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On derivations and Lie structure of semirings

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Abstract. In [9], Herstein introduced the notion of the Lie structure of associative rings and established the Lie type theory for rings. This paper extends these ring theoretical results and also extends some well known results of [3, 7, 8] in the framework of semirings, which are very important to investigate the Lie type theory of semirings and their higher commutators. Moreover, we characterize the Lie structure of semirings and thereby explore the action of derivations on Lie ideals of semirings.

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Contents

1. Introduction

In past few years, various authors explored the brief structure of non commutative rings by developing the Lie type theory of associative rings. In 1969, Herstein [9] introduced the notion of Lie structure for rings and obtained several results which are helpful for rings of operators on a Hilbert space. The purpose of this study is to extend these results for some more general structure, e.g., the algebraic structure of non-commutative semirings. But the problem arises when we replace rings by semirings, as semirings do not have additive inverses, so we impose the weaker version of additive inverses, i.e., the pseudo inverse introduced by Karvellas [10]. Recently, semirings have been studied by various researchers (cf. [5, 11, 12]). In this paper, we generalized some of the Herstein's results in the framework of additively regular semirings which are further used to study the Lie structure of prime semirings and some of its subsets. Moreover, the behavior of derivations on Lie ideals of semirings is studied. Consequently, this enables us to measure the size of the centralizer of Lie ideals for the case of semirings. We

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also investigate some properties regarding Lie semirings which are very useful to investigate the Lie derivations and higher derivations of Lie ideals of semirings.

The prime motivation for this paper is not only the intense desire to extend these well known results, in the field of semirings, but to explore the close characterization of Lie theory and derivations in semirings. It is natural to point out that the ideal theory, homomorphisms and the Jordan theory are easily accessible to analyse in comparison to the Lie theory. As in the Lie case, the center of algebraic structure comes in our way and thereby various well known results are untouched in the corresponding Lie theory of semirings which are true in the aforementioned theories. Henceforth, this paper delivers the suitable techniques which can be efficiently used more and more to study the enormous structure of Lie semirings.

2. Preliminaries and some examples

Recall from [6] that a non empty set $\mathscr{S}$ is called a semiring if $(\mathscr{S},+)$ is a commutative monoid; $(\mathscr{S},\cdot)$ is a semigroup; and both distributive laws of multiplication over addition hold with $0\cdot t=0=t\cdot 0, \forall\ t\in\mathscr{S}$. Further, an element $t\in\mathscr{S}$ is called additively regular if and only if $\exists$ some element t' of $\mathscr{S}$ with $t+t+t'=t$ and $t'+t'+t=t'$ and $\mathscr{S}$ is known as an additively regular semiring if and only if $\mathscr{S}=\mathscr{S}'=reg(\mathscr{S})$, where $reg(\mathscr{S})$ represents the set of all additively regular elements of $\mathscr{S}$. The element s' is the pseudo inverse [10] of s. For instance, if $\mathbb{B}=\{0,1\}$ is a boolean semiring with binary operations as $0+0=0$; $0+1=1=1+0$; $1+1=1$ and $0\cdot 0=1\cdot 0=0\cdot 1=0$; $1\cdot 1=1$, then $\mathbb{B}$ is an additively regular semiring, where $t'=t,\forall\ t\in\mathbb{B}$ is the pseudo inverse of $t\in\mathbb{B}$. One can easily check that the pseudo inverse of an element is always unique. In 1982, Bandelt and Petrich [1] considered an additively regular semiring $\mathscr{S}$ with conditions:

$(A_1):a_1(a_1+a_1')=a_1+a_1', \forall\ a_1\in\mathscr{S}$; $(A_2):s_1(a_1+a_1')=(a_1+a_1')s_1, \forall\ a_1,s_1\in\mathscr{S}$; $(A_3):a_1+(a_1+a_1')s_1=1, \forall\ a_1,s_1\in\mathscr{S}$ and investigated various results for this class of semirings. In addition, every Bandelt semiring [6] is an additively regular semiring with A_2-condition.

Further, $\mathscr{S}$ is said to be prime if $\mathcal{H}\mathcal{K}=(0)$ infers that either $\mathcal{H}=(0)$ or $\mathcal{K}=(0)$, where $\mathcal{H}$ and $\mathcal{K}$ are any two ideals of $\mathscr{S}$. A semiring which does not have any nilpotent ideals is called a semiprime semiring. Note that every prime semiring is also a semiprime semiring.

For given $a,b\in\mathscr{S}$, then $[a,b]$ (the Lie bracket) symbolizes the element $ab+b'a$ or $ab+ba'$. Indeed, for $\mathcal{H},\mathcal{K}\subseteq\mathscr{S}$, the Lie bracket $[\mathcal{H},\mathcal{K}]$ is an additive submonoid of $\mathscr{S}$ which is generated by all elements of the form $hk+k'h$ or $hk+kh'$, for $h\in\mathcal{H}$ and $k\in\mathcal{K}$ and $(\mathcal{H})$ denotes the ideal generated by $\mathcal{H}$. However, an additive submonoid $\mathscr{L}$ of $\mathscr{S}$ is called a Lie ideal if $[\mathscr{L},\mathscr{S}]\subseteq\mathscr{L}$. Note that $[\mathscr{L}_1,\mathscr{L}_2]$ is also a Lie ideal of $\mathscr{S}$, for $\mathscr{L}_1$ and $\mathscr{L}_2$ are Lie ideals of $\mathscr{S}$, because of the existence of the Jacobi identity $[r_1,[s_1,t_1]]+[s_1,[t_1,r_1]]=[[r_1,s_1],t_1]$. Throughout this study, $\mathscr{S}$ represents an additively regular semiring with A_2-condition and $\mathscr{L}$ is a Lie ideal of $\mathscr{S}$, unless otherwise mentioned. We now delay the discussion of higher commutators of $\mathscr{S}$ until later and proceed with some results which will be used frequently in the sequel.

For simplicity, we denote $u_\circ=u+u'$ and by A_2- condition, $u_\circ\in\mathscr{Z}(\mathscr{S}), \forall\ u\in\mathscr{S}$, where $\mathscr{Z}(\mathscr{S})$ represents the center of $\mathscr{S}$.

Lemma 2.1 ([6]). *Let $\mathscr{S}$ be an additively regular semiring. Then the following hold:*
(i) $u_1''=u_1$; *(ii)* $u_1'v_1'=(u_1'v_1)'=(u_1v_1')'=(u_1v_1)''=u_1v_1$; *(iii)* $(u_1v_1)'=u_1'v_1=u_1v_1'$; *(iv)* $(u_1+v_1)'=u_1'+v_1'$; *(v) If* $u_1+v_1=0$, *then* $v_1=u_1'$; *(vi)* $u_{1\circ}+u_{1\circ}=u_{1\circ}=u_{1\circ}'$; *(vii)* $u_1+u_{1\circ}=u_1$; *(viii)* $u_1'+u_{1\circ}=u_1'$; *(ix)* $u_{1\circ}v_1=u_1v_{1\circ}=(u_1v_1)_\circ=u_{1\circ}v_{1\circ}=v_{1\circ}u_{1\circ}=(v_1u_1)_\circ, \forall\ u_1,v_1\in\mathscr{S}$.

