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Distance$-k$ unique isolate perfect domination on graphs

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Abstract

A domination set D of a graph G is perfect if each vertex of $V(G)-D$ is dominated by exactly one vertex in D. A dominating set D is called k-perfect if for every $u \in V-D$ there exists a unique vertex $w \in D$ such that $d(u,D) = d(u,w) \leq k$. For an integer $k \geq 1$, $D \subseteq V(G)$ is a distance k-dominating set of G, if every vertex in $V(G)-D$ is within the distance k from some vertex $v \in D$. That is, $N_k[D] = V(G)$. A distance$-k$ perfect dominating set D of G is said to be a distance$-k$ UIPDS of G if $< D >$ has exactly one isolated vertex and D is $k-$perfect. This paper includes some properties of distance$-k$ UIPDS and gives the distance$-k$ UIPD number of paths, cycles, complete a-partite graphs, disconnected graphs and some directed graphs.

Keywords

Unique isolate dominating set, distance$-k$ unique isolate perfect dominating set, distance$-k$ unique isolate perfect domination number.

AMS Subject Classification

05C69.

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Contents

1. Introduction

In this paper, we consider finite non-trivial graphs with no loops and no multiple edges. Isolate domination was introduced by Sahul Hamid and S.Balamurugan in 2016 [4]. A dominating set S is called isolate dominating set if $\langle S \rangle$ has at least one isolate vertex. The minimum cardinality of a minimal isolate dominating set is called the isolate domination number γ_0.

In 2020, Sivagnanam Mutharasu and V. Nirmala [5] introduced the concept of unique isolate perfect domination in graphs. An isolate dominating set S of a graph G is called to be an UIPDS of G if there exists exactly one isolated vertex in $\langle S \rangle$ and the set S is a perfect dominating set. The minimum cardinality of a UIPDS of G is called UIPD number $\gamma_{0,p}^{U}(G)$.

By using the above concept 'UIPDS', we define a new parameter called "distance$-k$ Unique Isolate Perfect Domination (distance$-k$ UIPD)".

A dominating set S is a perfect dominating set if $|N(v) \cap S| = 1$ for each $v \in V-S$. For an integer $k \geq 1$, $D \subseteq V(G)$ is a distance k-dominating set of G, if every vertex in $V(G)-D$ is within the distance k from some vertex $v \in D$. That is, $N_k[D] = V(G)$. The minimum cardinality of a distance k-domination set is the distance-k domination number of G and it is denoted by $\gamma_k(G)$ [6].

A dominating set D is called k-perfect if for every $u \in V-D$ there exists a unique vertex $w \in D$ such that $d(u,D) = d(u,w) \leq k$.

From now on, in this paper we meant perfect as k-perfect. A distance$-k$ perfect dominating set D of G is said to be a distance$-k$ UIPDS of G if $\langle D \rangle$ has exactly one isolated vertex. A distance$-k$ UIPDS D is said to be minimal if no proper subset of D is an distance$-k$ UIPDS.The minimum(maximum) cardinality of a minimal distance$-k$ UIPDS of G is called distance$-k$ UIPD number $\gamma_{0,p,k}^{U}(G)$ (distance$-k$ upper isolate perfect domination number $\Gamma^{U}_{0,p,k}(G)$).

In this paper, we obtain some basic properties of distance−k UIPDS and also we obtain distance−k UIPD number of paths, cycles, complete a-partite graphs, disconnected graphs, unidirectional paths, unidirectional cycles and disconnected digraphs.

2. Distance−k unique isolate perfect domination on graphs

In this section, we obtain some basic properties of distance−k UIPDS. Also we obtain distance−k UIPD number of paths, cycles, complete a-partite graphs and disconnected graphs.

