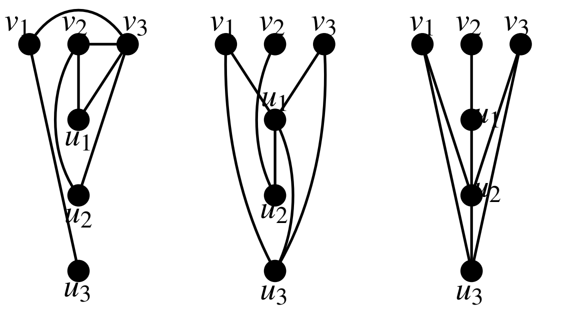


Figure 30. $G^{\{v_2,v_4\}}$

Subcase 2.b.b. $m = 3$

The graph $G = U(3,3)$ is given in figure 13. Now $p-2 = m+n-2 = 4$ and so $d_G(u)+d_G(v) = p-2 = 4$ implies that $uv \in \{v_1u_2, v_3u_2, v_2u_3, v_1v_3\}$. Clearly, $G^{\{v_1,u_2\}} \cong G^{\{v_3,u_2\}}$. The graphs $G^{\{v_1,u_2\}}$, $G^{\{v_2,u_3\}}$ and $G^{\{v_1,v_3\}}$ are given in figures 31 (a), 31 (b) and 31 (c), respectively. Now $G^{\{v_1,u_2\}} \cong G$ and an isomorphism f between them is given by $f(v_1) = u_1, f(v_2) = v_2, f(v_3) = u_2, f(u_1) = v_3, f(u_2) = v_1$ and $f(u_3) = u_1$. $G^{\{v_2,u_3\}} \cong G$ and an isomorphism between them is given by $g(v_1) = v_1, g(v_2) = u_3, g(v_3) = v_3, g(u_1) = u_1, g(u_2) = u_2$ and $g(u_3) = v_2$. Also $G^{\{v_1,v_3\}} \cong G$ and an isomorphism h between them is given by $h(v_1) = v_1, h(v_2) = u_3, h(v_3) = v_3, h(u_1) = u_2, h(u_2) = u_1$ and $f(u_3) = v_2$. Hence $\{v_1,u_2\}$, $\{v_1,v_3\}$ and $\{v_2,u_3\}$ are 2-vertex self switchings of $U(3,3)$.

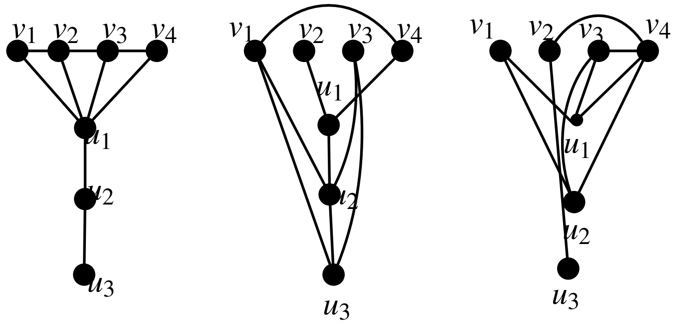


$(a).\ G^{\{v_1,u_2\}}$ $(b).\ G^{\{v_2,u_3\}}$ $(c)\ G^{\{v_1,v_3\}}$

Figure 31. $G^{\{v_1,u_2\}}, G^{\{v_2,u_3\}}, G^{\{v_1,v_3\}}$

Subcase 2.b.c. $m = 4$

The graph $G = U(4,3)$ is given in figure 32 (a). Here $p-2 = 5$ and so $d_G(u)+d_G(v) = p-2 = 5$ implies that $uv \in \{v_1v_3, v_2v_4, v_2u_2, v_3u_2\}$. Clearly $G^{\{v_1,v_3\}} \cong G^{\{v_2,v_4\}}$ and $G^{\{v_2,u_2\}} \cong G^{\{v_3,u_2\}}$. The graphs $G^{\{v_1,v_3\}}$ and $G^{\{v_2,u_2\}}$ are given in figure 32(b) and 32(c), respectively. The graph G has one vertex of degree 1 and three vertices of degree 2 whereas each of the graphs $G^{\{v_1,v_3\}}$ and $G^{\{v_2,u_2\}}$ has only two vertices of degree 2. This shows that $U(4,3)$ has no 2-vertex self switchings when $uv \notin E(U(4,3))$.