Example 2.2. *Consider $\mathscr{S}=\{0,1,a\}$ having all additively idempotent elements and the binary operations in it can be illustrated with the help of the Cayley tables given below:*

$\oplus$	0	1	a
0	0	1	a
1	1	1	a
a	a	a	a

$\otimes$	0	1	a
0	0	0	0
1	0	1	a
a	0	a	a

One can easily see that the pseudo inverse of a is $a' = a, \forall\ a \in \mathscr{S}$. Obviously, $\mathscr{S}$ is an additively regular semiring with A_2-condition and $\mathscr{L} = \{0, a\}$, is a Lie ideal of $\mathscr{S}$.

The forthcoming example demonstrates that every additively regular semiring may not satisfy A_2-condition.

Example 2.3. *Consider a Boolean semiring $\mathbb{B}$ and $\mathscr{S} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times 2}(\mathbb{B}) : a, b, c, d \in \mathbb{B} \right\}$ with usual addition and usual multiplication of matrices. Define pseudo inverse of an element of $\mathscr{S}$ by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}' = \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix}, \forall$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathscr{S}$. Then $\mathscr{S}$ is an additively regular semiring which does not satisfy A_2-condition.*

The following lemma is easy to prove, so we omit it.

Lemma 2.4. *If $u_1, v_1, w_1 \in \mathscr{S}$, then*
(i) $[u_1, v_1w_1] = [u_1, v_1]w_1 + v_1[u_1, w_1]$; (ii) $[u_1v_1, w_1] = u_1[v_1, w_1] + [u_1, w_1]v_1$; (iii) $[u_1 + v_1, w_1] = [u_1, w_1] + [v_1, w_1]$; (iv) $[u_1, [v_1, w_1]] + [v_1, [w_1, u_1]] = [[u_1, v_1], w_1]$ (Jacobi Identity).

3. Lie Ideals and Higher Commutators

We hereby introduce the notion of higher commutators of semirings. Also, some results regarding higher commutators of $\mathscr{S}$ are proved which play a significant part in characterizing the Lie structure of semirings. Throughout this section, $\mathscr{S}$ represents a prime additively regular semiring satisfying A_2-condition.

Proposition 3.1. *If $u_1 \in \mathscr{S}$ with $u_1[u_1, \mathscr{S}] = (0)$, then $u_1 \in \mathscr{Z}(\mathscr{S})$.*

Proof. By hypothesis,

$$u_1[u_1, r] = 0,\ \forall\ r \in \mathscr{S}. \tag{1}$$

Again by hypothesis, we have $u_1[u_1, rs] = 0, \forall\ r, s \in \mathscr{S}$ which implies that $u_1(u_1rs + rs'u_1) = 0, \forall\ r, s \in \mathscr{S}$. Further, Lemma 2.1 implies that

$$\begin{aligned} 0 &= u_1(u_1rs + rs'u_1 + r_\circ su_1) = u_1(u_1rs + rs'u_1 + rs_\circ u_1) \\ &= u_1(u_1rs + rs'u_1 + rsu_{1\circ}) = u_1(u_1rs + rs'u_1 + ru_{1\circ}s), \text{by A}_2\text{−condition} \\ &= u_1(u_1rs + rs'u_1 + r_\circ u_1s) = u_1((u_1r + r'u_1)s + r(u_1s + s'u_1)). \end{aligned}$$

Then by equation (1), we have $u_1r(u_1s + s'u_1) = 0, \forall\ r, s \in \mathscr{S}$, that is, $u_1\mathscr{S}[u_1, s] = (0), \forall\ s \in \mathscr{S}$. Therefore, primeness of $\mathscr{S}$ infers that either $u_1 = 0$ or $[u_1, \mathscr{S}] = (0)$ and hence $u_1 \in \mathscr{Z}(\mathscr{S})$. ■

Lemma 3.2. *If char $\mathscr{S} \neq 2$ and $[u_1, [u_1, \mathscr{S}]] = (0)$, for $u_1 \in \mathscr{S}$, then $u_1 \in \mathscr{Z}(\mathscr{S})$.*

Proof. For any $x_1 \in \mathscr{S}$, we have $[u_1, [u_1, x_1]] = 0$ which gives that

$$u_1[u_1, x_1] + [u_1, x_1]u_1' = 0.$$

Then by adding both sides $[u_1, x_1]u_1$, we obtain

$$u_1[u_1, x_1] + [u_1, x_1]u_1' + [u_1, x_1]u_1 = [u_1, x_1]u_1.$$

This infers that

$$u_1[u_1, x_1] + [u_1, x_1]u_{1\circ} = [u_1, x_1]u_1.$$

In view of A_2-condition, we obtain $u_1[u_1,x_1]+u_{1\circ}[u_1,x_1]=[u_1,x_1]u_1$. Thus,

$$u_1(u_1x_1+x_1'u_1)=(u_1x_1+x_1'u_1)u_1,\ \forall\, x_1\in\mathscr{S}. \tag{1}$$

Again hypothesis leads to $[u_1,[u_1,x_1y]]=0, \forall\, x_1,y\in\mathscr{S}$. Equivalently,

$$u_1(u_1x_1y+x_1'yu_1)+(u_1x_1y+x_1'yu_1)u_1'=0,\ \textit{for any}\ x_1,y\in\mathscr{S}.$$

Thus, by Lemma 2.1 and A_2-condition, we can replace $u_1x_1y+x_1'yu_1$ by $(u_1x_1+x_1'u_1)y+x_1(u_1y+y'u_1)$ which gives that

$$u_1((u_1x_1+x_1'u_1)y+x_1(u_1y+y'u_1))+((u_1x_1+x_1'u_1)y+x_1(u_1y+y'u_1))u_1'=0,\ \text{for any}\ x_1,y\in\mathscr{S}.$$

Further, by applying equation (1), we obtain $2(u_1x_1+x_1'u_1)(u_1y+y'u_1)=0, \forall\, x_1,y\in\mathscr{S}$. As the characteristic of $\mathscr{S}$ is other than 2, so we are left with

$$[u_1,x_1][u_1,y]=0,\ \forall\, x_1,y\in\mathscr{S}. \tag{2}$$

Replacing x_1 by x_1z in (2), where $z\in\mathscr{S}$, then by Lemma 2.4 and equation (2), we get that $[u_1,x_1]\mathscr{S}[u_1,y]=(0), \forall\, x_1,y\in\mathscr{S}$. Now, since $\mathscr{S}$ is prime, therefore $[u_1,x_1]=0$ or $[u_1,y]=0, \forall\, x_1,y\in\mathscr{S}$ and hence in both cases $u_1\in\mathscr{Z}(\mathscr{S})$. ■

Proposition 3.3. *If char $\mathscr{S}\neq 2$ and $[\mathscr{L},\mathscr{L}]=(0)$, then $\mathscr{L}\subseteq\mathscr{Z}(\mathscr{S})$.*

Proof. For any $l_1\in\mathscr{L}, s_1\in\mathscr{S}$, we have $[l_1,s_1]\in\mathscr{L}$. By hypothesis, $[l_1,[l_1,s_1]]=0, \forall\, l_1\in\mathscr{L}, s_1\in\mathscr{S}$, that is, $[l_1,[l_1,\mathscr{S}]]=(0), \forall\, l_1\in\mathscr{L}$. Lemma 3.2 concludes that $\mathscr{L}\subseteq\mathscr{Z}(\mathscr{S})$. ■

One can easily prove the upcoming two lemmas by using the similar arguments of [9, Lemma 1.8 and Lemma 1.9] with necessary variations.