Theorem 2.1. *For any graph G, we have $\gamma_{0,k}(G) \leq \gamma^{U}_{0,p,k}(G)$.*

Proof. Since every distance−k UIPDS of G is also a distance−k isolate dominating set of G, we have $\gamma_{0,k}(G) \leq \gamma^{U}_{0,p,k}(G)$. □

Remark 2.2. *If D is a distance−k UIPDS of a graph G, then the induced subgraph $< D >$ has exactly one isolated vertex and all other vertices of D has a neighbor in D.*

Lemma 2.3. *Let D be any distance−k UIPDS of a graph G such that every non-isolated vertex of $< D >$ has a distance−k private neighbor with respect to D. Then D is minimal.*

Proof. Let $v \in D$ and x be the isolated vertex in $< D >$. If $v = x$, then $D - \{v\}$ will not dominate the vertex v. If $v \neq x$, then there exists w such that w is the distance−k private neighbor of v in D. In this case, $D - \{v\}$ is not a distance−k UIPDS. Thus D is minimal. □

Remark 2.4. The converse of Lemma 2.3 is not true. For example, consider the Path P_n where $n = 4k+4$ with $V(P_n) = \{1, 2, \ldots, 4k+4\}$. Then $D = \{k+1, 3k+2, 3k+3, 3k+4\}$ is a minimum distance−k UIPDS of G but the vertex $3k+3$ has no private neighbor in D.

Lemma 2.5. *(a). If $k \geq rad(G)$ then there exists a vertex in G such that $\{v\}$ is a distance−k UIPDS and $\gamma^{U}_{0,p,k}(G) = 1$.*
(b). For any graph G which admits distance−k UIPDS, we have $\gamma^{U}_{0,p,k}(G) \neq 2$.

Proof. (a). Let r be the radius of G ($rad(G)$). Then there exists a vertex $v \in V(G)$ such that $d(v,w) \leq r$ for all $w \in V(G)$. Since $r \leq k$, $d(v,w) \leq k$ for all $w \in V(G)$. Thus $\{v\}$ is a distance−k UIPDS of G.
(b). Since any distance−k UIPDS D of a graph G contains exactly one isolated vertex in $< D >$, $\gamma^{U}_{0,p,k}(G) \neq 2$. □

Corollary 2.6. *Let $k > \lfloor n/2 \rfloor$ be an integer. Then the sun graph $Sun(n)$ admits distance−k UIPDS with $\gamma^{U}_{0,p,k}(Sun(n)) = 1$.*

Proof. Note that $rad(Sun(n)) = \lfloor n/2 \rfloor$. Thus by case a of the Lemma 2.5, the result is trivial. □

Lemma 2.7. *Let $a \geq 2$ be an integer and $G = K_{p_1,p_2,\ldots,p_a} = (P_1, P_2, \ldots, P_a)$ be a complete a−partite graph.*
(a) If $k = 1$, then G admits distance−k UIPDS if, and only if, $p_i = 1$ for some integer i with $1 \leq i \leq a$.
(b) If $k \geq 2$, then G admits distance−k UIPDS with distance−k UIPDN 1.

Proof. Assume that G admits distance−k UIPDS, say D.
(a) Suppose $k = 1$. On the contrary, assume that $p_i \geq 2$ for all $1 \leq i \leq a$.
Let x be the isolated vertex of $< D >$. Without loss of generality, assume that $x \in P_1$. Since $|P_1| \geq 2$, we can choose a vertex $y \in P_1$ such that $y \neq x$. Note that no vertex of $P_2 \cup P_3 \cup \ldots \cup P_a$ will be in D(otherwise x will not be isolated in $< D >$). Thus to dominate the vertex y, D must include y and hence $< D >$ has more than one isolated vertex, namely x and y, a contradiction.
(b) Suppose $k \geq 2$. Note that $rad(G) = 2$. Thus by Lemma 2.5, G admits a distance−k UIPDS with distance−k UIPDN 1. □

Lemma 2.8. *Let $n, i \geq 1$ be an integer. Then*
(a). $\gamma^{U}_{0,p,k}(P_n) = 2\lceil \frac{n-(2k+1)}{2k+2} \rceil$ if $n = (2k+1) + i(2k+2) + 1$.
(b). $\gamma^{U}_{0,p,k}(P_n) = 2\lceil \frac{n-(2k+1)}{2k+2} \rceil + 1$ if $n = (2k+1) + i(2k+2) + j$ $2 \leq j \leq 2k+1$.
(c). $\gamma^{U}_{0,p,k}(P_n) = 2\lceil \frac{n-(2k+1)}{2k+2} \rceil + 1$ if $n = (2k+1) + i(2k+2)$.