$(a)\ G = U(4,3)$ $(b)\ G^{\{v_1,v_3\}}$ $(c)\ G^{\{v_2,u_2\}}$

Figure 32. $G = U(4,3),\ G^{\{v_1,v_3\}}\ G^{\{v_2,u_2\}}$

Subcase 2.b.d. $m = 5$

The graph $G = U(5,3)$ is given in figure 33. Then $p-2 = 6$ and so $d_G(u)+d_G(v) = p-2 = 6$ implies that uv is v_2v_4. The graph $G^{\{v_2,v_4\}}$ is given in figure 34.

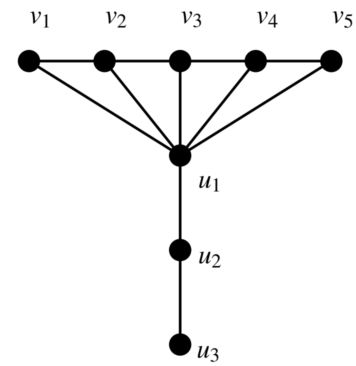


Figure 33. $G = U(4,3)$

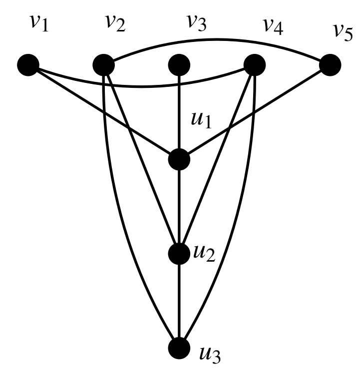


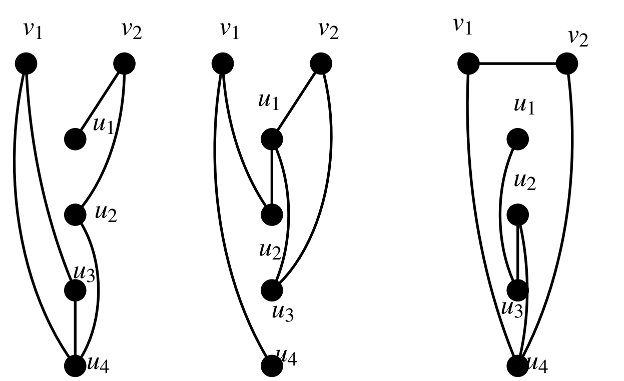
Figure 34. $G^{\{v_2,v_4\}}$

The graph G has three vertices of degree 2 whereas $G^{\{v_2,v_4\}}$ has two vertices of degree 2 and so $G^{\{v_2,v_4\}} \ncong G$. Hence, $G = U(5,3)$ has no 2-vertex self switchings.

Subcase 2.c. $n = 4$

Subcase 2.c.a. $m = 2$

The graph $G = U(2,4)$ is given in figure 18. Since $p = 6$, $d_G(u)+d_G(v) = p-2 = 4$ and so $uv \in \{v_1u_2, v_1u_3, v_2u_2, v_2u_3, u_1u_4\}$. Clearly, $G^{\{v_1,u_2\}} \cong G^{\{v_2,u_2\}}$ and $G^{\{v_1,u_3\}} \cong G^{\{v_2,u_3\}}$. The graphs $G^{\{v_1,u_2\}}, G^{\{v_1,u_3\}}$ and $G^{\{u_1,u_4\}}$ are given in figures 35 (a), 35 (b) and 35 (c), respectively.