Lemma 3.4. *If $\mathscr{J}\neq(0)$ is a left ideal of $\mathscr{S}$, then $\mathscr{J}+[\mathscr{S},\mathscr{S}]=\mathscr{S}$.*

Lemma 3.5. *If $\mathscr{L}\neq(0)$ with $u_1\mathscr{L}=(0)$ or $\mathscr{L}u_1=(0)$, for any $u_1\in\mathscr{S}$, then $u_1=0$.*

Now, we turn our attention to define the higher commutator of $\mathscr{S}$ and prove some basic lemmas which we need later to prove results concerning the Lie structure of higher commutators.

Definition 3.6. *The higher commutator of $\mathscr{S}$ is defined inductively by:*
(1) $\mathscr{S}^{(0)}=\mathscr{S}$, of weight 1;
(2) $\mathscr{S}^{(1)}=[\mathscr{S},\mathscr{S}]$, of weight 2

and a higher commutator of $\mathscr{S}$ of weight n is defined by $[\mathcal{P},\mathcal{Q}]$, where $\mathcal{P}$ is a higher commutator of $\mathscr{S}$ of weight p, $\mathcal{Q}$ is of weight q, with $p+q=n$.

For convenience, we give notation $\mathscr{S}^{(k)}$ for the following series of $\mathscr{S}$ defined as: $\mathscr{S}^{(0)}=\mathscr{S}$, $\mathscr{S}^{(1)}=[\mathscr{S},\mathscr{S}]$,..., $\mathscr{S}^{(k)}=[\mathscr{S}^{(k-1)},\mathscr{S}^{(k-1)}]$.

It is pertinent to mention that the higher commutator of weight 2 is only $[\mathscr{S},\mathscr{S}]=\mathscr{S}^{(1)}$, whereas the higher commutator of weight 3 is only $\mathscr{S}^{(3)}=[[\mathscr{S},\mathscr{S}],\mathscr{S}]$; there are two higher commutators of weight 4 viz, $[\mathscr{S},[\mathscr{S},[\mathscr{S},\mathscr{S}]]]$ and $[[\mathscr{S},\mathscr{S}],[\mathscr{S},\mathscr{S}]]$; three of weight 5 viz, $[\mathscr{S},[\mathscr{S},[\mathscr{S},[\mathscr{S},\mathscr{S}]]]]$, $[\mathscr{S},[[\mathscr{S},\mathscr{S}],[\mathscr{S},\mathscr{S}]]$ and $[[\mathscr{S},\mathscr{S}],[[\mathscr{S},\mathscr{S}],\mathscr{S}]]$ and so on.

The next lemma follows verbatim as Lemma 3 in [8].

Lemma 3.7. *A higher commutator of $\mathscr{S}$ contains $\mathscr{S}^{(k)}$, for some k.*

An application of the above lemma is

Corollary 3.8. *If $\mathcal{H}$ is a higher commutator of $\mathscr{S}$, then $(\mathcal{H})$, the ideal generated by $\mathcal{H}$, contains $\mathscr{S}^{(k)}$, for some k.*

Lemma 3.9. *Let $\mathcal{H}$ be a higher commutator of $\mathscr{S}$. Then $\mathcal{H}$ is a Lie ideal of $\mathscr{S}$.*

Proof. Let $\mathcal{H}$ be a higher commutator of $\mathscr{S}$ of weight n. We shall prove the result by using induction on n. For $n = 1$, clearly $\mathscr{S}$ is a Lie ideal of $\mathscr{S}$. Now, we suppose that it is true for $n = k-1$, that is, $\mathcal{H} = [\mathcal{P}, \mathcal{Q}]$, where $\mathcal{P}$ is a higher commutator of $\mathscr{S}$ of weight p and $\mathcal{Q}$ is of weight q, with $p + q = k - 1$, is a Lie ideal of $\mathscr{S}$. Further, consider $\mathcal{K} = [\mathcal{H}, \mathscr{S}]$ is a higher commutator of $\mathscr{S}$ of weight k. Then obviously, $\mathcal{K}$ is an additive submonoid of $\mathscr{S}$ and $[\mathcal{K}, \mathscr{S}] = [[\mathcal{H}, \mathscr{S}], \mathscr{S}] \subseteq [\mathcal{H}, \mathscr{S}] = \mathcal{K}$, as $[\mathcal{H}, \mathscr{S}] \subseteq \mathcal{H}$. Therefore, $\mathcal{K}$ is a Lie ideal of $\mathscr{S}$. This finishes the proof. ∎

Theorem 3.10. *If u_1 is any element of $\mathscr{S}$ which satisfies $[u_1, [\mathscr{S}, \mathscr{S}]] = (0)$, then $u_1 \in \mathscr{Z}(\mathscr{S})$.*

Proof. For any $x_1, y \in \mathscr{S}$, we have $[u_1, [x_1, y]] = 0$ leading to

$$u_1[x_1, y] + [x_1, y]u_1' + [x_1, y]u_1 = [x_1, y]u_1.$$

Now, by applying A_2- condition on this equality, we have

$$u_1[x_1, y] + (u_1 + u_1')[x_1, y] = [x_1, y]u_1$$

which infers that

$$u_1x_1y + u_1yx_1' = x_1yu_1 + y'x_1u_1, \ \forall\, x_1, y \in \mathscr{S}. \tag{1}$$

Again by hypothesis, we have $[u_1, [x_1, x_1y]] = 0, \forall\, x_1, y \in \mathscr{S}$. Thus,

$$\begin{aligned} 0 &= u_1(x_1x_1y + x_1y'x_1) + (x_1x_1y + x_1y'x_1)u_1' \\ &= u_1x_1(x_1y + y'x_1) + x_1(x_1y + y'x_1)u_1', \ \forall\, x_1, y \in \mathscr{S}. \end{aligned}$$

By equation (1), we obtain that $u_1x_1(x_1y + y'x_1) + x_1u_1'(x_1y + y'x_1) = 0, \forall\, x_1, y \in \mathscr{S}$ which is equivalent to

$$(u_1x_1 + x_1u_1')(x_1y + y'x_1) = 0, \ \forall\, x_1, y \in \mathscr{S}. \tag{2}$$

Changing y with yu_1 in equation (2) and applying A_2-condition, we have

$$\begin{aligned} 0 &= (u_1x_1 + x_1u_1')(x_1yu_1 + yu_1'x_1) = (u_1x_1 + x_1u_1')(x_1yu_1 + x_1y_\circ u_1 + yu_1'x_1) \\ &= (u_1x_1 + x_1u_1')(x_1yu_1 + y_\circ x_1u_1 + yu_1'x_1) = (u_1x_1 + x_1u_1')((x_1y + y'x_1)u_1 + y(x_1u_1 + u_1'x_1)). \end{aligned}$$

Moreover, by equation (2), we obtain $(u_1x_1 + x_1u_1')y(x_1u_1 + u_1'x_1) = 0, \forall\, x_1, y \in \mathscr{S}$. Equivalently,

$$(u_1x_1 + x_1u_1')\mathscr{S}(x_1u_1 + u_1'x_1) = (0), \ \forall\, x_1 \in \mathscr{S}$$

and in that case $(\mathscr{S}(x_1u_1 + u_1'x_1))^2 = (0)$, which is a contradiction, as $\mathscr{S}$ does not have any non-zero nilpotent ideal. Thus, $u_1x_1 + x_1u_1' = 0, \forall\, x_1 \in \mathscr{S}$, that is, $u_1x_1 = x_1u_1, \forall x_1 \in \mathscr{S}$. Hence $u_1 \in \mathscr{Z}(\mathscr{S})$. ∎