Proof. Let D be a minimum distance−k UIPDS and x be the isolated vertex in $< D >$.
Note that x can dominate a maximum of $(2k+1)$ vertices including it.$- - - - - >$(1)
Also any other vertex of D has a neighbor in D. Further two adjacent vertices of D can dominate a maximum of $(2k+2)$ vertices including them.$- - - - - >$(2)
Case a: $n = (2k+1) + i(2k+2) + 1$ for some $i \geq 1$.
From equations (1) and (2), to dominate $(2k+1) + i(2k+2)$ vertices, D must have $1 + 2i$ vertices in it. To dominate the remaining 1 vertex, D must include one more vertex in it. Thus $|D| \geq 1 + 2i + 1 = 2\lceil \frac{n-(2k+1)}{2k+2} \rceil$.
Consider the set $D = \{k+1\} \cup \{(2b+1)(k+1) - 1, (2b+1)(k+1) : b = 1, 2, 3, \ldots, i\} \cup \{(2i+1)(k+1)+1\}$ with $2i+2 = 2\lceil \frac{n-(2k+1)}{2k+2} \rceil$ elements.
Let $v \in V - D$.
Subcase a.1: Suppose $1 \leq v \leq k$ or $k+2 \leq v \leq 2k+1$.
Then v is dominated only by $k+1 \in D$.
Subcase a.2: Suppose $v = ((2b+1)(k+1) - 1) - a$: $1 \leq a \leq k$; $1 \leq b \leq i$.
Then v is dominated only by $(2b+1)(k+1) - 1 \in D$.
Subcase a.3: Suppose $v = ((2b+1)(k+1)) + a$: $1 \leq a \leq k$; $1 \leq b \leq i-1$.
Then v is dominated only by $(2b+1)(k+1) \in D$.
Subcase a.4: Suppose $v = ((2b+1)(k+1)+1) + a$: $1 \leq a \leq k$; $b = i$.
Then v is dominated only by $(2b+1)(k+1) \in D$.
Clearly for each $b = 1, 2, 3, \ldots, i$ the vertex $(2b+1)(k+1) - 1 \in D$ is adjacent with another vertex of D, namely $(2b+$

$1)(k+1)$.
Also the vertex $(2i+1)(k+1)+1 \in D$ is adjacent with $(2i+1)(k+1) \in D$. Further the vertex $(k+1) \in D$ is not adjacent with any vertices of D.
Thus $< D >$ has exactly one isolated vertex.
Hence D is a distance$-k$ dominating set.
Therefore D is a distance $k-$ UIPDS.
Case b: $n = (2k+1)+i(2k+2)+2$ for some $i \geq 1$.
We prove this case by using induction on i.
When $i = 1$, $n = 4k+5$.We claim that $\gamma^U_{0,p,k}(P_n) = 5$.
Let D be a minimum distance$-k$ UIPDS of P_n and x be an isolated in D.
Here, x will dominate a maximum of $2k+1$ vertices including x, let $N_k[x] = A$.
Since $|V(P_n)| > 2k+1$, by the definition of UIPD, D must have at least two adjacent vertices in D, say y and z.
Note that y and z together will dominate a maximum of $(2k+2)$ vertices including y and z, let $N_k[\{y,z\}] = B$.
Note that there are at least two undominated vertices, say u, v which lies outside of $A \cup B$ as given below.

