MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **12(02)**(2024), 163–185.
http://doi.org/10.26637/mjm1202/003

Modeling and optimal control of the dynamics of narcoterrorism in the Sahel

MATHIEU ROMARIC POODA*[1], YACOUBA SIMPORE[3] AND OUMAR TRAORE[1,2]

[1] *Laboratoire de Sciences et Technologie, Université Thomas SANKARA, 12 BP 417 Ouagadougou 12, Burkina Faso.*
[2] *Laboratoire d'Analyse Mathématiques et d'Informatique, Université Joseph KI-ZERBO, 03 BP 7021 Ouagadougou 03, Burkina Faso, Email: oumar.traore@uts.bf.*
[3] *Laboratoire d'Analyse Mathématiques et d'Informatique, Université Joseph KI-ZERBO, 03 BP 7021 Ouagadougou 03, Burkina Faso. DeustoTech Fundación Deusto Avda Universidades, 24, 48007, Bilbao, Basque Country, Spain, Email:simplesaint@gmail.com.*

*R*eceived 17 August 2023; *A*ccepted 21 March 2024

Abstract. This work explores some aspects of modeling and controlling narcoterrorism in the Sahel. We examine the multidimensional factors underlying this dynamic, identifying interactions and recruitment within the narcoterrorist class. We then develop a preventive model and decision-support tools to optimize resource allocation and formulate more effective counter-narcotics and brigandage policies. This research will certainly contribute to the fight against narcoterrorism in the Sahel by proposing solutions based on rigorous scientific approaches and assessing the benefits and limitations of optimal modeling and control.

AMS Subject Classifications: 49K15, 93B05, 93C15, 93D23.

Keywords: narcoterrorism, local and global asymptotic stability, global threshold, optimal control, and numerical simulation.

Contents

*Corresponding author. Email address: **math7roma8@gmail.com** (Mathieu Romaric POODA)

1. Introduction and Background

The African continent offers Latin American and South American drug traffickers an uncontrolled transit route, with its porous borders, ideal location close to Europe, and fragile, corrupted states. According to the United Nations Office on Drugs and Crime (UNODC), the market value of cocaine transiting West Africa each year was estimated at US dollars 1.25 billion in 2013. The map below illustrates drug trafficking and transit zones from Latin and South America to Europe via West Africa and the Sahel, updated in February 2013 by the United Nations Office on Drugs and Crime (UNODC).

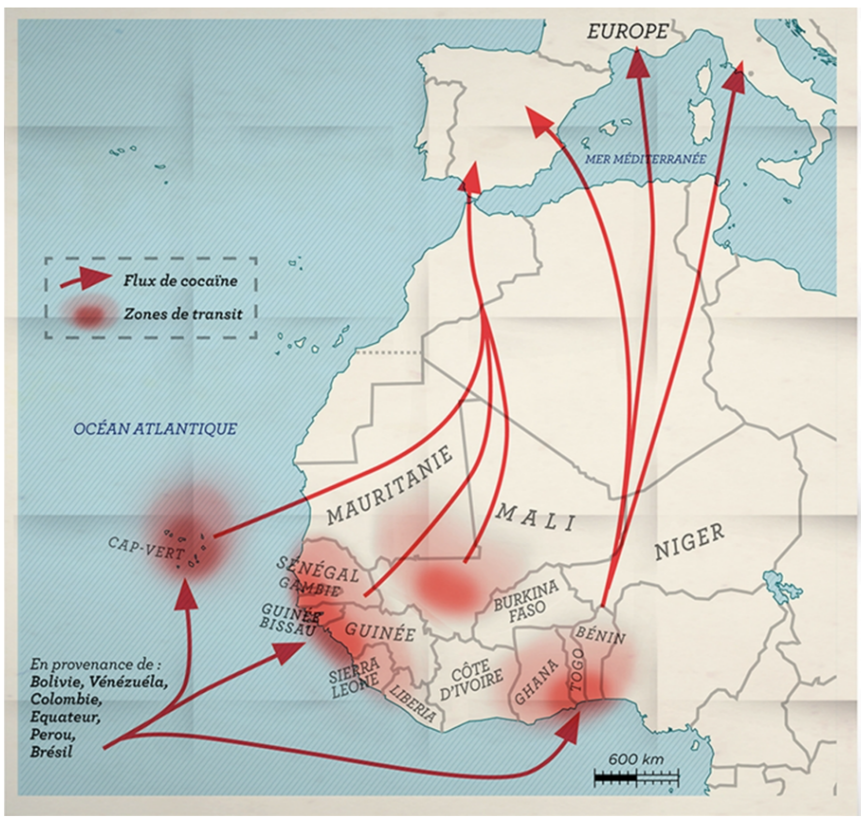

Figure 1: Map of drug trafficking and transit zones to Europe via West Africa and the Sahel.

In recent years, narcoterrorism has become a major problem in the Sahel. This deadly combination of drug trafficking and terrorist activity creates a complex and constantly evolving security and humanitarian crisis, requiring innovative approaches to understanding and controlling the threat. What are the dynamics of narcoterrorism in the Sahel? What factors have encouraged the development of narcoterrorism in the Sahel over the last few decades? Are there effective, targeted, and optimal strategies for eradicating narcoterrorism in the Sahel? It would be interesting to find answers to these questions and develop decision-making tools for political decision-makers and defense and security forces in the fight against drug trafficking, terrorism, and insecurity in general. In this spirit, we have decided to tackle this problem using a mathematical approach that is intended to be modest, as we do not claim to be able to say that mathematics can answer all these questions.

The rise of narcoterrorism in the Sahel can be explained by several factors. These include geographical and demographic factors. The Sahel's vast, sparsely populated territory, porous borders, and proximity to major drug-producing regions such as Latin America and West Africa make it an attractive transit route for drug traffickers. Added to this is the weakness of governance and security structures in the Sahel, which is said to benefit transnational criminal networks transporting illicit drugs, notably cocaine, heroin, and cannabis, across the region. We also have ideological terrorism and insurgency movements in the Sahel. The Sahel is indeed experiencing an increase in terrorist and insurgent activity, mainly perpetrated by groups such as Al-Qaeda in the Islamic Maghreb (AQIM), Boko Haram, and the Islamic State in the Greater Sahara (ISGS). These extremist groups exploit the region's socio-economic and political vulnerabilities, including poverty, unemployment, poor governance, and community tensions, to recruit fighters, finance their activities, and carry out attacks. The presence of drug-trafficking networks is an additional source of revenue for these terrorist groups. Another factor would be the financing of terrorism, as terrorist groups engage in a variety of criminal activities, including protecting drug convoys, taxing drug traffickers, and drug trafficking itself. Profits from the drug trade would enable these groups to continue their operations, buy weapons and recruit new members. The convergence of these criminal and terrorist activities creates a complex and dangerous environment that challenges the security forces and governments of the Sahel countries.

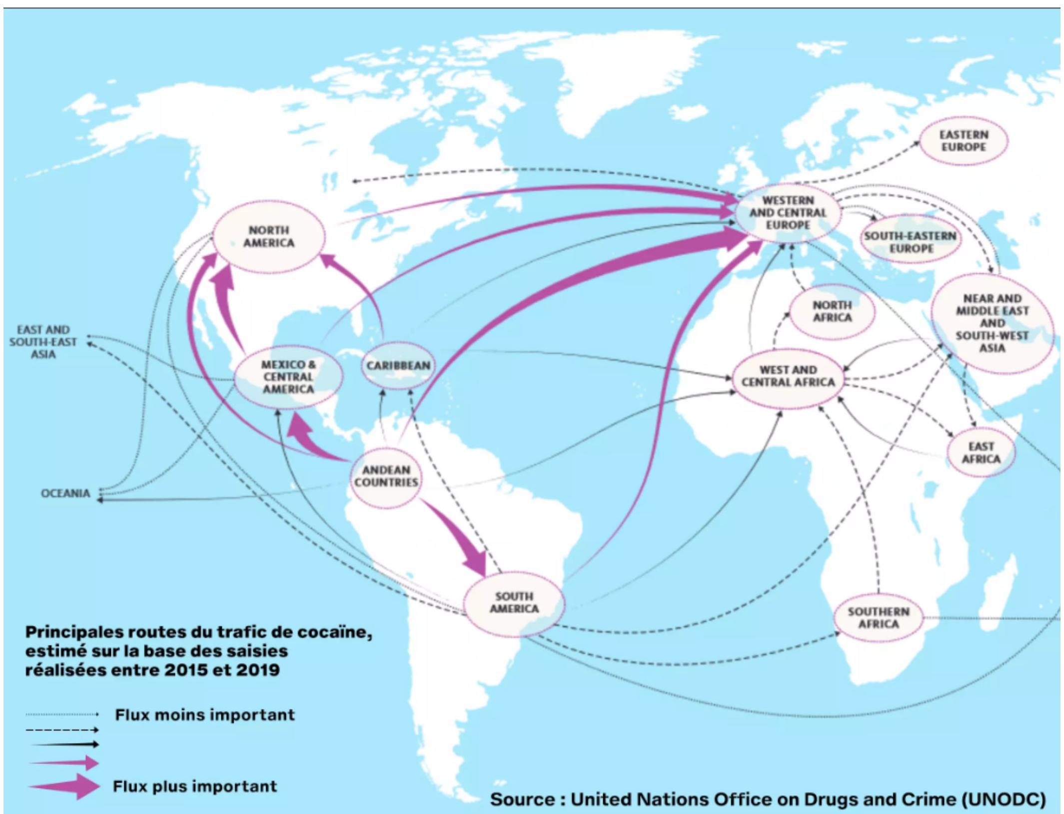

Figure 2: Map of the main cocaine trafficking flows.