Remark 3.11. *Let $\mathscr{Z}(\mathscr{S})$ be the center of $\mathscr{S}$. We define the extended centroid of $\mathscr{L}$ by the set $\mathscr{Z}_{\mathscr{S}(\mathscr{L})} = \{s \in \mathscr{S} : sl = ls, \forall\, l \in \mathscr{L}\}$. Also, one can easily check that $\mathscr{Z}(\mathscr{S}) \subseteq \mathscr{Z}_{\mathscr{S}(\mathscr{L})}$.*

Theorem 3.12. *Let char $\mathscr{S} \neq 3$ and $\mathcal{P}$ be an additive submonoid of $\mathscr{S}$. If $[[p, [p, \mathcal{P}]], s] = (0), \forall\, p \in \mathcal{P}, s \in \mathscr{S}$, then $[\mathcal{P}, [\mathcal{P}, \mathcal{P}]] \subseteq \mathscr{Z}(\mathscr{S})$.*

Proof. The given hypothesis infers that

$$[[k,[k,m_1]],y]=0,\ \forall\, k,m_1\in\mathcal{P},y\in\mathscr{S}. \tag{1}$$

Again by hypothesis, $[[l_1+k,[l_1+k,m_1]],y]=0$, for any $l_1,m_1,k\in\mathcal{P},y\in\mathscr{S}$. Then by Lemma 2.4, we obtain

$$0=[[l_1+k,[l_1,m_1]+[k,m_1]],y]=[[l_1+k,[l_1,m_1]],y]+[[l_1+k,[k,m_1]],y]$$

for any $l_1,m_1,k\in\mathcal{P},y\in\mathscr{S}$. Further, by Lemma 2.4 and equation (1), we get that

$$[[k,[l_1,m_1]],y]+[[l_1,[k,m_1]],y]=0,\ \forall\, l_1,m_1,k\in\mathcal{P},y\in\mathscr{S}. \tag{2}$$

By using Jacobi identity, we can substitute $[[l_1,[k,m_1]]+[[l_1,k],m_1]$ in place of $[[k,[l_1,m_1]]$ in equation (2) and thus we have

$$2[[l_1,[k,m_1]],y]+[[l_1,k],m_1],y]=0,\ \forall\, l_1,m_1,k\in\mathcal{P},y\in\mathscr{S}. \tag{3}$$

By interchanging l_1 and m_1 in equation (3), we have

$$2[[m_1,[k,l_1]],y]+[[m_1,k],l_1],y]=0,\forall\, l_1,m_1,k\in\mathcal{P},y\in\mathscr{S}$$

which is equivalent to

$$2[[[l_1,k],m_1],y]+[[l_1,[k,m_1]],y]=0,\ \forall\, l_1,m_1,k\in\mathcal{P},y\in\mathscr{S}. \tag{4}$$

Further, by adding equations (3) and (4), we get that

$$3([[[l_1,k],m_1],y]+[[l_1,[k,m_1]],y])=0,\ \forall\, l_1,m_1,k\in\mathcal{P},y\in\mathscr{S}.$$

Again using Jacobi identity, we have $3[[[l_1,m_1],k],y]=0,\forall\, l_1,m_1,k\in\mathcal{P},y\in\mathscr{S}$. As $char\ \mathscr{S}\neq 3$, so we are left with $[[[l_1,m_1],k],y]=0,\forall\, l_1,m_1,k\in\mathcal{P},y\in\mathscr{S}$. Therefore, $[\mathcal{P},[\mathcal{P},\mathcal{P}]]\subseteq\mathscr{Z}(\mathscr{S})$. ■

The next corollary is an important outcome of the previous result.

Corollary 3.13. *If $char\ \mathscr{S}\neq 3$ and $[[l_1,[l_1,\mathscr{L}]],\mathscr{L}]=(0)$, for any $l_1\in\mathscr{L}$, then $[\mathscr{L},[\mathscr{L},\mathscr{L}]]\subseteq\mathscr{Z}_{\mathscr{S}(\mathscr{L})}$.*

Theorem 3.14. *If $char\ \mathscr{S}\neq 2$ and $[[\mathscr{L},[\mathscr{L},\mathscr{L}]],\mathscr{L}]=(0)$, then $[\mathscr{L},[\mathscr{L},\mathscr{L}]]\subseteq\mathscr{Z}(\mathscr{S})$.*

Proof. Since $\mathscr{L}$ is a Lie ideal of $\mathscr{S}$, therefore $[\mathscr{L},[\mathscr{L},\mathscr{L}]]$ is also a Lie ideal of $\mathscr{S}$. Thus, $[l_1,s_1]\in[\mathscr{L},[\mathscr{L},\mathscr{L}]],\ \forall\ l_1\in[\mathscr{L},[\mathscr{L},\mathscr{L}]]$ and $s_1\in\mathscr{S}$. Moreover, by hypothesis $[l_1,[l_1,s_1]]=0,\ \forall\ l_1\in[\mathscr{L},[\mathscr{L},\mathscr{L}]],s_1\in\mathscr{S}$ and hence Lemma 3.2, concludes that $[\mathscr{L},[\mathscr{L},\mathscr{L}]]\subseteq\mathscr{Z}(\mathscr{S})$. ■

4. Lie structure of $\mathscr{S}$

The idea behind the results proved in this section was first brought to the author's attention during the study of the Lie structure of rings given by Herstein [7–9]. Throughout this section, $\mathscr{L}$ denotes a 2-Lie ideal (that is, a Lie ideal having property $2l_1m_1\in\mathscr{L},\forall\, l_1,m_1\in\mathscr{L}$) of $\mathscr{S}$. We now begin this section with an example:

Example 4.1. *Consider $\mathscr{S}=\mathbb{Z}\times\mathbb{Z}^+=\{(u_1,r_1):u_1\in\mathbb{Z},r_1\in\mathbb{Z}^+\}$, where $\mathbb{Z}^+$ is the set of all positive integers with binary operations $\oplus$ and $\odot$ by $(u_1,r_1)\oplus(v,s)=(u_1+v,lcm(r_1,s))$ and $(u_1,r_1)\odot(v,s)=(u_1v,gcd(r_1,s)),\ \forall\ (u_1,r_1),(v,s)\in\mathscr{S}$. Further, define the pseudo inverse of an element (u_1,r_1) of $\mathscr{S}$ by $(u_1,r_1)'=(-u_1,r_1)$. Then clearly, $\mathscr{S}$ is an additively regular semiring with A_2-condition. Indeed, the set $\mathscr{L}=\{(0,s):s\in\mathbb{Z}^+\}$ is a 2-Lie ideal of $\mathscr{S}$.*

We now introduce a more general result which is a generalization of [9, Lemma 1.3].