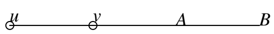

Figure.2.1

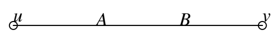

Figure.2.4

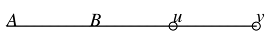

Figure.2.2

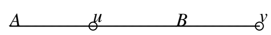

Figure.2.5

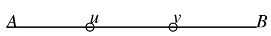

Figure.2.3

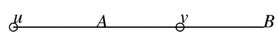

Figure.2.6

Subcase b.1: Suppose u and v are as shown in Figure.2.1.
If either u or v alone in D(without loss of generality, let it be v), then $< D >$ has two isolated vertices, namely x and v, a contradiction.
Suppose $u, v \notin D$. Then there must exists a vertex $w \in D$ such that $3 \leq w \leq k+1$ and u and v are dominated by w.
In this case, D has two isolated vertices, namely w and x, a contradiction.
Thus $u, v \in D$ and so $|D| \geq 5$.
As proved in subcase b.1, we can prove that $|D| \geq 5$ if u and v are as shown in Figure.2.2 and Figure.2.3.
Subcase b.2: Suppose u and v are as shown in Figure.2.4.
In this case, the two vertices u and v are of distance greater than or equal to $2k+2$. Thus a vertex cannot dominate both u and v. Therefore D must include two more vertices in it and so $|D| \geq 5$.
As proved in subcase b.2, we can prove that $|D| \geq 5$ if u and v are as shown in Figure.2.5 and Figure.2.6.
Thus $\gamma^U_{0,p,k}(P_n) \geq 5$ when $i = 1$. Note that, when k is even $\{k+1, 3(k+1)-1, 3(k+1), 4k+4, 4k+5\}$ is a minimum distance$-k$ UIPDS and when k odd then $\{k+1, 3(k+1)-1, 3(k+1), 4k+3, 4k+4\}$ is a minimum distance$-k$ UIPDS.
Thus $\gamma^U_{0,p,k}(P_n) \leq 5$.
Hence we proved the result for $i = 1$.
Assume the result for all $i \leq a$. Next we prove the result for $i = a+1$.
By mathematical induction, we have $\gamma^U_{0,p,k}(P_n) = 2\lceil \frac{n-(2k+1)}{2k+2} \rceil + 1 = 2a+3$ if $n = (2k+1)+a(2k+2)+2$ for some $i \geq 1$.$- - - - - >(1)$.
Suppose $n = (2k+1)+(a+1)(2k+2)+2 = (2k+1)+a(2k+2)+2k+4$.
By (1), to dominate the first $(2k+1)+a(2k+2)+2$ vertices, we need $2a+3$ vertices in D. Thus to dominate the remaining $(2k+2)$ vertices, D must have at least two more vertices in it and so $\gamma^U_{0,p,k}(P_n) \geq 2a+5$ when $i = a+1$. Note that, when k is even $D = \{k+1\} \cup \{(2b+1)(k+1)-1, (2b+1)(k+1) : b = 1,2,3,\ldots,i\} \cup \{n-\lceil \frac{j}{2} \rceil, n-(\lceil \frac{j}{2} \rceil - 1)$ is a minimum distance$-k$ UIPDS and when k is odd $D = \{k+1\} \cup \{(2b+1)(k+1)-1, (2b+1)(k+1) : b = 1,2,3,\ldots,i\} \cup \{n-(\lceil \frac{j}{2} \rceil + 1), n-\lceil \frac{j}{2} \rceil$ is a minimum distance$-k$ UIPDS. Hence $\gamma^U_{0,p,k}(P_n) \leq 2a+5$. Thus $\gamma^U_{0,p,k}(P_n) = 2a+5 = 2\lceil \frac{n-(2k+1)}{2k+2} \rceil + 1$.
When $n = (2k+1)+i(2k+2)+j$ for some $3 \leq j \leq 2k+1$, as proved in case b, we can prove that $\gamma^U_{0,p,k}(P_n) = 2\lceil \frac{n-(2k+1)}{2k+2} \rceil + 1$.
Case c: $n = (2k+1)+i(2k+2)$ for some $i \geq 1$.
As proved in above cases, we can prove $\gamma^U_{0,p,k}(P_n) = 2\lceil \frac{n-(2k+1)}{2k+2} \rceil + 1$. Note that $D = \{k+1\} \cup \{(2b+1)(k+1)-1, (2b+1)(k+1) : b = 1,2,3,\ldots,i\}$. □

As proved the above result, we can prove the following result.