$(a)\ G^{\{v_1,u_2\}}$ $(b)\ G^{\{v_1,u_3\}}$ $(c)\ G^{\{u_1,u_4\}}$

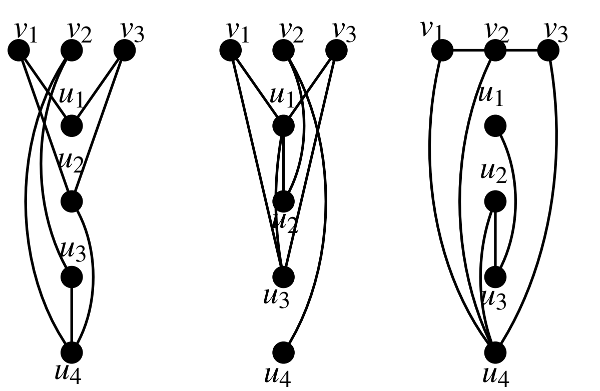
Figure 35. $G^{\{v_1,u_2\}}, G^{\{v_1,u_3\}}, G^{\{u_1,u_4\}}$

Now $G^{\{v_1,u_2\}} \cong G$ and an isomorphism f between them is given by $f(v_1) = v_1, f(v_2) = u_3, f(u_1) = u_4, f(u_2) = u_2, f(u_3) = v_2$ and $f(u_4) = u_1$. Hence, $\{v_1, u_2\}$ and $\{v_2, u_2\}$ are 2-vertex self switchings of $U(2,4)$. Also $G^{\{v_1,u_3\}} \cong G$ and an isomorphism g between them is given by $g(v_1) = v_2, f(v_2) = u_3, f(u_1) = u_1, f(u_2) = u_2, f(u_3) = v_1$ and $f(u_4) = u_4$. Hence, $\{v_1, u_3\}$ and $\{v_2, u_3\}$ are 2-vertex self switchings of $U(2,4)$. Moreover $G^{\{u_1,u_4\}} \cong G$ and an isomorphism h between them is given by $h(v_1) = v_1, h(v_2) = v_2, h(u_1) = u_4, h(u_2) = u_2, h(u_3) = u_3$ and $h(u_4) = u_1$. Hence, $\{u_1, u_4\}$ is a 2-vertex self switchings of $U(2,4)$. Thus, $U(2,4)$ has five 2-vertex self switchings when $uv \notin E(U(2,4))$.

Subcase 2.c.b. $m = 3$

The graph $G = U(3,4)$ is given in figure 19. Now $p-2 = m+n-2 = 5$ and so $d_G(u) + d_G(v) = p-2 = 5$ implies that $uv \in \{v_2u_2, v_2u_3, u_1u_4\}$. The graphs $G^{\{v_2,u_2\}}, G^{\{v_2,u_3\}}$ and $G^{\{u_1,u_4\}}$ are given in figures 36.

The graph $G^{\{v_2,u_2\}}$ has no vertex of degee 1 and so $\{v_2, u_2\}$ is not a 2-vertex self switching of G. Now $G^{\{v_2,u_3\}} \cong G$ and an isomorphism f between them is given by $f(v_1) = v_1, f(v_2) = u_3, f(v_3) = v_3, f(u_1) = u_1, f(u_2) = u_2, f(u_3) = v_2$ and $f(u_4) = u_4$. Hence, $\{v_2, u_3\}$ is a 2-vertex self switching of $U(3,4)$. Also $G^{\{u_1,u_4\}} \cong G$ and so $\{u_1, u_4\}$ is a 2-vertex self switching of $U(3,4)$. Thus $U(3,4)$ has two 2-vertex self switchings when $uv \notin E(U(3,4))$.