Theorem 4.2. *If char $\mathscr{S} \neq 2$, then either $\mathscr{L}$ is contained in $\mathscr{Z}(\mathscr{S})$ or $\mathscr{L}$ contains a non-zero ideal of $\mathscr{S}$.*

Proof. In case $[\mathscr{L}, \mathscr{L}] = (0)$, then by Proposition 3.3, we have $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{S})$. But, if $\mathscr{L} \nsubseteq \mathscr{Z}(\mathscr{S})$, then again by Proposition 3.3, $[\mathscr{L}, \mathscr{L}] \neq (0)$, so we shall prove the containment of the ideal $2\mathscr{S}[\mathscr{L}, \mathscr{L}]\mathscr{S}$ of $\mathscr{S}$ in $\mathscr{L}$. We claim that $2\mathscr{S}[\mathscr{L}, \mathscr{L}] \subseteq \mathscr{L}$. For this, let $2s[l_1, m_1] \in 2\mathscr{S}[\mathscr{L}, \mathscr{L}]$, for any $l_1, m_1 \in \mathscr{L}, s \in \mathscr{S}$. Then

$$\begin{aligned}
2s[l_1, m_1] &= 2(sl_1m_1 + sm_1'l_1) \\
&= 2(sl_1m_1 + s_\circ l_1m_1 + sm_1'l_1), \text{by Lemma 2.1} \\
&= 2(sl_1m_1 + l_1s_\circ m_1 + sm_1'l_1), \text{by } A_2\text{-condition} \\
&= 2(sl_1m_1 + l_1sm_1 + l_1s'm_1 + s'm_1l_1), \text{by Lemma 2.1} \\
&= 2((l_1sm_1 + s'm_1l_1) + (l_1s'm_1 + sl_1m_1)) = 2[l_1, sm_1] + 2[l_1, s']m_1 \in \mathscr{L}.
\end{aligned}$$

Hence, $2\mathscr{S}[\mathscr{L}, \mathscr{L}] \subseteq \mathscr{L}$. This infers that

$$[2r[l_1, m_1], s] \in \mathscr{L}, \forall\, l_1, m_1 \in \mathscr{L}, r, s \in \mathscr{S}$$

which is equivalent to

$$2r[l_1, m_1]s + 2s'r[l_1, m_1] \in \mathscr{L}. \tag{1}$$

Also, since $2sr[l_1, m_1] \in \mathscr{L}$, therefore equation (1) gives

$$2r[l_1, m_1]s + 2s'r[l_1, m_1] + 2sr[l_1, m_1] \in \mathscr{L}.$$

Equivalently, $2r[l_1, m_1]s + 2s_\circ r[l_1, m_1] \in \mathscr{L}$. Thus, we obtain

$$2r(l_1m_1 + m_1'l_1)s + 2(s + s')r(l_1m_1 + m_1'l_1) \in \mathscr{L}.$$

In other words, $2r(l_1m_1 + m_1'l_1)s + 2(s_\circ rl_1m_1 + s_\circ rm_1'l_1) \in \mathscr{L}$. Then, A_2-condition yields

$$2r(l_1m_1 + m_1'l_1)s + 2r_\circ l_1m_1s + 2r_\circ m_1'l_1s \in \mathscr{L}.$$

Therefore, Lemma 2.1 concludes that $2r(l_1m_1 + m_1'l_1)s \in \mathscr{L}$, $\forall\, l_1, m_1 \in \mathscr{L}, r, s \in \mathscr{S}$ which gives $2\mathscr{S}[\mathscr{L}, \mathscr{L}]\mathscr{S} \subseteq \mathscr{L}$. The theorem is thereby established. ■

Definition 4.3. *[2] A semiring $\mathscr{S}$ is called ideal-simple (id-simple for short), if $\mathscr{S}$ is non-trivial and $\mathscr{I} = \mathscr{S}$, whenever $\mathscr{I}$ is a non-zero ideal of $\mathscr{S}$ such that $\mathscr{I}$ contains atleast two elements respectively.*

The above result immediately implies the following theorem which is a generalization of [9, Theorem 1.2]

Theorem 4.4. *If $\mathscr{S}$ is an id-simple semiring with char $\mathscr{S} \neq 2$, then either $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{S})$ or $\mathscr{L}$ coincides with $\mathscr{S}$.*

Lemma 4.5. *If char $\mathscr{S} \neq 2$ and $\mathscr{L} \nsubseteq \mathscr{Z}(\mathscr{S})$, then there exists an ideal $\mathscr{I}$ of $\mathscr{S}$ such that $[\mathscr{I}, \mathscr{S}] \subseteq \mathscr{L}$.*

Proof. As proved in the Theorem 4.2, the non-zero ideal $2\mathscr{S}[\mathscr{L}, \mathscr{L}]\mathscr{S}$ of $\mathscr{S}$ is contained in $\mathscr{L}$, then it follows easily that $[2\mathscr{S}[\mathscr{L}, \mathscr{L}]\mathscr{S}, \mathscr{S}] \subseteq \mathscr{L}$. Hence proved. ■

In the rest of this section, $\mathscr{S}$ denotes a prime semiring with *char* $\mathscr{S} \neq 2$. It is easy to observe that every id-simple semiring is a prime semiring. Therefore, all the forthcoming results of this section are also true for an id-simple semiring.

Lemma 4.6. *If $\mathscr{L} \nsubseteq \mathscr{Z}(\mathscr{S})$ and $u_1, v \in \mathscr{S}$ with $u_1\mathscr{L}v = (0)$, then either $u_1 = 0$ or $v = 0$.*

Proof. By the above lemma, there exists an ideal $\mathscr{J}$ of $\mathscr{S}$ with $[\mathscr{J},\mathscr{S}] \subseteq \mathscr{L}$. Let $l_1 \in \mathscr{L}, i \in \mathscr{J}, s \in \mathscr{S}$. Then $[iu_1l_1, s] \in [\mathscr{J},\mathscr{S}] \subseteq \mathscr{L}$. Henceforth,

$$\begin{aligned} 0 &= u_1[iu_1l_1, s]v = u_1[iu_1, s]l_1v + u_1iu_1[l_1, s]v = u_1[iu_1, s]l_1v, \text{ as } u_1\mathscr{L}v = (0) \\ &= u_1(iu_1s + s'iu_1)l_1v = u_1iu_1sl_1v, \text{ as } u_1\mathscr{L}v = (0). \end{aligned}$$

This shows that $u_1\mathscr{J}u_1\mathscr{S}\mathscr{L}v = (0)$. If $u_1 \neq 0$ and by using the fact that $\mathscr{S}$ is prime, we obtain $\mathscr{L}v = (0)$, then by Lemma 3.5, $v = 0$. ■

Theorem 4.7. *If* $[u_1, [\mathscr{L},\mathscr{L}]] = (0)$, *for any* $u_1 \in \mathscr{S}$, *then either* $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{S})$ *or* $[u_1,\mathscr{L}] = (0)$.

Proof. Assume that $\mathscr{L} \nsubseteq \mathscr{Z}(\mathscr{S})$. For any $l_1, m_1 \in \mathscr{L}$, hypothesis gives that $[u_1, [l_1, m_1]] = 0$ which leads to

$$u_1(l_1m_1 + m_1'l_1) + (l_1m_1 + m_1'l_1)u_1' + (l_1m_1 + m_1'l_1)u_1 = (l_1m_1 + m_1'l_1)u_1.$$

This infers that

$$u_1(l_1m_1 + m_1'l_1) = (l_1m_1 + m_1'l_1)u_1. \tag{1}$$

Again by hypothesis, we have

$$[u_1, [l_1, 2l_1m_1]] = 0, \forall\, l_1, m_1 \in \mathscr{L}.$$

Since $char\ \mathscr{S} \neq 2$, therefore $[u_1, [l_1, l_1m_1]] = 0, \forall\, l_1, m_1 \in \mathscr{L}$ which is equivalent to

$$u_1l_1(l_1m_1 + m_1'l_1) + l_1(l_1m_1 + m_1'l_1)u_1' = 0, \forall\, l_1, m_1 \in \mathscr{L}.$$