The map above shows the scale of the threat. In November 2009, the image of the charred wreckage of a Boeing 727 found north of Gao in Mali revealed the scale of a hitherto unknown phenomenon. The plane, coming from Venezuela near the Colombian border, was carrying a cargo of several tonnes of cocaine. The media went so far as to popularise the concept of "air cocaine", while government intelligence services became aware of the imminence of the new threat looming on the horizon as a result of the convergence between extremist movements in the Sahel and drug traffickers in South America.

There is growing interest in the modelling and optimal control of the dynamics of narcoterrorism in the Sahel. These approaches, which combine mathematical tools, advanced simulation methods, and empirical data, provide a better understanding of the mechanisms underlying this complex dynamic. They also offer the possibility of formulating more effective and targeted control strategies. In this study, we seek to explore the different aspects of modeling and optimal control of narcoterrorism in the Sahel. We examine the multidimensional factors that drive this dynamic. By identifying the interactions as in evolution studies [3], [11], [6], [5] and recruitment within the narcoterrorist class we can better understand the mechanisms by which narcoterrorism spreads in the region. Building on this knowledge and adapting it to the specific context of the Sahel, we are developing a preventive model and decision-support tools to optimize resource allocation and formulate more effective counter-narcotics and counter-brigandage policies. This research aims to contribute to the fight against narcoterrorism in the Sahel by proposing solutions based on rigorous scientific approaches. Finally, by assessing the advantages and limitations of modelling and optimal control, we hope that this work will be useful to political decision-makers, security forces, and international players involved in the region. The specifics of the model are described in more detail in the next paragraph.

2. Model formulation

In order to facilitate the description of this model, we have divided the total population (N), into seven classes. Thus, we have the class of non-combatant civilians (C), the class of volunteers for the defence of the homeland and self-defence groups (V), the class of defence and security forces (A), the class of people discharged from the ranks of the defence and security forces (R), the class of brigands (B), the class of narcoterrorists (T), and the class of prisoners (P). The sum of the fighting classes $(A + V + B + T)$ is also referred to as I.

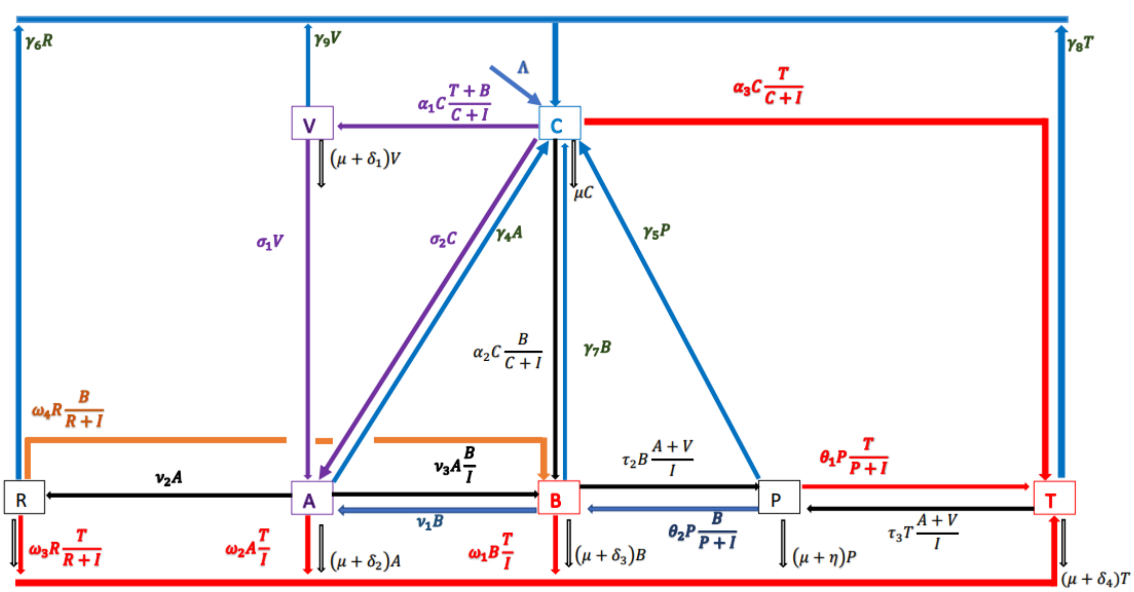

Figure 3: Diagram of the dynamics of narcoterrorism in the Sahel.

Here Λ denotes the population renewal constant, γ_i the rate of return to non-combatant civilian life for individuals in classes A, P, R, B, T, and V respectively for i ranging from 4 to 8. The probability of dying as a result of combat is denoted by δ_i, i ranging from 1 to 4 for individuals in classes V, A, B, and T respectively and ζ_i the intensity of the combat or nuisance force of individuals in classes B and T over those in classes V and A for i ranging from 1 to 2 respectively but also those of individuals in classes V and A over individuals in classes B and T respectively for i ranging from 3 to 4, η is the probability of dying as a result of the conditions of detention and μ is the natural mortality rate for all individuals in the population. The strength or capacity of recruitment into the narcoterrorist class of individuals in classes B, A, and R is respectively by ω_i, where i ranges from 1 to 3, and the strength of recruitment into brigandage of individuals in class R by ω_4. In the same way, for individuals in class C, α_1 denotes the intensity of the force of determination in defense of the homeland, α_2 that of the force of attraction in brigandage, α_3 the intensity of the force of attraction in narcoterrorism activities, σ_1 and σ_2 are the rates of recruitment into class A of individuals in classes V and C respectively. It is assumed that these rates (σ_1 and σ_2) are fixed by a given State in its defense strategy, but it is also assumed that a slight disturbance could occur during this recruitment which would mean that individuals from class B could be recruited with a probability ν_1. Furthermore, ν_2 designates the rate of radiation or desertion in class A and ν_3 the intensity of the conversion force in the brigandage of individuals in class A. The parameters, θ_1 and θ_2 are the capacities of recruitment of prisoners by the narcoterrorists and the brigands respectively, τ_2 and τ_3 the operational capacities of the classes V and A to be able to put in prison the individuals of the classes B and T respectively. It is assumed that these prisoners can be recruited as a result of prison breaks, prison attacks, or just contacts before the end of their sentence. Last but not least, it should be noted that recruitment is modelled on a contact or contagion process in epidemiology, taking into account in some cases the dissuasive presence of defence and security forces as well as self-defence groups and volunteers for the defence of the homeland. The equation of the model is formulated as follows:

$$\frac{dC}{dt} = \Lambda + \gamma_4 A + \gamma_5 P + \gamma_6 R + \gamma_7 B + \gamma_8 T + \gamma_9 V - \left(\alpha_1 \frac{T+B}{C+I} + \alpha_2 \frac{B}{C+I} + \alpha_3 \frac{T}{C+I} + \sigma_2 + \mu\right) C \tag{2.1}$$

$$\frac{dR}{dt} = \nu_2 A - \left(\omega_3 \frac{T}{R+I} + \omega_4 \frac{B}{R+I} + \gamma_6 + \mu\right) R \tag{2.2}$$

$$\frac{dA}{dt} = \sigma_1 V + \sigma_2 C + \nu_1 B - \left(\nu_3 \frac{B}{I} + \omega_2 \frac{T}{I} + \gamma_4 + \nu_2 + \mu + \zeta_1 \frac{T+B}{I}\right) A \tag{2.3}$$

$$\frac{dV}{dt} = \alpha_1 C \frac{T+B}{C+I} - \left(\gamma_9 + \sigma_1 + \mu + \zeta_2 \frac{T+B}{I}\right) V \tag{2.4}$$

$$\frac{dB}{dt} = \alpha_2 \frac{CB}{C+I} + \omega_4 \frac{RB}{R+I} + \nu_3 \frac{AB}{I} + \theta_2 \frac{PB}{P+I} - \left(\omega_1 \frac{T}{I} + \tau_2 \frac{A+V}{I} + \gamma_7 + \nu_1 + \mu + \zeta_3 \frac{A+V}{I}\right) B \tag{2.5}$$

$$\frac{dT}{dt} = \alpha_3 C \frac{T}{C+I} + \omega_1 B \frac{T}{I} + \omega_2 A \frac{T}{I} + \omega_3 R \frac{T}{R+I} + \theta_1 P \frac{T}{P+I} - \left(\tau_3 \frac{A+V}{I} + \gamma_8 + \mu + \zeta_4 \frac{A+V}{I}\right) T \tag{2.6}$$

$$\frac{dP}{dt} = \tau_2 B \frac{A+V}{I} + \tau_3 T \frac{A+V}{I} - \left(\theta_1 \frac{T}{P+I} + \theta_2 \frac{B}{P+I} + \gamma_5 + \mu + \eta\right) P \tag{2.7}$$

with non-negative initial conditions given by:

$$C(0) > 0; V(0) \geq 0; A(0) > 0; R(0) \geq 0; B(0) \geq 0; P(0) \geq 0; T(0) \geq 0, N(0) \leqslant \frac{\Lambda}{\mu}. \tag{2.8}$$

The parameters of the system $(2.1) - (2.7)$ are assumed to be all non-negative.