Lemma 2.9. *Let C_n be a cycle of n vertices for $n \geq 1$. Then*
(a). $\gamma^U_{0,p,k}(C_n) = 2\lceil \frac{n-(2k+1)}{2k+2} \rceil$ if $n = (2k+1)+i(2k+2)+1$.
(b). $\gamma^U_{0,p,k}(C_n) = 2\lceil \frac{n-(2k+1)}{2k+2} \rceil + 1$ if $n = (2k+1)+i(2k+2)+j$, $2 \leq j \leq 2k$.
(c). $\gamma^U_{0,p,k}(C_n) \leq 2\lceil \frac{n-(2k+1)}{2k+2} \rceil + 2$ if $n = (2k+1)+i(2k+2)+2k+1$.
(d). $\gamma^U_{0,p,k}(C_n) = 2\lceil \frac{n-(2k+1)}{2k+2} \rceil + 1$ if $n = (2k+1)+i(2k+2)$.

Theorem 2.10. *Let $n \geq 2$ be an integer and let G be a disconnected graph with n components $G_1, G_2, \ldots, G_n$ such that the first r components $G_1, G_2, \ldots, G_r$ admit distance$-k$ UIPD. Then $\gamma^U_{0,p,k}(G) = \min_{1 \leq i \leq r} \{t_i\}$,*
where $t_i = \gamma^U_{0,p,k}(G_i) + \sum_{j=1, j \neq i}^{n} \gamma_{t,p,k}(G_j)$ for $1 \leq i \leq r$ and $|V(G_j)| \geq 2$.

Proof. Without loss of generality, let $t_1 = \min_{1 \leq i \leq r} \{t_i\}$.
Let D be a $\gamma^U_{0,p,k}$- set of G_1 and D_i be a $\gamma_{t,p,k}$- set of G_i for each i with $2 \leq i \leq n$. Then $D \cup (\bigcup_{i=2}^{n} D_i)$ is a distance$-k$ UIPDS of G with cardinality $\gamma^U_{0,p,k}(G_1) + \sum_{i=2}^{n} \gamma_{t,p,k}(G_i)$ and so $\gamma^U_{0,p,k}(G) \leq \gamma^U_{0,p,k}(G_1) + \sum_{i=2}^{n} \gamma_{t,p,k}(G_i) = t_1$.
Let D be a minimal distance$-k$ UIPDS of G. Then D must intersect $V(G_i)$ for each $1 \leq i \leq n$. Further, there exists an

integer j such that $D \cap V(G_j)$ is a minimal distance$-k$ UIPDS of G_j and $1 \le j \le r$. Also for each $1 \le i \le n, i \ne j$, the set $D \cap V(G_i)$ is a minimal distance$-k$ total perfect dominating set of G_i.

Therefore $|D| \ge \gamma^U_{0,p,k}(G_j) + \sum_{i=1, i \ne j}^{n} \gamma_{t,p,k}(G_i) \ge t_1$ and hence $\gamma^U_{0,p,k}(G) = \min_{1 \le i \le r}\{t_i\}$. □

3. Distance-k unique isolate perfect domination on digraphs

In this section, we obtain some basic properties of distance$-k$ UIPDS of digraphs and also we obtain distance$-k$ UIPD number of unidirectional paths, unidirectional cycles and disconnected digraphs.

Remark 3.1. *If D is a distance$-k$ UIPDS of a digraph $\overrightarrow{G}$, then the induced subgraph $< D >$ has exactly one isolated vertex and all other vertices of D has a neighbor in D.*

Lemma 3.2. *(a). If a digraph $\overrightarrow{G}$ has at least one vertex u such that $N^+_k[u] = V(\overrightarrow{G})$, then $\overrightarrow{G}$ admits distance$-k$ UIPD with $\gamma^U_{0,p,k}(\overrightarrow{G}) = 1$.*

(b). For any digraph $\overrightarrow{G}$ which admits distance$-k$ UIPDS, we have $\gamma^U_{0,p,k}(\overrightarrow{G}) \ne 2$.

Proof. (a). The set $D = \{v\}$ is distance$-k$ UIPDS of $\overrightarrow{G}$ with $\gamma^U_{0,p,k}(\overrightarrow{G}) = 1$.