(a) $G^{\{v_2,u_2\}}$ (b) $G^{\{v_2,u_3\}}$ (c) $G^{\{u_1,u_4\}}$

Figure 36. $G^{\{v_2,u_2}, G^{\{v_2,u_3\}}, G^{\{u_1,u_4\}}$

Subcase 2.c.c. $m \geq 4$

The graph $G = U(m,4)$ is given in figure 37. Here $p-2 = m+2$ and so $d_G(u) + d_G(v) = p-2 = m+2$ implies that uv is u_1u_4. The graph $G^{\{u_1,u_4\}}$ is given in figure 37. Clearly, $G^{\{u_1,u_4\}} \cong G$ and an isomorphism f between them is given by $f(v_1) = v_1, f(v_2) = v_2, ..., f(v_m) = v_m, f(u_1) = u_4, f(u_2) = u_2, f(u_3) = u_3$ and $f(u_4) = u_1$ and is shown in figure 38. Hence, $U(m,4)$ has one 2-vertex self switching when $uv \notin E(U(m,4))$.

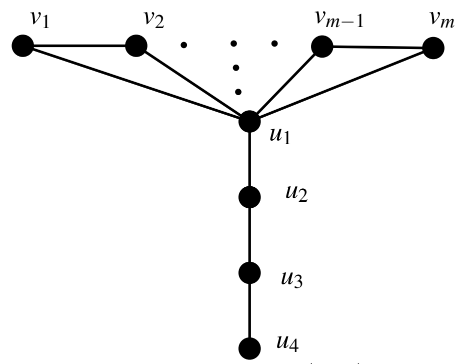


Figure 37. $G = U(m,4)$

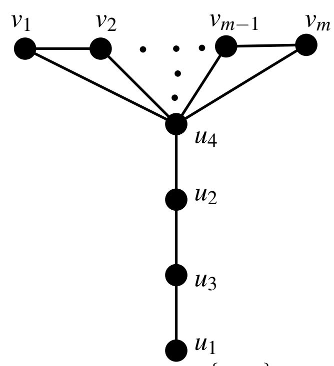


Figure 38. $G^{\{u_1,u_4\}}$

Subcase 2.4. $n = 5$
Subcase 2.4.a. $m = 2$

The graph $G = U(2,5)$ is given in figure 39. Then $p-2 = 5$ and so $d_G(u) + d_G(v) = p-2 = 5$ implies that $uv \in \{u_1u_3, u_1u_4\}$. The vertices u_1 and u_3 are adjacent to the vertex v_2 of degree 2 in $U(2,5)$ and thereby Theorem 2.2, $\{u_1, u_3\}$ is not a 2-vertex switching of $U(2,5)$. The graph $G^{\{u_1,u_4\}}$ is given in figure 40.

Clearly, $G^{\{u_1,u_4\}} \cong G$ and an isomorphism f between them is given by $f(v_1) = v_1, f(v_2) = v_2, f(u_1) = u_4, f(u_2) = u_3, f(u_3) = u_2, f(u_4) = u_1$ and $f(u_5) = u_5$. Hence $U(2,5)$ has one 2-vertex self switching when $uv \notin E(U(2,5))$.

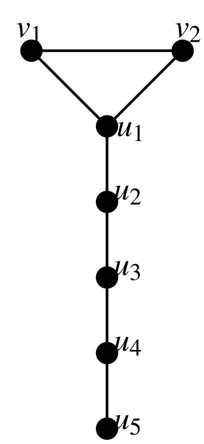


Figure 39. $G = U(2,5)$

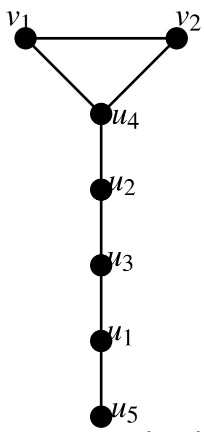


Figure 40. $G^{\{u_1,u_4\}}$

Subcase 2.d.b. $m \geq 3$

The graph $G = U(m,5)$ is given in figure 41. Clearly, $d_G(u)+d_G(v) \in \{3,4,5,6,m+2,m+3\}$. Since $p = m+n$, $p-2 = m+3$ and so $uv \in \{u_1u_3, u_1u_4\}$. The vertices u_1 and u_3 are adjacent to the vertex v_2 of degree 2 in $U(m,5)$ and thereby Theorem 2.2, $\{u_1,u_3\}$ is not a 2-vertex switching of $U(m,5)$. The graph $G^{\{u_1,u_4\}}$ is also given in figure 42.