3. Mathematical analysis of the model

3.1. Existence and uniqueness of solution

The $(2.1) - (2.7)$ model is described by a system of first order nonlinear differential equations. It is rewritten as follows:

$$X'(t) = f(X(t)) \tag{3.1}$$

where $X(t)$ is a column vector of the number of individuals by class, and $f : \mathbb{R}^7 \to \mathbb{R}^7$ is a function. More precisely,

$$X(t) = \begin{bmatrix} C(t) \\ R(t) \\ A(t) \\ V(t) \\ B(t) \\ T(t) \\ P(t) \end{bmatrix} \tag{3.2}$$

and

$$f(x) = \begin{bmatrix} \Lambda + \gamma_4 x_3 + \gamma_5 x_7 + \gamma_6 x_2 + \gamma_7 x_5 + \gamma_8 x_6 + \gamma_9 x_4 - \left(\alpha_1 \dfrac{x_6 + x_7}{x_1 + x_8} + \alpha_2 \dfrac{x_5}{x_1 + x_8} + \alpha_2 \dfrac{x_5}{x_1 + x_8} + \sigma_2 + \mu\right) x_1 \\ \nu_2 x_3 - \left(\omega_3 \dfrac{x_6}{x_2 + x_8} + \omega_4 \dfrac{x_5}{x_2 + x_8} + \gamma_6 + \mu\right) x_2 \\ \sigma_1 x_4 + \sigma_2 x_1 + \nu_1 x_5 - \left(\nu_3 \dfrac{x_5}{x_8} + \omega_2 \dfrac{x_6}{x_8} + \gamma_4 + \nu_2 + \mu + \zeta_1 \dfrac{x_6 + x_5}{x_8}\right) x_3 \\ \alpha_1 x_1 \dfrac{x_6 + x_5}{x_1 + x_8} - \left(\gamma_9 + \sigma_1 + \mu + \zeta_2 \dfrac{x_6 + x_5}{x_8}\right) x_4 \\ \alpha_2 \dfrac{x_1 x_5}{x_1 + x_8} + \omega_4 \dfrac{x_2 x_5}{x_2 + x_8} + \nu_3 \dfrac{x_3 x_5}{x_8} + \theta_2 \dfrac{x_7 x_5}{x_7 + x_8} - \left(\omega_1 \dfrac{x_6}{x_8} + \tau_2 \dfrac{x_3 + x_4}{x_8} + \gamma_7 + \nu_1 + \mu + \zeta_3 \dfrac{x_3 + x_4}{x_8}\right) x_5 \\ \alpha_3 x_1 \dfrac{x_6}{x_1 + x_8} + \omega_1 x_5 \dfrac{x_6}{x_8} + \omega_2 x_3 \dfrac{x_6}{x_8} + \omega_3 x_2 \dfrac{x_6}{x_2 + x_8} + \theta_1 x_7 \dfrac{x_6}{x_7 + x_8} - \left(\tau_3 \dfrac{x_3 + x_4}{x_8} + \gamma_8 + \mu + \zeta_4 \dfrac{x_3 + x_4}{x_8}\right) x_6 \\ \tau_2 x_5 \dfrac{x_3 + x_4}{x_8} + \tau_3 x_6 \dfrac{x_3 + x_4}{x_8} - \left(\theta_1 \dfrac{x_6}{x_7 + x_8} + \theta_2 \dfrac{x_5}{x_7 + x_8} + \gamma_5 + \mu + \eta\right) x_7 \end{bmatrix} \tag{3.3}$$

with

$$x = (x_1, x_2, x_3, x_4, x_5, x_6, x_7) \in \mathbb{R}^7$$

and

$$\begin{cases} x_8 = x_3 + x_4 + x_5 + x_6 \\ x_9 = x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7. \end{cases}$$

The function f is clearly locally lipschitzian with respect to x. We then deduce the existence and the uniqueness of the maximal solution to the Cauchy problem associated to the differential equation $(2.1) - (2.7)$ related to the initial condition (2.8).

3.2. Positivity of the solutions

For this model of the dynamics of mafia terrorism to be realistic, it is necessary to show that all state variables remain positive at all times.

Proposition 3.1. ***(Positivity)*** *The positive orthan* $\mathbb{R}^7_{\geq 0}$ *is positively invariant for the system* $(2.1) - (2.7)$*, and the initial condition* (2.8) *ensures the positivity of the solutions of the system* $(2.1) - (2.7)$ *for any time* $t > 0$*.*

Proof: We use the barrier theorem [2].

Let us show that the set $\left\{C \geq 0\right\}$ is positively invariant. Let $x = \left(C, R, A, V, B, T, P\right)$ and consider L an application defined by

$$L(x) = -C \tag{3.4}$$

The application L thus defined is differentiable and we have:

$$\nabla L(x) = (-1, 0, 0, 0, 0, 0, 0) \neq 0_{\mathbb{R}^7}. \tag{3.5}$$

The vector field for $\{C=0\}$ is given by

$$X(x)=\begin{bmatrix} \Lambda+\gamma_4A+\gamma_5P+\gamma_6R+\gamma_7B+\gamma_8T+\gamma_9V \\ \nu_2A-\Big(\omega_3\dfrac{T}{R+I}+\omega_4\dfrac{B}{R+I}+\gamma_6+\mu\Big)R \\ \sigma_1V+\nu_1B-\Big(\nu_3\dfrac{B}{I}+\omega_2\dfrac{T}{I}+\gamma_4+\nu_2+\mu+\zeta_1\dfrac{T+B}{I}\Big)A \\ -\Big(\gamma_9+\sigma_1+\mu+\zeta_2\dfrac{T+B}{I}\Big)V \\ \omega_4R\dfrac{B}{R+I}+\nu_3A\dfrac{B}{I}+\theta_2P\dfrac{B}{P+I}-\Big(\omega_1\dfrac{T}{I}+\tau_2\dfrac{A+V}{I}+\gamma_7+\nu_1+\mu+\zeta_3\dfrac{A+V}{I}\Big)B \\ \omega_1B\dfrac{T}{I}+\omega_2A\dfrac{T}{I}+\omega_3R\dfrac{T}{R+I}+\theta_1P\dfrac{T}{P+I}-\Big(\tau_3\dfrac{A+V}{I}+\gamma_8+\mu+\zeta_4\dfrac{A+V}{I}\Big)T \\ \tau_2B\dfrac{A+V}{I}+\tau_3T\dfrac{A+V}{I}-\Big(\theta_1\dfrac{T}{P+I}+\theta_2\dfrac{B}{P+I}+\gamma_5+\mu+\eta\Big)P \end{bmatrix} \tag{3.6}$$

From (3.5) and (3.6), we have

$$\langle X(x),\nabla L(x)\rangle=-\Big(\Lambda+\gamma_4A+\gamma_5P+\gamma_6R+\gamma_7B+\gamma_8T+\gamma_9V\Big)\le 0 \tag{3.7}$$

From (3.5) and (3.7) we deduce that $\{C\ge 0\}$ is positively invariant by application of the barrier theorem. Similarly, we show that $\{R\ge 0\}$, $\{A\ge 0\}$, $\{V\ge 0\}$, $\{B\ge 0\}$, $\{T\ge 0\}$, and $\{P\ge 0\}$ are positively invariant. Therefore, $\mathbb{R}^7_{\ge 0}$ is positively invariant.
Also by the initial condition (2.8), we have $x(0)\in\mathbb{R}^7_{\ge 0}$. Since $\mathbb{R}^7_{\ge 0}$ is positively invariant, then this ensures that all solutions of the system $(2.1)-(2.7)$ stay positive for all time $t>0$ $\square$.

3.3. Invariant region

Theorem 3.2. *For initial conditions* (2.8)*, the solutions of the system* $(2.1)-(2.7)$ *are contained in the positively invariant, compact and attractive region*

$$\Psi=\left\{\Big(C,R,A,V,B,T,P\Big)\in\mathbb{R}^7_{\ge 0}:N(t)\le\frac{\Lambda}{\mu}\right\} \tag{3.8}$$

Proof: Summing the equations (2.1) to (2.7), we find :

$$\frac{dN}{dt}=\Lambda-\mu N-\delta_1V-\delta_2A-\delta_3B-\delta_4T-\eta P,$$

with $\delta_1=\zeta_1\dfrac{T+B}{I}$, $\delta_2=\zeta_2\dfrac{T+B}{I}$, $\delta_3=\zeta_3\dfrac{A+V}{I}$, and $\delta_4=\zeta_4\dfrac{A+V}{I}$.
Since A,V,B,T,F,P are positive functions and using the positivity of the functions δ_1, δ_2, δ_3, δ_4, given that the constants ζ_1, ζ_2, ζ_3, ζ_4 and η are strictly positive as well, we get:

$$\frac{dN}{dt}\le\quad\Lambda-\mu N.$$

Then

$$\frac{d}{dt}\Big(N-\frac{\Lambda}{\mu}\Big)\le\quad-\mu\Big(N-\frac{\Lambda}{\mu}\Big).$$

So the Gromwall inequality gives

$$N(t)-\frac{\Lambda}{\mu}\leq\left(N(0)-\frac{\Lambda}{\mu}\right)e^{-\mu t}.$$

Thus

$$N(t)\leq\frac{\Lambda}{\mu}+\left(N(0)-\frac{\Lambda}{\mu}\right)e^{-\mu t}.$$

Since $N(0)\leq\frac{\Lambda}{\mu}$, then $0\leq N(t)\leq\frac{\Lambda}{\mu}$.