Then by using equation (1), we get

$$u_1l_1(l_1m_1 + m_1'l_1) + l_1u_1'(l_1m_1 + m_1'l_1) = 0$$

which concludes that

$$(u_1l_1 + l_1u_1')(l_1m_1 + m_1'l_1) = 0,\ \forall\, l_1, m_1 \in \mathscr{L}. \tag{2}$$

The replacement of m_1 with $2m_1n$, where $n \in \mathscr{L}$, in equation (2) gives

$$2(u_1l_1 + l_1u_1')(l_1m_1n + m_1'nl_1) = 0.$$

But $char\ \mathscr{S} \neq 2$ gives

$$\begin{aligned} 0 &= (u_1l_1 + l_1u_1')(l_1m_1n + m_1'nl_1) = (u_1l_1 + l_1u_1')(l_1m_1n + m_{1\circ}nl_1 + m_1n'l_1) \\ &= (u_1l_1 + l_1u_1')((l_1m_1 + m_1'l_1)n + m_1(l_1n + n'l_1)) \\ &= (u_1l_1 + l_1u_1')(l_1m_1 + m_1'l_1)n + (u_1l_1 + l_1u_1')m_1(l_1n + n'l_1), \end{aligned}$$

then equation (2) yields $(u_1l_1 + l_1u_1')m_1(l_1n + n'l_1) = 0, \forall\, l_1, m_1, n \in \mathscr{L}$. In other words,

$$(u_1l_1 + l_1u_1')\mathscr{L}(l_1n + n'l_1) = (0),\ \forall\, l_1, n \in \mathscr{L}.$$

By the above lemma, for any $l_1 \in \mathscr{L}$, we obtain either $u_1l_1 + l_1u_1' = 0$ or $l_1n + n'l_1 = 0, \forall\, n \in \mathscr{L}$. Now, if $[l_1,\mathscr{L}] = (0), \forall\, l_1 \in \mathscr{L}$, then $[\mathscr{L},\mathscr{L}] = (0)$ and hence Proposition 3.3 yields $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{S})$, which is absurd.

Therefore, there exists some $k \in \mathscr{L}$ with $[k,\mathscr{L}] \neq (0)$, then $[u_1, k] = 0$. We now claim that $[u_1,\mathscr{L}] = (0)$. For this, if possible, let $j(\neq k) \in \mathscr{L}$ with $[u_1, j] \neq 0$. Thus, $[j,\mathscr{L}] = (0)$. This infers that $[j + k,\mathscr{L}] \neq (0)$ and $[u_1, j + k] \neq 0$ hold simultaneously, which is not true. Hence, $[u_1,\mathscr{L}] = (0)$. ■

An application of the above theorem is as follows:

Corollary 4.8. *If $u_1 \in \mathscr{S}$ satisfies $[u_1, [\mathscr{L}, \mathscr{L}]] = (0)$, then either $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{S})$ or u_1 commutes with every element of $\mathscr{L}$.*

The upcoming theorem is a partial extension of [8, Theorem 1].

Theorem 4.9. *If $[u_1, [u_1, \mathscr{L}]] = (0)$, for any $u_1 \in \mathscr{S}$, then either $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{S})$ or $[u_1, \mathscr{L}] = (0)$.*

Proof. Let $\mathscr{L} \nsubseteq \mathscr{Z}(\mathscr{S})$. By given hypothesis, $[u_1, [u_1, x_1]] = 0, \forall\, x_1 \in \mathscr{L}$. This infers that $u_1[u_1, x_1] + [u_1, x_1]u_1' + [u_1, x_1]u_1 = [u_1, x_1]u_1, \forall\, x_1 \in \mathscr{L}$. Further, A_2-condition implies that

$$u_1(u_1x_1 + x_1'u_1) = (u_1x_1 + x_1'u_1)u_1,\ \forall\, x_1 \in \mathscr{L}. \tag{1}$$

Again by using hypothesis, we obtain $[u_1, [u_1, 2l_1m_1]] = 0$, for any $l_1, m_1 \in \mathscr{L}$. Then $char\ \mathscr{S} \neq 2$ gives that $[u_1, [u_1, l_1m_1]] = 0$, for any $l_1, m_1 \in \mathscr{L}$. Thus,

$$\begin{aligned}
0 &= u_1(u_1l_1m_1 + l_1'm_1u_1) + (u_1l_1m_1 + l_1'm_1u_1)u_1' \\
&= u_1(u_1l_1m_1 + l_{1\circ}m_1u_1 + l_1m_1'u_1) + (u_1l_1m_1 + l_{1\circ}m_1u_1 + l_1'm_1u_1)u_1' \\
&= u_1(u_1l_1m_1 + l_{1\circ}u_1m_1 + l_1m_1'u_1) + (u_1l_1m_1 + l_{1\circ}u_1m_1 + l_1'm_1u_1)u_1'
\end{aligned}$$

which gives that

$$u_1((u_1l_1+l_1'u_1)m_1+l_1(u_1m_1+m_1'u_1))+((u_1l_1+l_1'u_1)m_1+l_1(u_1m_1+m_1'u_1))u_1' = 0, \text{ for any } l_1, m_1 \in \mathscr{L}.$$

By using equation (1), we are left with $2(u_1l_1 + l_1'u_1)(u_1m_1 + m_1'u_1) = 0, \forall\, l_1, m_1 \in \mathscr{L}$. Since $char\ \mathscr{S} \neq 2$, so we obtain

$$(u_1l_1 + l_1'u_1)(u_1m_1 + m_1'u_1) = 0,\ \forall\, l_1, m_1 \in \mathscr{L}$$

or

$$[u_1, l_1][u_1, m_1] = 0,\ \forall\, l_1, m_1 \in \mathscr{L}. \tag{2}$$

Further, by replacing l_1 by $2l_1w$, for any $w \in \mathscr{L}$ in the above equation, we have

$$2[u_1, l_1w][u_1, m_1] = 0,\ \forall\, l_1, m_1 \in \mathscr{L}.$$

Again, since $char\ \mathscr{S} \neq 2$, so we left with $[u_1, l_1w][u_1, m_1] = 0, \forall\, l_1, m_1 \in \mathscr{L}$. By Lemma 2.4 and equation (2), we deduce that $[u_1, l_1]w[u_1, m_1] = 0, \forall\, l_1, m_1, w \in \mathscr{L}$. Equivalently, $[u_1, l_1]\mathscr{L}[u_1, m_1] = (0), \forall\, l_1, m_1 \in \mathscr{L}$. Lemma 4.6, infers that either $[u_1, l_1] = 0$ or $[u_1, m_1] = 0, \forall\, l_1, m_1 \in \mathscr{L}$. Both cases implies that $[u_1, \mathscr{L}] = (0)$. ■

Corollary 4.10. *Let $[u_1, [u_1, \mathscr{L}]] = (0)$, for any $u_1 \in \mathscr{S}$. Then either $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{S})$ or u_1 commutes with every element of $\mathscr{L}$.*

Now, we divert our attention to the study of the Lie structure of higher commutators as a Lie ideal of $\mathscr{S}$ and hence as a Lie subsemiring of $\mathscr{S}$.