(b). Since any distance$-k$ UIPDS D of a digraph $\overrightarrow{G}$ contains exactly one isolated vertex in $< D >$, $\gamma^U_{0,p,k}(\overrightarrow{G}) \ne 2$. □

Lemma 3.3. *Let $n, i \ge 1$ be an integers. Then*

(a). $\gamma^U_{0,p,k}(\overrightarrow{P}_n) = 2\lceil \frac{n-(k+1)}{k+2} \rceil$ if $n = (k+1) + i(k+2) + 1$.

(b). $\gamma^U_{0,p,k}(\overrightarrow{P}_n) = 2\lceil \frac{n-(k+1)}{k+2} \rceil + 1$ if $n = (k+1) + i(k+2) + j$, $2 \le j \le k+1$.

(c). $\gamma^U_{0,p,k}(\overrightarrow{P}_n) = 2\lceil \frac{n-(k+1)}{k+2} \rceil + 1$ if $n = (k+1) + i(k+2)$.

Proof. Let D be a minimum distance$-k$ UIPDS and x be the isolated vertex in $< D >$.

Note that x can dominate a maximum of $(k+1)$ vertices including it.$- - - - - >$(1)

Also any other vertex of D has a neighbor in D. Further two adjacent vertices of D can dominate a maximum of $(k+2)$ vertices including them.$- - - - - >$(2)

Case a: $n = (k+1) + i(k+2) + 1$ for some $i \ge 1$.

From equations (1) and (2), to dominate $(k+1) + i(k+2)$ vertices, D must have $1 + 2i$ vertices in it. To dominate the remaining 1 vertex, D must include one more vertex in it. Thus $|D| \ge 1 + 2i + 1 = 2\lceil \frac{n-(k+1)}{k+2} \rceil$.

Consider the set $D = \{1\} \cup \{b(k+2), b(k+2)+1 : b = 1, 2, 3, \ldots, i\} \cup \{i(k+2)+2\}$ with $2i + 2 = 2\lceil \frac{n-(k+1)}{k+2} \rceil$ elements.

Let $v \in V - D$.

Subcase a.1: Suppose $2 \le v \le k$.

Then v is dominated only by $1 \in D$.

Subcase a.2: Suppose $v = b(k+2) + a$: $2 \le a \le k$; $1 \le b \le i-1$.

Then v is dominated only by $b(k+2) + 2 \in D$.

Subcase a.3: Suppose $v = i(k+2) + a$: $2 \le a \le k$.

Then v is dominated only by $i(k+2) + 1 \in D$.

Clearly $(b(k+2), b(k+2)+1) \in E(G)$ forall $b = 1, 2, 3, \ldots, i$ and $(i(k+2)+1, i(k+2)+2) \in E(G)$.

Thus $< D >$ has exactly one isolated vertex. Therefore D is a distance$-k$ UIPDS and $|D| \ge 2i + 2$.

Hence $|D| = 2i + 2$.

Case b: Suppose $n = (k+1) + i(k+2) + 2$ for some $i \ge 1$.

We prove this case by using induction on i.

When $i = 1$, $n = 2k + 5$. We claim that $\gamma^U_{0,p,k}(\overrightarrow{P}_n) = 5$.

Let D be a minimum distance$-k$ UIPDS of $\overrightarrow{P}_n$ and x be an isolated in D.

Here, x will dominate a maximum of $k+1$ vertices including x, let $N_k[x] = A$;$A = \{x, x+1, \ldots, x+k\}$.

Since $\left|V(\overrightarrow{P}_n)\right| > k+1$, by the definition of UIPD, D must have at least two adjacent vertices in D, say y and $y+1$.

Note that y and $y+1$ together will dominate a maximum of $(k+2)$ vertices including y and $y+1$, let $N_k[\{y, y+1\}] = B$; $B = \{y, y+1, \ldots, y+k+1\}$.

Note that there are at least two undominated vertices, say u, v which lies outside of $A \cup B$ as given below.

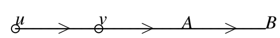

Figure.3.1

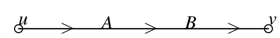

Figure.3.4

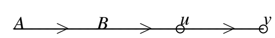

Figure.3.2

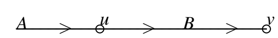

Figure.3.5

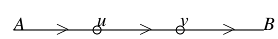

Figure.3.3

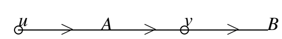

Figure.3.6

Subcase b.1: Suppose u and v are as shown in Figure.3.1.