Therefore, all feasible solutions of the model $(2.1)-(2.7)$ converge in the region Ψ. □

4. Equilibrium without terrorist, nor brigand (x^*), and basic reproduction rumber $\mathcal{R}_0$

4.1. Equilibrium without terrorist, nor brigand x^*

The uninfected compartments are C, R, A, V and the infected compartments are B, T, P. Given that we are at equilibrium without narcoterrorist nor brigand then we can discard the P compartment and the infected compartments being B, T, then an equilibrium solution with B=T=0 has the form:

$$x^*=\Big(C^*,R^*,A^*,0,0,0\Big) \tag{4.1}$$

with

$$\begin{cases} C^*=\dfrac{\Lambda(\gamma_6+\mu)(\gamma_4+\nu_2+\mu)}{\mu\big[(\gamma_6+\mu)(\gamma_4+\mu+\nu_2+\sigma_2)+\sigma_2\nu_2\big]}\\[2ex] R^*=\dfrac{\Lambda\nu_2\sigma_2}{\mu\big[(\gamma_6+\mu)(\gamma_4+\mu+\nu_2+\sigma_2)+\sigma_2\nu_2\big]}\\[2ex] A^*=\dfrac{\Lambda\sigma_2(\gamma_6+\mu)}{\mu\big[(\gamma_6+\mu)(\gamma_4+\mu+\nu_2+\sigma_2)+\sigma_2\nu_2\big]}\end{cases}$$

4.2. Matrix of next generation $\mathcal{K}$, and basic reproduction number $\mathcal{R}_0$

The Jacobian matrix of the system $(2.1)-(2.7)$ is decomposed into $J_x(x^*)=D\mathcal{F}(x^*)+D\mathcal{V}(x^*)$ with

$$\mathcal{F}=\begin{bmatrix}0\\0\\0\\0\\ \alpha_2C\dfrac{B}{C+I}+\omega_4R\dfrac{B}{R+I}+\nu_3A\dfrac{B}{I}\\ \alpha_3C\dfrac{T}{C+I}+\omega_1B\dfrac{T}{I}+\omega_2A\dfrac{T}{I}+\omega_3R\dfrac{T}{R+I}\end{bmatrix}$$

and

$$\mathcal{V}=\begin{bmatrix} \Lambda+\gamma_4 A+\gamma_6 R+\gamma_7 B+\gamma_8 T+\gamma_9 V-\Big(\alpha_3\dfrac{T}{C+I}+\alpha_1\dfrac{T+B}{C+I}+\alpha_2\dfrac{B}{C+I}+\sigma_2+\mu\Big)C \\ \nu_2 A-\Big(\omega_3\dfrac{T}{R+I}+\omega_4\dfrac{B}{R+I}+\gamma_6+\mu\Big)R \\ \sigma_1 V+\sigma_2 C+\nu_1 B-\Big(\nu_3\dfrac{B}{I}+\omega_2\dfrac{T}{I}+\gamma_4+\nu_2+\mu+\zeta_1\dfrac{T+B}{I}\Big)A \\ \alpha_1 C\dfrac{T+B}{C+I}-\Big(\gamma_9+\sigma_1+\mu+\zeta_2\dfrac{T+B}{I}\Big)V \\ -\Big(\omega_1\dfrac{T}{I}+\tau_2\dfrac{A+V}{I}+\gamma_7+\nu_1+\mu+\zeta_3\dfrac{A+V}{I}\Big)B \\ -\Big(\tau_3\dfrac{A+V}{I}+\gamma_8+\mu+\zeta_4\dfrac{A+V}{I}\Big)T \end{bmatrix}$$

$$D\mathcal{F}(x^*)=\begin{bmatrix}0 & 0\\ 0 & \mathbb{F}\end{bmatrix} \quad ; \quad D\mathcal{V}(x^*)=\begin{bmatrix}J_1 & J_2\\ 0 & \mathbb{V}\end{bmatrix} \quad with \quad \mathbb{F}=\left[\frac{\partial \mathcal{F}_i(x^*)}{\partial x_j}\right]_{5\le i,j\le 6};$$

$$J_1=\left[\frac{\partial \mathcal{V}_i(x^*)}{\partial x_j}\right]_{1\le i,j\le 4} \quad ; \quad J_2=\left[\frac{\partial \mathcal{V}_i(x^*)}{\partial x_j}\right]_{1\le i\le 4;\, 5\le j\le 6} \quad and \ \ \mathbb{V}=\left[\frac{\partial \mathcal{V}_i(x^*)}{\partial x_j}\right]_{5\le i,j\le 6}.$$

Let:

$$g=\alpha_2\frac{C^*}{C^*+A^*}+\omega_4\frac{R^*}{R^*+A^*}+\nu_3;$$

$$h=\alpha_3\frac{C^*}{C^*+A^*}+\omega_2+\omega_3\frac{R^*}{R^*+A^*}.$$

We get

$$\mathbb{F}=\begin{bmatrix}g & 0\\ 0 & h\end{bmatrix};$$

$$J_1=\begin{bmatrix} -(\sigma_2+\mu) & \gamma_6 & \gamma_4 & \gamma_9 \\ 0 & -(\gamma_6+\mu) & \nu_2 & 0 \\ \sigma_2 & 0 & -(\gamma_4+\mu+\nu_2) & \sigma_1 \\ 0 & 0 & 0 & -(\gamma_9+\mu+\sigma_1) \end{bmatrix}$$

and

$$J_2=\begin{bmatrix} \varpi_1 & \varpi_2 \\ -\omega_4\dfrac{R^*}{R^*+A^*} & \varpi_3 \\ \nu_1-\nu_3 & -\omega_2 \\ \alpha_1\dfrac{C^*}{C^*+A^*} & \alpha_1\dfrac{C^*}{C^*+A^*} \end{bmatrix},$$

with

$$\begin{aligned}\varpi_1 &= \gamma_7 - \alpha_1 \frac{C^*}{C^* + A^*} - \alpha_2 \frac{C^*}{C^* + A^*} \\ \varpi_2 &= \gamma_8 - \alpha_3 \frac{C^*}{C^* + A^*} - \alpha_1 \frac{C^*}{C^* + A^*} \\ \varpi_3 &= -\omega_3 \frac{R^*}{R^* + A^*}\end{aligned}$$

Note that J_1 is a non-singular Metzler matrix $(see \quad [1])$.

$$\mathbb{V} = \begin{bmatrix} -d & 0 \\ 0 & -e \end{bmatrix}$$

with

$$\begin{aligned} d &= \gamma_7 + \mu + \tau_2 + \nu_1 + \zeta_3 \\ e &= \gamma_8 + \mu + \tau_3 + \zeta_4 \end{aligned}$$

We also note that $\mathbb{V}$ is a Metzler-Hurwitz matrix and

$$\mathbb{V}^{-1} = \begin{bmatrix} -\frac{1}{d} & 0 \\ 0 & -\frac{1}{e} \end{bmatrix}$$

$$\mathbb{V}^{-1} = \begin{bmatrix} -\frac{1}{d} & 0 \\ 0 & -\frac{1}{e} \end{bmatrix} \Rightarrow \mathcal{K} = -\mathbb{F}\mathbb{V}^{-1} = \begin{bmatrix} \frac{g}{d} & 0 \\ 0 & \frac{h}{e} \end{bmatrix}$$

where

$$\begin{cases} \dfrac{g}{d} = \left(\dfrac{1}{\gamma_7 + \tau_2 + \nu_1 + \mu + \zeta_3}\right)\left(\alpha_2 \dfrac{\gamma_4 + \nu_2 + \mu}{\gamma_4 + \nu_2 + \sigma_2 + \mu} + \omega_4 \dfrac{\nu_2}{\gamma_6 + \nu_2 + \mu} + \nu_3\right) \\ \\ \dfrac{h}{e} = \left(\dfrac{1}{\gamma_8 + \tau_3 + \mu + \zeta_4}\right)\left(\alpha_3 \dfrac{\gamma_4 + \nu_2 + \mu}{\gamma_4 + \nu_2 + \sigma_2 + \mu} + \omega_2 + \omega_3 \dfrac{\nu_2}{\gamma_6 + \nu_2 + \mu}\right) \end{cases}$$

and

$$\mathcal{R}_0 = \rho(\mathcal{K}) = \max\left\{\frac{g}{d}; \frac{h}{e}\right\} \tag{4.2}$$

Theorem 4.1. *The equilibrium without terrorist, nor brigand x^*, is locally asymptotically stable if $\mathcal{R}_0 < 1$ and is unstable if $\mathcal{R}_0 > 1$.*

See [14, 16].