Theorem 4.11. *If $a \in \mathscr{S}$ satisfies $[a, [a, (\mathcal{H})]] = (0)$, where $(\mathcal{H})$ is an ideal generated by $\mathcal{H}$, for some higher commutator $\mathcal{H}$ of $\mathscr{S}$, then either $(\mathcal{H}) \subseteq \mathscr{Z}(\mathscr{S})$ or $a \in \mathscr{Z}(\mathscr{S})$.*

Proof. Let $\mathcal{H}$ be a higher commutator of $\mathscr{S}$. Then clearly $(\mathcal{H})$ is a Lie ideal of $\mathscr{S}$. Suppose that $(\mathcal{H}) \nsubseteq \mathscr{Z}(\mathscr{S})$. Thus, by Theorem 4.9, we have

$$[a, (\mathcal{H})] = (0). \tag{1}$$

By Corollary 3.8, $(\mathcal{H}) \supseteq \mathscr{S}^{(k)}$, for some k. Thus, equation (1) implies that $[a, \mathscr{S}^{(k)}] = (0)$, that is, $[a, [\mathscr{S}^{(k-1)}, \mathscr{S}^{(k-1)}]] = (0)$. By Theorem 4.7 and the same argument as above we can say that $[a, \mathscr{S}^{(k-1)}] = (0)$. Now, repeating the same process and using Theorem 3.10, we end up with $a \in \mathscr{Z}(\mathscr{S})$. ■

5. Derivations in semirings

Throughout this section, $\mathscr{S}$ represents a prime semiring with $char\ \mathscr{S} \neq 2$.

Definition 5.1. *[4] An additive map $d : \mathscr{S} \to \mathscr{S}$ is called a derivation of $\mathscr{S}$, if $(xy)^d = x^d y + xy^d$, $\forall\, x, y \in \mathscr{S}$.*

Definition 5.2. *Let $\mathcal{T}$ be any arbitrary subset of $\mathscr{S}$. Then the centralizer of $\mathcal{T}$ in $\mathscr{S}$ is $\mathcal{C}_{\mathscr{S}}(\mathcal{T}) = \{x \in \mathscr{S} : [x, \mathcal{T}] = (0)\}$.*

Lemma 5.3. *$\mathcal{C}_{\mathscr{S}}(\mathscr{L})$ is a Lie ideal and subsemiring of $\mathscr{S}$.*

Proof. Let $t_1 \in \mathcal{C}_{\mathscr{S}}(\mathscr{L})$. For any $s \in \mathscr{S}$, $[[t_1, s], l_1] = [t_1, [s, l_1]] + [s, [l_1, t_1]]$, by Jacobi identity. This gives that $[[t_1, s], l_1] = 0$, as $[t_1, \mathscr{L}] = (0)$. Thus, $[t_1, s] \in \mathcal{C}_{\mathscr{S}}(\mathscr{L})$. This concludes that $\mathcal{C}_{\mathscr{S}}(\mathscr{L})$ is a Lie ideal of $\mathscr{S}$. Now, let $t_1, t_2 \in \mathcal{C}_{\mathscr{S}}(\mathscr{L})$. Then $[t_1 t_2, l_1] = t_1[t_2, l_1] + [t_1, l_1]t_2 = 0$, $\forall\, l_1 \in \mathscr{L}$ which yields $t_1 t_2 \in \mathcal{C}_{\mathscr{S}}(\mathscr{L})$. This proves the lemma. ■

Observe that the centralizer of a Lie ideal of $\mathscr{S}$ is a 2-Lie ideal of $\mathscr{S}$.

Theorem 5.4. *If $\mathscr{L} \nsubseteq \mathscr{Z}(\mathscr{S})$, then $\mathcal{C}_{\mathscr{S}}(\mathscr{L}) \subseteq \mathscr{Z}(\mathscr{S})$.*

Proof. By the above lemma, $\mathcal{C}_{\mathscr{S}}(\mathscr{L})$ is a Lie ideal and subsemiring of $\mathscr{S}$. We now claim that $\mathcal{C}_{\mathscr{S}}(\mathscr{L})$ can not contain a non-zero ideal of $\mathscr{S}$. On contrary, let $\mathscr{I}$ be a non-zero ideal of $\mathscr{S}$ such that $\mathscr{I} \subseteq \mathcal{C}_{\mathscr{S}}(\mathscr{L})$, i.e., $[\mathscr{I}, \mathscr{L}] = (0)$. This concludes that $[\mathscr{S}\mathscr{I}, \mathscr{L}] = (0)$ which implies that $[si, l_1] = 0$, $\forall\, s \in \mathscr{S}, i \in \mathscr{I}, l_1 \in \mathscr{L}$. Thus, $s[i, l_1] + [s, l_1]i = 0$ leading to $[s, l_1]i = 0$, $\forall\, s \in \mathscr{S}, i \in \mathscr{I}, l_1 \in \mathscr{L}$. Hence, $[s, l_1]\mathscr{I} = (0)$, $\forall\, s \in \mathscr{S}, l_1 \in \mathscr{L}$. This deduces that $[s, l_1]\mathscr{S}\mathscr{I} = (0)$, $\forall\, s \in \mathscr{S}, l_1 \in \mathscr{L}$. Primeness of $\mathscr{S}$ yields $[s, l_1] = 0$, $\forall\, s \in \mathscr{S}, l_1 \in \mathscr{L}$. Therefore, $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{S})$, which is absurd. This concludes that $\mathcal{C}_{\mathscr{S}}(\mathscr{L})$ does not contain any non-zero ideal of $\mathscr{S}$. By Theorem 4.2, we get that $\mathcal{C}_{\mathscr{S}}(\mathscr{L}) \subseteq \mathscr{Z}(\mathscr{S})$. ■

The next result can be directly deduced as an outcome of Theorem 4.7.

Theorem 5.5. *If $\mathscr{L}$ is a 2-Lie ideal of $\mathscr{S}$ such that $\mathscr{L} \nsubseteq \mathscr{Z}(\mathscr{S})$, then $\mathcal{C}_{\mathscr{S}}([\mathscr{L}, \mathscr{L}]) = \mathcal{C}_{\mathscr{S}}(\mathscr{L})$.*

Theorem 5.6. *If d is a derivation of $\mathscr{S}$ such that $\mathscr{L}^d = (0)$, then either $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{S})$ or $d = 0$.*

Proof. As $\mathscr{L}^d = (0)$, therefore $0 = [l_1, s]^d = (l_1 s + s' l_1)^d = l_1^d s + l_1 s^d + (s')^d l_1 + s' l_1^d$, $\forall\, l_1 \in \mathscr{L}, s \in \mathscr{S}$. This infers

$$l_1 s^d + (s')^d l_1 = 0,\ \forall\, l_1 \in \mathscr{L}, s \in \mathscr{S}. \tag{1}$$

In equation (1), the replacement of s with sm_1, where $m_1 \in \mathscr{L}$, gives $l_1(s^d m_1 + s m_1^d) + ((s')^d m_1 + s' m_1^d) l_1 = 0$, $\forall\, l_1 \in \mathscr{L}, s \in \mathscr{S}$. The given hypothesis concludes that

$$l_1 s^d m_1 + (s')^d m_1 l_1 = 0,\ \forall\, l_1, m_1 \in \mathscr{L}, s \in \mathscr{S}. \tag{2}$$

By applying Lemma 2.1 on equation (1), we get $l_1 s^d = s^d l_1$ and then using this in equation (2), we have $s^d l_1 m_1 + (s')^d m_1 l_1 = 0$, $\forall\, l_1, m_1 \in \mathscr{L}, s \in \mathscr{S}$. In other words,

$$s^d [l_1, m_1] = 0,\ \forall\, l_1, m_1 \in \mathscr{L}, s \in \mathscr{S}. \tag{3}$$