If $v \in D$, then $(v, x) \in E(\overrightarrow{P_n})$, x is not a isolated vertex in D, a contradiction.

If $u \in D$ and $v \notin D$, then u and x are two isolated vertices in $< D >$, a contradiction. Therefore this case does not exist.

Subcase b.2: Suppose u and v are as shown in Figure.3.2.

If we take either u or v alone in D, then $< D >$ has two isolated vertices namely x and u or v, a contradiction. Thus $u, v \in D$ and so $|D| \ge 5$.

Subcase b.3: Suppose u and v are shown in Figure.3.3.

W.K.T $x, x+1, x+2, \ldots, x+k+1$ are all vertices in A and $x \in D$.

To dominate u and v, either $x+1$ or $x+2$ or $\ldots$ or $x+k+1$ or $u \in D$. Suppose $x+1 \in D$, since $(x, x+1) \in E(\overrightarrow{P_n})$, a contradiction to x is isolated vertex in D.

Therefore $x+1 \notin D$.

Suppose a vertex $x+i \in D$ $\forall i = 2, 3, \ldots, k+1$ then x and $x+i$ are two isolated vertex in $< D >$, a contradiction. Thus $\{x+i, x+i+1\} \in D$ and so $|D| \ge 5$.

Subcase b.4: Suppose u and v are shown in Figure.3.4 and Figure.3.6, to dominate u, D must include u, a contradiction to x is isolated vertex in $< D >$. Thus these two cases does not exist.
Subcase b.5: Suppose u and v as shown in Figure.3.5, to dominate v, either $y+2$ or $y+3$ or $\ldots$ or $y+k+1$ or $v \in D$.
If $v \in D$, then there exist two isolated vertices in $< D >$ namely x and v, a contradiction.
We must include one more vertices $y+k+1$, and so $(y+k+1, v) \in E(\overrightarrow{P_n})$. Thus $\gamma^U_{0,p,k}(\overrightarrow{P_n}) \geq 5$.
If $y+2 \in D$, then v dominated by $y+2 \in D$, then v dominated by $y+2$. Further to dominate u, as we proved in subcase 3 we must include at least one vertex in D.
Thus $\gamma^U_{0,p,k}(\overrightarrow{P_n}) \geq 5$.
Note that $\{1, k+2, k+3, 2k+4, 2k+5\}$ is a minimum distance$-k$ UIPDS and so $\gamma^U_{0,k}(\overrightarrow{P_n}) \leq 5$. Hence we proved the result for $i = 1$.
Assume the result for all $i \leq a$. Next we prove the result for $i = a+1$.
By mathematical induction, we have $\gamma^U_{0,k}(\overrightarrow{P_n}) = 2\lceil \frac{n-(k+1)}{k+2} \rceil + 1 = 2a+3$ if $n = (k+1) + a(k+2) + 2$ for some $i \geq 1$.$-----> (1)$
Suppose $n = (k+1) + (a+1)(k+2) + 2 = (k+1) + a(k+2) + (k+2) + 2$.
By (1), to dominate the first $(k+1) + a(k+2) + 2$ vertices, we need $2a+3$ vertices in D. Thus to dominate the remaining $(k+2)$ vertices, D must have at least two more vertices in it and so $\gamma^U_{0,p,k}(\overrightarrow{P_n}) \geq 2a+5$ when $i = a+1$.
Note that $D = \{1\} \cup \{b(k+2), b(k+2)+1 : b = 1,2,3,\ldots, i+1\}$ is a minimum distance$-k$ UIPDS and so $\gamma^U_{0,p,k}(\overrightarrow{P_n}) \leq 2a+5$. Thus $\gamma^U_{0,p,k}(\overrightarrow{P_n}) = 2a+5 = 2\lceil \frac{n-(k+1)}{k+2} \rceil + 1$.
When $n = (k+1) + i(k+2) + j$ for some $3 \leq j \leq k+1$, as proved in case b, we can prove that $\gamma^U_{0,p,k}(\overrightarrow{P_n}) = 2\lceil \frac{n-(k+1)}{k+2} \rceil + 1$.
Case c: Suppose $n = (k+1) + i(k+2)$.
As proved in above cases, we can prove $\gamma^U_{0,p,k}(\overrightarrow{P}_n) = 2\lceil \frac{n-(k+1)}{k+2} \rceil + 1$. Note that $D = \{1\} \cup \{b(k+2), b(k+2)+1 : b = 1,2,3,\ldots, i\}$ is a minimum distance$-k$ UIPDS. □

As proved the above result, we can prove the following result.