Theorem 4.2. *The equilibrium without terrorist, nor brigand x^*, is globally asymptotically stable if $\mathcal{R}_0 < 1$ and is unstable if $\mathcal{R}_0 > 1$.*

Proof: From Theorem 4.1 when $\mathcal{R}_0 < 1$ the states $B, T \to 0$ when $t \to \infty$. Identifying B and T with zero, it comes that $(C, R, A, V, B, T, P) \to x^*$ when $t \to \infty$ since x^* is the unique point in the positively invariant, compact and attractive solution region Ψ, such that $B = T = 0$. □

5. Global thresholds

5.1. A sufficient condition for the eradication of narcoterrorism

The result we set out in this section highlights the fact that, when the recruitment capacity or the sum of the forces of association with individuals in the narcoterrorist class is lower than the forces of exit from this class, then we will see an eradication of narcoterrorism. It's worth noting that when we talk about the forces of attraction in narcoterrorism activities, we're alluding in this study to the ability of narcoterrorists to offer a certain improvement in living conditions in financial terms.

Theorem 5.1. *Let $\lambda_2 = \alpha_3 + \omega_1 + \omega_2 + \omega_3 + \theta_1$, $\lambda_3 = (\tau_3 + \zeta_4)\kappa + \gamma_8$ with κ the infimum of $\dfrac{A+V}{I}$. So for all $\mathcal{R}_2 = \dfrac{\lambda_2}{\lambda_3} < 1$, we have $\lim\limits_{t\to\infty} T(t) = 0$.*

Proof: From the equation (2.6) we have:

$$
\begin{aligned}
\frac{dT}{dt} &= \alpha_3 \frac{C}{C+I} T + \omega_1 B \frac{T}{I} + \omega_2 A \frac{T}{I} + \omega_3 R \frac{T}{R+I} + \theta_1 P \frac{T}{P+I} - \left(\tau_3 \frac{A+V}{I} + \gamma_8 + \zeta_4 \frac{A+V}{I}\right) T \\
&= (\alpha_3 \frac{C}{C+I} + \omega_1 \frac{B}{I} + \omega_2 \frac{A}{I} + \omega_3 \frac{R}{R+I} + \theta_1 \frac{P}{P+I}) T - \left(\tau_3 \frac{A+V}{I} + \gamma_8 + \zeta_4 \frac{A+V}{I}\right) T \\
&\le (\alpha_3 \frac{C}{C+I} + \omega_1 \frac{B}{I} + \omega_2 \frac{A}{I} + \omega_3 \frac{R}{R+I} + \theta_1 \frac{P}{P+I}) T - \left(\tau_3 \kappa + \gamma_8 + \zeta_4 \kappa\right) T \\
&\le (\alpha_3 + \omega_1 + \omega_2 + \omega_3 + \theta_1) T - \left((\tau_3 + \zeta_4)\kappa + \gamma_8\right) T = (\lambda_2 - \lambda_3) T.
\end{aligned}
$$

It follows from the last inequality that T decreases exponentially to zero as soon as $\lambda_2 < \lambda_3$.□

Thus $\mathcal{R}_2 = \dfrac{\lambda_2}{\lambda_3} < 1$, gives a sufficient condition of the stabilization or eradication of narcoterrorism. This result reflects the fact that the greater the nuisance capacity of the defense and security forces, as well as their attractiveness in other legal activities, the more the narcoterrorist class tends towards elimination.

5.2. A sufficient condition of the eradication of brigandage.

The result that we also present in this section highlights the fact that, when the recruitment capacity or the sum of the forces of association with individuals in the bandit class is less than the forces of exit from this class, banditry is eradicated or stabilized.

Theorem 5.2. *Let $\lambda_5 = \alpha_2 + \omega_4 + \nu_3 + \theta_2$ and $\lambda_6 = \tau_2\kappa + \gamma_7 + \nu_1 + \zeta_3\kappa$ with κ respective infimum of $\dfrac{A+V}{I}$ and of. So for all $\mathcal{R}_4 = \dfrac{\lambda_5}{\lambda_6} < 1$, we have $\lim\limits_{t\to\infty} B(t) = 0$.*

Proof: From the equation (2.5) we have:

$$\begin{aligned}
\frac{dB}{dt} &= \alpha_2 C\frac{B}{C+I} + \omega_4 R\frac{B}{R+I} + \nu_3 A\frac{B}{I} + \theta_2 P\frac{B}{P+I} - \left(\tau_2\frac{A+V}{I} + \gamma_7 + \nu_1 + \zeta_3\frac{A+V}{I}\right)B \\
&\leq \big(\alpha_2 + \omega_4 + \nu_3 + \theta_2\big)B - \Big(\tau_2\kappa + \gamma_7 + \nu_1 + \zeta_3\kappa\Big)B \\
&\leq \big(\lambda_5 - \lambda_6\big)B.
\end{aligned}$$

6. Numerical simulation

To highlight the results of our analysis, we carry out a numerical simulation in this section. This simulation is carried out in Matlab using the difference method, in particular an explicit Euler scheme. Figure 4 shows that for a value of $\mathcal{R}_0$ less than 1 we have a complete elimination or stabilization at zero of classes T, B, and P but also of class V. The latter result can be explained by the fact that, in this model, class V is linked to classes T and B. On the other hand, Figure 5 shows the persistence of narcoterrorism and brigandage for a value of $\mathcal{R}_0$ strictly greater than 1. For this simulation, we consider the initial states C(0)=100000, R(0)=80, A(0)=1000, V(0)=2000, B(0)=110, T(0)=110, P(0)=80 and the parameter values defined in the table below:

Table 1: Parameter values estimeted

Parameters	value for extinction	value for persistence
Λ	36900	36900
γ_4	0.047	0.047
γ_5	0.0016	0.0016
γ_6	0.00149	0.00149
γ_7	0.0046	0.00046
γ_8	0.0000011	0.0000011
γ_9	0.011	0.011
θ_1	0.0032	0.22
θ_2	0.0032	0.24
η	0.19	0.19
ζ_1	0.27	0.27
ζ_2	0.27	0.27
ζ_3	0.37	0.37
ζ_4	0.37	0.37
μ	0.148	0.148
ν_1	0.02	0.02
ν_2	0.01	0.001
ν_3	0.02	0.02
τ_1	0.2	0.02
τ_2	0.125	0.0125
τ_3	0.125	0.125
σ_1	0.012	0.012
σ_2	0.006	0.006
α_1	0.2	0.2
α_2	0.31	0.78
α_3	0.31	0.48
ω_1	0.02	0.1
ω_2	0.02	0.147
ω_3	0.02	0.58
ω_4	0.04	0.5