Clearly, $G^{\{u_1,u_4\}} \cong G$ and an isomorphism f between them is given by $f(v_1) = v_1, f(v_2) = v_2, ..., f v_m) = v_m, f(u_1) = u_4, f(u_2) = u_3, f(u_3) = u_2, f(u_4) = u_1$ and $f(u_5) = u_5$. Hence $U(m,5)$ has one 2-vertex self switching when $uv \notin E(U(m,5))$.

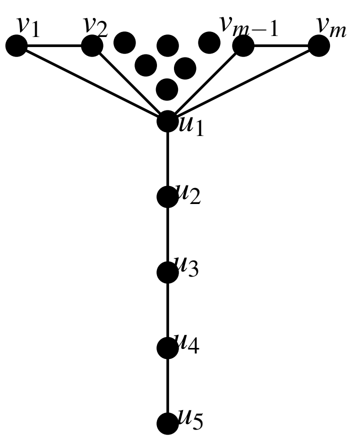


Figure 41. $G = U(m,5)$

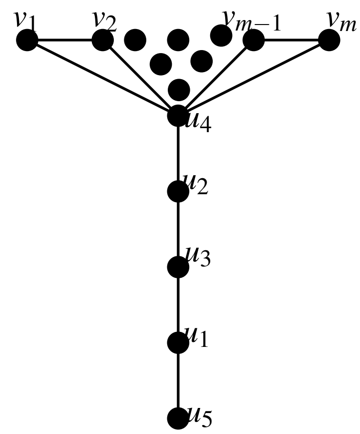


Figure 42. $G^{\{u_1,u_4\}}$

Subcase 2.e. $n \geq 6$

Clearly, for $2 \leq m \leq 4$, $d_G(u)+d_G(v) \in \{3,4,5,m+2,m+3\}$ and for $m \geq 5$, $d_G(u)+d_G(v) \in \{3,4,5,6,m+2,m+3\}$. Since $p = m+n$ and $n \geq 6$, $p-2 \geq m+4$. Also for $2 \leq m \leq 4$, $p-2 \geq 6$, and for $m \geq 5$, $p-2 \geq 9$ and thereby no possible $uv \notin E(U(m,n))$ exists and so $ss_2(U(m,n)) = 0$ for $uv \notin E(U(m,n))$.

From subcases 1.a.a and 2.a.a, $ss_2(U(2,2)) = 2$, from subcases 1.a.b and 2.a.b, $ss_2(U(3,2)) = 3$, from subcases 1.a.c, 2.a.c, 2.a.d and 2.a.e, $ss_2(U(m,2))$=1 for all $m \geq 4$, from subcases 1.b.a and 2.b.a, $ss_2(U(2,3)) = 3$, from subcases 1.b.b and 2.b.b, $ss_2(U(3,3)) = 6$, from subcases 1.b.c, 2.b.c and 2.b.d, $ss_2(U(m,3)) = 1$ for all $m \geq 4$, from subcases 1.c.a and 2.c.a, $ss_2(U(2,4)) = 5$, from subcases 1.c.b and 2.c.b, $ss_2(U(3,4)) = 2$, from subcases 1.c.c and 2.c.c, $ss_2(U(m,4)) = 1$ for all $m \geq 4$, from subcase 2.d.a, $ss_2(U(2,5)) = 1$ and from subcases 2.d.b and 2.e, $ss_2(U(m,n)) = 0$ for all other values of u and n. Hence the theorem. ☐

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ISSN(P):2319 − 3786
Malaya Journal of Matematik
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