By putting st in place of s, with $t \in \mathscr{S}$, we get that $(s^d t + s t^d)[l_1, m_1] = 0$. Then equation (3) yields $s^d \mathscr{S}[l_1, m_1] = (0)$. By the primeness of $\mathscr{S}$, either $d = 0$ or $[l_1, m_1] = 0$, $\forall\, l_1, m_1 \in \mathscr{L}$. By Proposition 3.3, we get the desired conclusion. ■

Proposition 5.7. *If $\mathscr{L}$ is a 2-Lie ideal of $\mathscr{S}$ and d is a non-zero derivation of $\mathscr{S}$ with $\mathscr{L} \nsubseteq \mathscr{Z}(\mathscr{S})$. Suppose that $a\mathscr{L}^d = (0)$ or $\mathscr{L}^d a = (0)$, then $a = 0$.*

Proof. As $d \neq 0$ and $\mathscr{L} \nsubseteq \mathscr{Z}(\mathscr{S})$, so by the above theorem, $\mathscr{L}^d \neq (0)$. Since $[l_1, s]l_1 = l_1(sl_1) + (s'l_1)l_1 \in [\mathscr{L}, \mathscr{S}] \subseteq \mathscr{L}$, for any $l_1 \in \mathscr{L}, s \in \mathscr{S}$, therefore $0 = a([l_1, s]l_1)^d = a([l_1, s]^d l_1 + [l_1, s]l_1^d)$. The given hypothesis infers that $a[l_1, s]l_1^d = 0, \forall\, l_1 \in \mathscr{L}, s \in \mathscr{S}$. Replacing s by $m_1^d r$, with $m_1 \in \mathscr{L}, r \in \mathscr{S}$, we get that $0 = a[l_1, m_1^d r]l_1^d = am_1^d[l_1, r]l_1^d + a[l_1, m_1^d]rl_1^d, \forall\, l_1, m_1 \in \mathscr{L}, r \in \mathscr{S}$. This concludes that $a(l_1 m_1^d + m_1^d l_1')rl_1^d = 0, \forall\, l_1, m_1 \in \mathscr{L}, r \in \mathscr{S}$, as $am_1^d = 0$. This implies that $al_1 m_1^d r l_1^d = 0, \forall\, l_1, m_1 \in \mathscr{L}, r \in \mathscr{S}$. Equivalently, $al_1 \mathscr{L}^d \mathscr{S} l_1^d = (0), \forall\, l_1 \in \mathscr{L}$. Primeness of $\mathscr{S}$ gives that for any $l_1 \in \mathscr{L}$, either $al_1\mathscr{L}^d = (0)$ or $l_1^d = 0$. But $\mathscr{L}^d \neq (0)$, so there exists some $n \in \mathscr{L}$ such that $n^d \neq 0$ and $an\mathscr{L}^d = (0)$. We further claim that $ap\mathscr{L}^d = (0), \forall\, p \in \mathscr{L}$. If possible, let $p(\neq n) \in \mathscr{L}$ with $ap\mathscr{L}^d \neq (0)$. This deduces that $p^d = 0$. Thus, $a(p+n)\mathscr{L}^d = ap\mathscr{L}^d + an\mathscr{L}^d \neq (0)$ and $(p+n)^d = n^d \neq 0$ hold simultaneously and it leads to a contradiction. Hence, $ap\mathscr{L}^d = (0), \forall\, p \in \mathscr{L}$, equivalently $a\mathscr{L}\mathscr{L}^d = (0)$. In view of Lemma 4.6, we obtain $a = 0$. ■

Finally, we give an extension of [3, Theorem 1].

Theorem 5.8. *If d is a non-zero derivation of $\mathscr{S}$ and $\mathscr{L}$ is a 2-Lie ideal of $\mathscr{S}$ with $\mathscr{L}^{d^2} = (0)$, then $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{S})$.*

Proof. As we have proved earlier in Theorem 4.2, the ideal $2\mathscr{S}[\mathscr{L}, \mathscr{L}]\mathscr{S} \subseteq \mathscr{L}$, therefore

$$(2s[l_1, m_1]n)^{d^2} = 0, \ \forall\ l_1, m_1, n \in \mathscr{L}, s \in \mathscr{S}.$$

This gives that

$$0 = ((2s[l_1, m_1])^d n + 2s[l_1, m_1]n^d)^d = (2s[l_1, m_1])^{d^2} n + (2s[l_1, m_1])^d n^d + (2s[l_1, m_1])^d n^d + 2s[l_1, m_1]n^{d^2}.$$

Since $2\mathscr{S}[\mathscr{L}, \mathscr{L}] \subseteq \mathscr{L}$, $n \in \mathscr{L}$ and $char\ \mathscr{S} \neq 2$, therefore given hypothesis leads to

$$(s[l_1, m_1])^d n^d = 0, \ \forall\ l_1, m_1, n \in \mathscr{L}, s \in \mathscr{S}.$$

This infers that

$$s^d[l_1, m_1]n^d + s[l_1, m_1]^d n^d = 0, \ \forall\, l_1, m_1, n \in \mathscr{L}, s \in \mathscr{S}. \tag{1}$$

Replacing s by sr, with $r \in \mathscr{S}$ and obtain

$$s^d r[l_1, m_1]n^d + s(r^d[l_1, m_1]n^d + r[l_1, m_1]^d n^d) = 0.$$

Then equation (1) implies that $s^d r[l_1, m_1]n^d = 0$ which is equivalent to $s^d\mathscr{S}[l_1, m_1]n^d = (0), \forall\, l_1, m_1, n \in \mathscr{L}, s \in \mathscr{S}$. By primeness of $\mathscr{S}$, we get that

$$[l_1, m_1]n^d = 0, \ \forall\, l_1, m_1, n \in \mathscr{L}. \tag{2}$$

Further, replacing m_1 by $2m_1 t$, with $t \in \mathscr{L}$, we have $2([l_1, m_1]tn^d + m_1[l_1, t]n^d) = 0$. By using $char\ \mathscr{S} \neq 2$ and equation (2), we are left with $[l_1, m_1]\mathscr{L}n^d = (0), \forall\, l_1, m_1, n \in \mathscr{L}$. Then Lemma 4.6 gives either $[l_1, m_1] = 0$ or $n^d = 0, \forall\, l_1, m_1, n \in \mathscr{L}$. Then Proposition 3.3 and Theorem 5.6 concludes that $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{S})$. ■

6. Conclusions

This paper characterized the Lie structure of semirings and action of derivations on Lie ideals of semirings. It is observed that for a prime semiring $\mathscr{S}$, with $char\ \mathscr{S} \neq 2$ and $[a, [\mathscr{L}, \mathscr{L}]] = (0)$, for any Lie ideal $\mathscr{L}$ of $\mathscr{S}$ and $a \in \mathscr{S}$, either $\mathscr{L} \subseteq \mathscr{Z}(\mathscr{S})$ or $[a, \mathscr{L}] = (0)$ and thereby partially generalized Herstein's theorems in the framework of additively regular semirings and their higher commutators. Moreover, an extension to Herstein's result: "For a ring $\mathcal{R}$ with $char\ \mathcal{R} \neq 2$, any Lie ideal $\mathscr{L}$ is either contained in the center of $\mathcal{R}$ or contains a non-zero ideal of $\mathcal{R}$" is established which also enable us to extend Bergen's theorem for derivations.

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