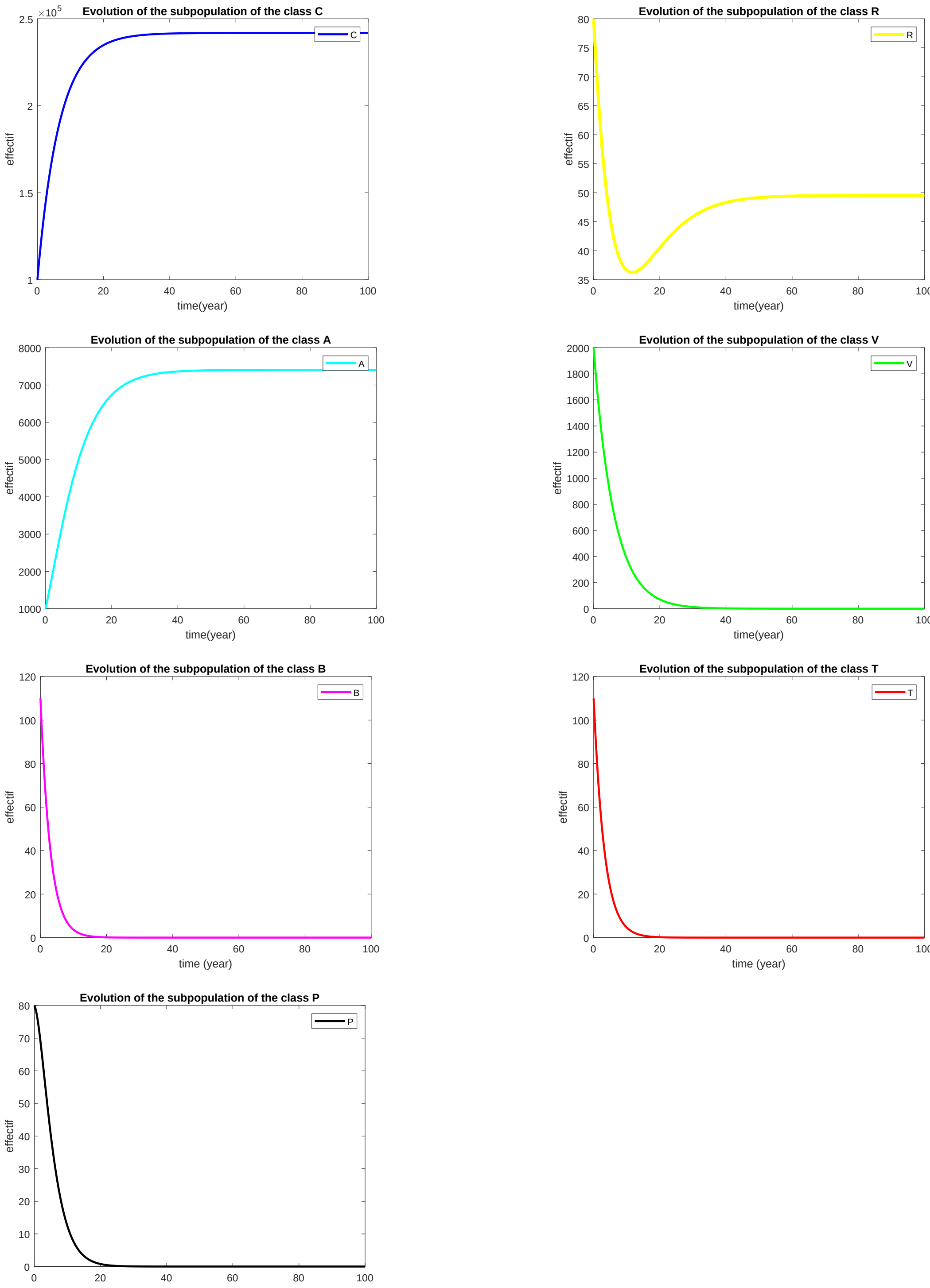

Figure 4: Evolution of the different classes of the model $(2.1)-(2.7)$ with the extinction values. We get $\mathcal{R}_0 = 0.7656$, which is less than unity.

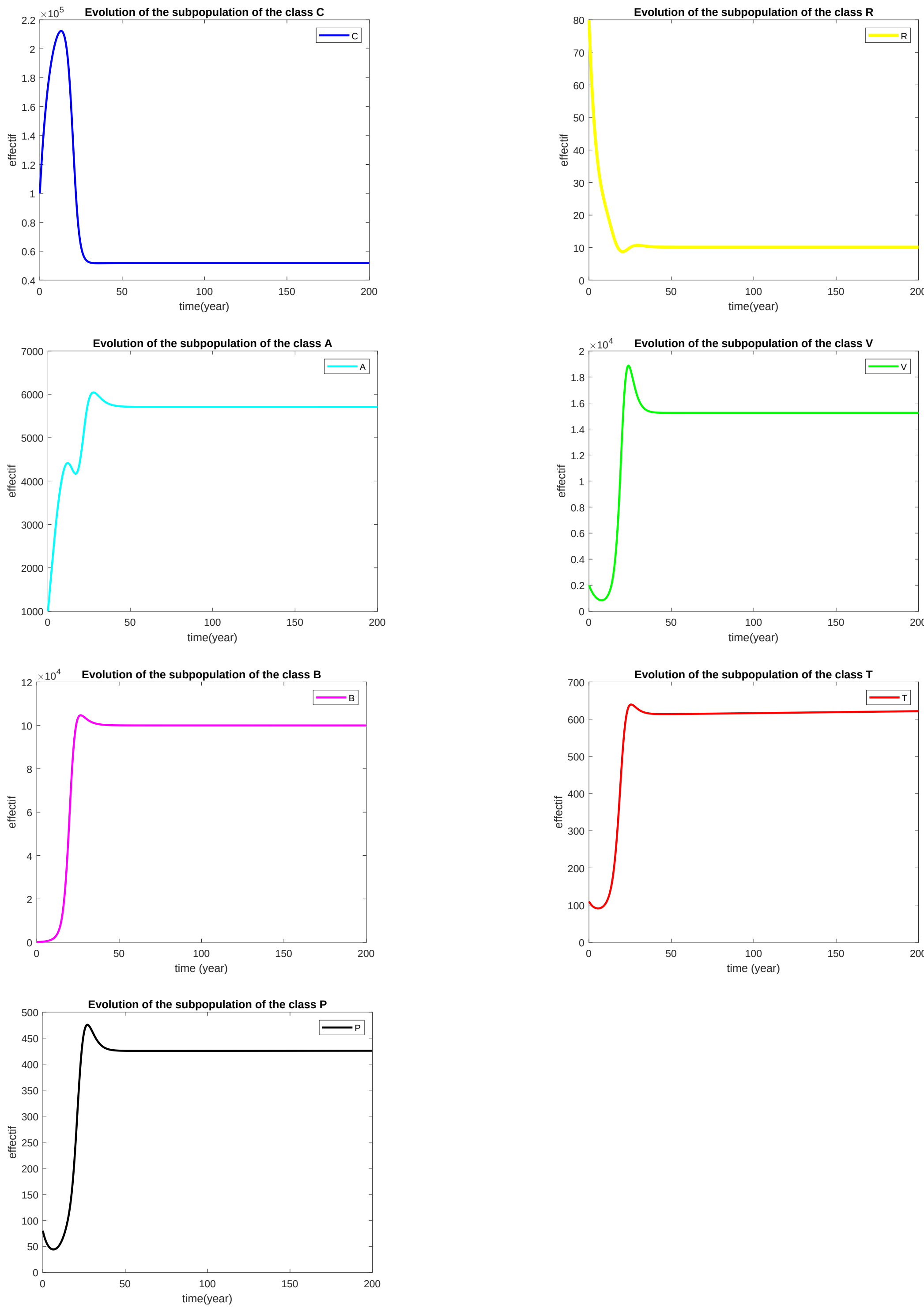

Figure 5: Evolution of the different classes of the model $(2.1) - (2.7)$ with persistence values. We obtain $\mathcal{R}_0 = 1.4966$, which is greater than unity.

7. Optimal control analysis

7.1. Strategy to fight against narcoterrorism and brigandage

In light of the results of the analysis, optimal control theory is applied to the $(2.1) - (2.7)$ model to fight narcoterrorism and banditry. Thus, two time-dependent control variables are introduced: $u_1(t)$ and $u_2(t)$, which are several strategies described in detail as follows:

(*i*) $u_1(t)$ is a strategy to fight against drug trafficking, organized crime, brigandage, and corruption. It also integrates all police actions of proximity, investigation, and protection. By ensuring a better territorial network and better training and equipment for the defense and security forces, as well as for volunteers for the defense of the country. In addition to these actions, this strategy could also integrate all the actions of accompaniment and reintegration of the accused or prisoners into active life. Note that the closer the u_2 strategy is to 1, the more efficient it is.

(*ii*) $u_2(t)$ is a strategy to combat narcoterrorism. It places particular emphasis on the fight against drug trafficking, which is the main source of funding for this type of terrorism. In addition, this strategy integrates all actions aimed at increasing the firepower of defense and security forces, while developing operational intelligence that is better adapted and better than that of narco-terrorists, so as to be able to carry out well-coordinated and well-calculated actions to minimize narco-terrorist attacks. Note that the closer u_2 is to 1, the more efficient it is.

7.2. Mathematical analysis of strategy optimality

Let's put

$$c_i(t) = 1 - u_i(t), \qquad \forall i \in \{1, 2\}. \tag{7.1}$$

Consequently, the optimal control model with the two aforementioned time-dependent variables is given by the following differential equations

$$\begin{cases} \dfrac{dC}{dt} &= \Lambda + \gamma_4 A + \gamma_5 P + \gamma_6 R + \gamma_7 B + \gamma_8 T + \gamma_9 V - \left(\alpha_1 \dfrac{T+B}{C+I} + c_1 \alpha_2 \dfrac{B}{C+I} + c_2 \alpha_3 \dfrac{T}{C+I} + \sigma_2 + \mu\right) C \\ \dfrac{dR}{dt} &= \nu_2 A - \left(c_2 \omega_3 \dfrac{T}{R+I} + c_1 \omega_4 \dfrac{B}{R+I} + \gamma_6 + \mu\right) R \\ \dfrac{dA}{dt} &= \sigma_1 V + \sigma_2 C + \nu_1 B - \left(c_1 \nu_3 \dfrac{B}{I} + c_2 \omega_2 \dfrac{T}{I} + \gamma_4 + \nu_2 + \mu + \zeta_1 \dfrac{T+B}{I}\right) A \\ \dfrac{dV}{dt} &= \alpha_1 C \dfrac{T+B}{C+I} - \left(\gamma_9 + \sigma_1 + \mu + \zeta_2 \dfrac{T+B}{I}\right) V \\ \dfrac{dB}{dt} &= c_1 \left(\alpha_2 \dfrac{CB}{C+I} + \omega_4 \dfrac{RB}{R+I} + \nu_3 \dfrac{AB}{I} + \theta_2 \dfrac{PB}{P+I}\right) - \left(c_2 \omega_1 \dfrac{T}{I} + \tau_2 \dfrac{A+V}{I} + \gamma_7 + \nu_1 + \mu + \zeta_3 \dfrac{A+V}{I}\right) B \\ \dfrac{dT}{dt} &= c_2 \left(\alpha_3 C \dfrac{T}{C+I} + \omega_1 B \dfrac{T}{I} + \omega_2 A \dfrac{T}{I} + \omega_3 R \dfrac{T}{R+I} + \theta_1 P \dfrac{T}{P+I}\right) - \left(\tau_3 \dfrac{A+V}{I} + \gamma_8 + \mu + \zeta_4 \dfrac{A+V}{I}\right) T \\ \dfrac{dP}{dt} &= \tau_2 B \dfrac{A+V}{I} + \tau_3 T \dfrac{A+V}{I} - \left(c_2 \theta_1 \dfrac{T}{P+I} + c_1 \theta_2 \dfrac{B}{P+I} + \gamma_5 + \mu + \eta\right) P \end{cases} \tag{7.2}$$