Lemma 3.4. *Let $\overrightarrow{C}_n$ be an unidirectional cycle of n vertices for $n \geq 1$. Then*
(a). $\gamma^U_{0,p,k}(\overrightarrow{C}_n) = 2\lceil \frac{n-(k+1)}{k+2} \rceil$ *if* $n = (k+1) + i(k+2) + 1$.
(b). $\gamma^U_{0,p,k}(\overrightarrow{C}_n) = 2\lceil \frac{n-(k+1)}{k+2} \rceil + 1$ *if* $n = (k+1) + i(k+2) + j$, $2 \leq j \leq k+1$.
(c). $\gamma^U_{0,p,k}(\overrightarrow{C}_n) = 2\lceil \frac{n-(k+1)}{k+2} \rceil + 1$ *if* $n = (k+1) + i(k+2)$.

Theorem 3.5. *Let $n \geq 2$ be an integer and let $\overrightarrow{G}$ be a disconnected directed graph with n components $\overrightarrow{G}_1, \overrightarrow{G}_2, \ldots, \overrightarrow{G}_n$ such that the first r components $\overrightarrow{G}_1, \overrightarrow{G}_2, \ldots, \overrightarrow{G}_r$ admit distance$-k$ UIPD. Then $\gamma^U_{0,p,k}(\overrightarrow{G}) = \min_{1\leq i\leq r}\{t_i\}$, where $t_i = \gamma^U_{0,p,k}(\overrightarrow{G}_i) + \sum_{j=1, j\neq i}^{n} \gamma_{t,p,k}(\overrightarrow{G}_j)$ for $1 \leq i \leq r$ and $|V(\overrightarrow{G}_j)| \geq 2$.*

Proof. Without loss of generality, let $t_1 = \min_{1\leq i\leq r}\{t_i\}$.
Let D be a $\gamma^U_{0,p,k}$- set of $\overrightarrow{G}_1$ and D_i be a $\gamma_{t,p,k}$- set of $\overrightarrow{G}_i$ for each i with $2 \leq i \leq n$.
Then $D \cup (\bigcup_{i=2}^{n} D_i)$ is a distance$-k$ UIPDS of $\overrightarrow{G}$ with cardinality $\gamma^U_{0,p,k}(\overrightarrow{G}_1) + \sum_{i=2}^{n} \gamma_{t,p,k}(\overrightarrow{G}_i)$ and so $\gamma^U_{0,p,k}(\overrightarrow{G}) \leq \gamma^U_{0,p,k}(\overrightarrow{G}_1) + \sum_{i=2}^{n} \gamma_{t,p,k}(\overrightarrow{G}_i) = t_1$.
Let D be a minimal distance$-k$ UIPDS of $\overrightarrow{G}$. Then D must intersect $V(\overrightarrow{G}_i)$ for each $1 \leq i \leq n$.
Further, there exists an integer j such that $D \cap V(\overrightarrow{G}_j)$ is a minimal distance$-k$ UIPDS of $\overrightarrow{G}_j$ and $1 \leq j \leq r$. Also for each $1 \leq i \leq n, i \neq j$, the set $D \cap V(\overrightarrow{G}_i)$ is a minimal distance$-k$ total perfect dominating set of $\overrightarrow{G}_i$.
Therefore $|D| \geq \gamma^U_{0,p,k}(\overrightarrow{G}_j) + \sum_{i=1, i\neq j}^{n} \gamma_{t,p,k}(\overrightarrow{G}_i) \geq t_1$ and hence $\gamma^U_{0,p,k}(\overrightarrow{G}) = \min_{1\leq i\leq r}\{t_i\}$.

□

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