with initial conditions given by (2.8). This system can be rewritten in matrix form as follows:

$$X'(t) = g(t, X, c) \tag{7.3}$$

where X is defined in (3.2), $c = (c_1(t), c_2(t)) \in \mathbb{R}^2$ verifies (7.1), and $g : \mathbb{R} \times^7 \times \mathbb{R}^2 \to \mathbb{R}^7$ is a non-linear function written as in (3.3) but introducing the control c in order to verify (7.2). The purpose of introducing the two control variables is to find the optimal solution required to minimize the number of individuals in both the narcoterrorists class and the brigands class. Consequently, the objective function for this control problem is given by

$$\mathscr{J}(u_1,u_2)=\min_{0\leqslant u_1,u_2\leqslant 1}\int_0^{T_f}\left(j(t)+\frac{1}{2}k(t)\right)dt \tag{7.4}$$

where

$$j(t)=w_1B(t)+w_2T(t)+w_3P(t)$$
$$k(t)=\left[w_4u_1^2(t)+w_5u_2^2(t)\right]$$

where the constants $w_i, i=1,2,...,5$ are positive weights required to balance the corresponding terms of the objective function. We choose quadratic costs on the controls, where $\frac{1}{2}w_4u_1^2(t)$, $\frac{1}{2}w_5u_2^2(t)$, are the total costs of implementing the preventive measure and the military-police response to manage the active cases of narcoterrorism and brigands over the time interval $[0,T_f]$. More precisely, we are looking for the optimal dual control $u^*=\left(u_1^*,u_2^*\right)$ such that

$$\mathscr{J}(u_1^*,u_2^*)=\min\left\{\mathscr{J}(u_1,u_2):u_1,u_2\in\mathcal{U}\right\}, \tag{7.5}$$

where, $\mathcal{U}$ is the non-empty control set defined by

$$\mathcal{U}=\left\{(u_1,u_2)\;\middle|\;\begin{array}{c}u_i(t)\text{ is a piecewise continuous function on }[0,T_f]\\ \text{and}\quad 0\leqslant u_i\leqslant 1,\quad \forall\in t\in[0,T_f],\qquad i=1,2\end{array}\right\} \tag{7.6}$$

Thus, to determine the necessary conditions that the optimal control must satisfy, we use the Pontryagin maximum principle $[12]$, which transforms the control problem (7.5) subject to the model (7.2) into a problem of pointwise minimization of a Hamiltonian $\mathcal{H}$. This Hamiltonian is given by

$$\begin{aligned}\mathcal{H}=\;&w_1B+w_2T+w_3P+\frac{1}{2}\left[w_4u_1^2(t)+w_5u_2^2(t)\right]\\
&+\lambda_1\left[\Lambda+\gamma_4A+\gamma_5P+\gamma_6R+\gamma_7B+\gamma_8T+\gamma_9V-\left(\alpha_1\frac{T+B}{C+I}+c_1\alpha_2\frac{B}{C+I}+c_2\alpha_3\frac{T}{C+I}+\sigma_2+\mu\right)C\right]\\
&+\lambda_2\left[\nu_2A-\left(c_2\omega_3\frac{T}{R+I}+c_1\omega_4\frac{B}{R+I}+\gamma_6+\mu\right)R\right]\\
&+\lambda_3\left[\sigma_1V+\sigma_2C+\nu_1B-\left(c_1\nu_3\frac{B}{I}+c_2\omega_2\frac{T}{I}+\gamma_4+\nu_2+\mu+\zeta_1\frac{T+B}{I}\right)A\right]\\
&+\lambda_4\left[\alpha_1C\frac{T+B}{C+I}-\left(\gamma_9+\sigma_1+\mu+\zeta_2\frac{T+B}{I}\right)V\right]\\
&+\lambda_5\left[c_1\left(\alpha_2\frac{CB}{C+I}+\omega_4\frac{RB}{R+I}+\nu_3\frac{AB}{I}+\theta_2\frac{PB}{P+I}\right)-\left(c_2\omega_1\frac{T}{I}+\tau_2\frac{A+V}{I}+\gamma_7+\nu_1+\mu+\zeta_3\frac{A+V}{I}\right)B\right]\\
&+\lambda_6\left[c_2\left(\alpha_3C\frac{T}{C+I}+\omega_1B\frac{T}{I}+\omega_2A\frac{T}{I}+\omega_3R\frac{T}{R+I}+\theta_1P\frac{T}{P+I}\right)-\left(\tau_3\frac{A+V}{I}+\gamma_8+\mu+\zeta_4\frac{A+V}{I}\right)T\right]\\
&+\lambda_7\left[\tau_2B\frac{A+V}{I}+\tau_3T\frac{A+V}{I}-\left(c_2\theta_1\frac{T}{P+I}+c_1\theta_2\frac{B}{P+I}+\gamma_5+\mu+\eta\right)P\right]\end{aligned} \tag{7.7}$$

where $\lambda_i, i = 1, 2, ..., 7$, represent the adjoint variables associated with the state variables of the model (7.2). The standard existence results for the minimizing control problem, as they appeared in [7] are adapted as follows.

Theorem 7.1. *There exists an optimal control* $\left(u_1^*, u_2^*\right) \in \mathcal{U}$ *satisfying* (7.4) *subject to the control system* (7.2) *with non-negative initial conditions given by* (2.8).

Proof: The existence of optimal control is obtained thanks to Fleming and Rishel [7]. Thanks to a result of Lukes's [10] which ensures the existence of solutions for the state system (7.2) with constant coefficients, the set of controls and corresponding solutions is non-empty. In addition, the set of controls $\mathcal{U}$ is a closed convex set by definition, and the vector field of the system (7.2) is bounded. Also, the integrand of the objective function is convex, and $g(t, X, c)$ in (7.3) is convex concerning c. On the other hand, there exist $a_1, a_2 > 0$ and $\beta > 1$ such that

$$w_1 B + w_2 T + w_3 P + \frac{1}{2}\left[w_4 u_1^2(t) + w_5 u_2^2(t)\right] \geq a_1 \left(|u_1|^2 + |u_2|^2\right)^{\frac{\beta}{2}} - a_2$$

since the state variables are bounded. Then, we deduce the existence of an optimal control (u_1^*, u_2^*) that minimizes the objective function $\mathscr{J}(u_1, u_2)$. $\square$

Theorem 7.2. *Given that* $\left(u_1^*, u_2^*\right)$ *minimizes the objective functional* (7.4) *subject to the corresponding state system* (7.2)*, then the adjoint variables* $\lambda_i, i = 1, 2, ..., 7$*, satisfy the following system:*

$$\begin{aligned}
\frac{d\lambda_1}{dt} &= (\lambda_1 - \lambda_4)\alpha_1 \frac{(T+B)I}{(C+I)^2} + (\lambda_1 - \lambda_5)c_1\alpha_2 \frac{BI}{(C+I)^2} + (\lambda_1 - \lambda_6)c_2\alpha_3 \frac{TI}{(C+I)^2} + (\lambda_1 - \lambda_3)\sigma_2 + \lambda_1\mu \\
\frac{d\lambda_2}{dt} &= (\lambda_2 - \lambda_5)c_1\omega_4 \frac{BI}{(R+I)^2} + (\lambda_2 - \lambda_6)c_2\omega_3 \frac{TI}{(R+I)^2} + (\lambda_2 - \lambda_1)\gamma_6 + \lambda_2\mu \\
\frac{d\lambda_3}{dt} &= (\lambda_3 - \lambda_1)\gamma_4 + (\lambda_4 - \lambda_1)\alpha_1 \frac{(T+B)C}{(C+I)^2} + (\lambda_5 - \lambda_1)c_1\alpha_2 \frac{BC}{(C+I)^2} + (\lambda_6 - \lambda_1)c_2\alpha_3 \frac{TC}{(C+I)^2} \\
&\quad + (\lambda_6 - \lambda_2)c_2\omega_3 \frac{TR}{(R+I)^2} + (\lambda_5 - \lambda_2)c_1\omega_4 \frac{BR}{(R+I)^2} + (\lambda_3 - \lambda_5)c_1\nu_3 \frac{B(V+T+B)}{I^2} \\
&\quad + (\lambda_3 - \lambda_6)c_2\omega_2 \frac{T(V+T+B)}{I^2} + \lambda_3\zeta_1 \frac{(T+B)(V+T+B)}{I^2} + \lambda_3\mu - \lambda_4\zeta_2 \frac{(T+B)V}{I^2} \\
&\quad + (\lambda_5 - \lambda_7)c_1\theta_2 \frac{PB}{(P+I)^2} + (\lambda_5 - \lambda_7)\tau_2 \frac{B(T+B)}{I^2} + (\lambda_6 - \lambda_5)c_2\omega_1 \frac{TB}{I^2} + (\lambda_3 - \lambda_2)\nu_2 \\
&\quad + \lambda_5\zeta_3 \frac{B(T+B)}{I^2} + (\lambda_6 - \lambda_7)\theta_1 \frac{TP}{(P+I)^2} + (\lambda_6 - \lambda_7)\tau_3 \frac{T(T+B)}{I^2} + \lambda_6\zeta_4 \frac{T(T+B)}{I^2}
\end{aligned} \tag{7.8}$$

$$\begin{aligned}
\frac{d\lambda_4}{dt} &= (\lambda_3-\lambda_1)\gamma_9 + (\lambda_4-\lambda_1)\alpha_1\frac{(T+B)C}{(C+I)^2} + (\lambda_5-\lambda_1)c_1\alpha_2\frac{BC}{(C+I)^2} + (\lambda_6-\lambda_1)c_2\alpha_3\frac{TC}{(C+I)^2}\\
&+(\lambda_6-\lambda_2)c_2\omega_3\frac{TR}{(R+I)^2} + (\lambda_5-\lambda_2)c_1\omega_4\frac{BR}{(R+I)^2} + (\lambda_5-\lambda_3)c_1\nu_3\frac{BA}{I^2} + (\lambda_4-\lambda_3)\sigma_1\\
&+(\lambda_6-\lambda_3)c_2\omega_2\frac{TA}{I^2} - \lambda_3\zeta_1\frac{(T+B)A}{I^2} + \lambda_4\mu + \lambda_4\zeta_2\frac{(T+B)(A+T+B)}{I^2}\\
&+(\lambda_5-\lambda_7)c_1\theta_2\frac{PB}{(P+I)^2} + (\lambda_5-\lambda_7)\tau_2\frac{B(T+B)}{I^2} + (\lambda_6-\lambda_5)c_2\omega_1\frac{TB}{I^2}\\
&+\lambda_5\zeta_3\frac{B(T+B)}{I^2} + (\lambda_6-\lambda_7)\theta_1\frac{TP}{(P+I)^2} + (\lambda_6-\lambda_7)\tau_3\frac{T(T+B)}{I^2} + \lambda_6\zeta_4\frac{T(T+B)}{I^2}
\end{aligned}$$

$$\begin{aligned}
\frac{d\lambda_5}{dt} &= -w_1 + (\lambda_5-\lambda_1)\gamma_7 + (\lambda_6-\lambda_2)c_2\omega_3\frac{TR}{(R+I)^2} + (\lambda_6-\lambda_1)c_2\alpha_3\frac{TC}{(C+I)^2} + (\lambda_5-\lambda_3)\nu_1\\
&+(\lambda_1-\lambda_4)\alpha_1\frac{C(C+A+V)}{(C+I)^2} + (\lambda_1-\lambda_5)c_1\alpha_2\frac{C(C+A+V+T)}{(C+I)^2} + (\lambda_2-\lambda_5)c_1\omega_4\frac{R(R+A+V+T)}{(R+I)^2}\\
&+\lambda_5\mu + (\lambda_3-\lambda_6)c_2\omega_2\frac{TA}{I^2} + (\lambda_3-\lambda_5)c_1\nu_3\frac{A(A+V+T)}{I^2} + \lambda_3\zeta_1\frac{A(A+V)}{I^2} + \lambda_4\zeta_2\frac{V(A+V)}{I^2}\\
&+(\lambda_5-\lambda_7)\tau_2\frac{(A+V)(A+V+T)}{I^2} + \lambda_5\zeta_3\frac{(A+V)(A+V+T)}{I^2} + (\lambda_5-\lambda_7)c_1\theta_2\frac{P(P+A+V+T)}{(P+I)^2}\\
&+(\lambda_6-\lambda_5)c_2\omega_1\frac{TB}{I^2} + (\lambda_6-\lambda_7)\theta_1\frac{TP}{(P+I)^2} + (\lambda_7-\lambda_6)\tau_3\frac{T(A+V)}{I^2} - \lambda_6\zeta_4\frac{T(A+V)}{I^2}\\
&+(\lambda_5-\lambda_6)c_2\omega_1\frac{T(A+V+T)}{I^2}
\end{aligned}$$

$$\begin{aligned}
\frac{d\lambda_6}{dt} &= -w_2 + (\lambda_6-\lambda_1)\gamma_8 + (\lambda_1-\lambda_4)\alpha_1\frac{C(C+A+V)}{(C+I)^2} + (\lambda_5-\lambda_1)c_1\alpha_2\frac{BC}{(C+I)^2} + (\lambda_5-\lambda_2)c_1\omega_4\frac{BR}{(R+I)^2}\\
&+(\lambda_2-\lambda_9)c_2\omega_3\frac{R(R+A+V+B)}{(R+I)^2} + (\lambda_5-\lambda_3)c_1\nu_3\frac{BA}{I^2} + (\lambda_3-\lambda_6)c_2\omega_2\frac{A(A+V+B)}{I^2} + \lambda_3\zeta_1\frac{A(A+V)}{I^2}\\
&+\lambda_4\zeta_2\frac{V(A+V)}{I^2} + (\lambda_5-\lambda_7)c_1\theta_2\frac{PB}{(P+I)^2} + (\lambda_5-\lambda_6)c_2\omega_1\frac{B(A+V+B)}{I^2} - \lambda_5\zeta_3\frac{B(A+V)}{I^2}\\
&+(\lambda_7-\lambda_5)\tau_2\frac{B(A+V)}{I^2} + (\lambda_7-\lambda_6)c_2\theta_1\frac{P(P+A+V+B)}{(P+I)^2} + (\lambda_6-\lambda_7)\tau_3\frac{(A+V)(A+V+B)}{I^2} + \lambda_6\mu\\
&+\lambda_6\zeta_4\frac{(A+V)(A+V+B)}{I^2} + (\lambda_1-\lambda_6)c_2\alpha_3\frac{C(C+A+V+B)}{(C+I)^2}
\end{aligned}$$

$$\frac{d\lambda_7}{dt} = -w_3 + (\lambda_7-\lambda_1)\gamma_5 + (\lambda_7-\lambda_5)c_1\theta_2\frac{BI}{(P+I)^2} + (\lambda_7-\lambda_6)c_2\theta_1\frac{TI}{(P+I)^2} + \lambda_7\mu + \lambda_7\eta$$

with transversality conditions

$$\lambda_i(T_f) = 0, \quad i = 1, 2, ..., 7.$$

Further, the optimal control $\left(u_1^*, u_2^*\right)$ is given as follows

$$\begin{cases} u_1^* = \max\left\{0, \min\left\{1, \dfrac{(\lambda_5-\lambda_1)\alpha_2\dfrac{BC}{C+I}+(\lambda_5-\lambda_2)\omega_4\dfrac{BR}{R+I}+(\lambda_5-\lambda_3)\nu_3\dfrac{BA}{I}+(\lambda_5-\lambda_7)\theta_2\dfrac{BP}{P+I}}{w_4}\right\}\right\} \\ \\ u_2^* = \max\left\{0, \min\left\{1, \dfrac{(\lambda_6-\lambda_1)\alpha_3\dfrac{TC}{C+I}+(\lambda_6-\lambda_2)\omega_3\dfrac{TR}{R+I}+(\lambda_6-\lambda_3)\omega_2\dfrac{TA}{I}+(\lambda_6-\lambda_5)\omega_1\dfrac{TB}{I}+(\lambda_6-\lambda_7)\theta_1\dfrac{TP}{P+I}}{w_5}\right\}\right\} \end{cases} \tag{7.9}$$

Proof:

As mentioned earlier, the characterization of the optimal solution is obtained by applying the Pontryagin's maximum principle to the Hamiltonian of the system $\mathcal{H}$. The system of ordinary differential equations (7.8) governing the adjoint variables is derived by differentiating the Hamiltonian. Further, the control characterizations in (7.9) are derived by solving, on the interior of the control set $\mathcal{U}$, the partial differentials of the Hamiltonian $\mathcal{H}$ with respect to each of the controls u_1 and u_2. Hence, by standard arguments involving control bounds, it follows that:

$$u_1^* = \begin{cases} 0 & \text{if } r_1^* \leq 0 \\ r_1^* & \text{if } 0 < r_1^* < 1 \\ 1 & \text{if } r_1^* \geq 1 \end{cases}$$

$$u_2^* = \begin{cases} 0 & \text{if } r_2^* \leq 0 \\ r_2^* & \text{if } 0 < r_2^* < 1 \\ 1 & \text{if } r_2^* \geq 1 \end{cases}$$

where,

$$\begin{cases} r_1^* = \dfrac{(\lambda_5-\lambda_1)\alpha_2\dfrac{BC}{C+I}+(\lambda_5-\lambda_2)\omega_4\dfrac{BR}{R+I}+(\lambda_5-\lambda_3)\nu_3\dfrac{BA}{I}+(\lambda_5-\lambda_7)\theta_2\dfrac{BP}{P+I}}{w_4} \\ \\ r_2^* = \dfrac{(\lambda_6-\lambda_1)\alpha_3\dfrac{TC}{C+I}+(\lambda_6-\lambda_2)\omega_3\dfrac{TR}{R+I}+(\lambda_6-\lambda_3)\omega_2\dfrac{TA}{I}+(\lambda_6-\lambda_5)\omega_1\dfrac{TB}{I}+(\lambda_6-\lambda_7)\theta_1\dfrac{TP}{P+I}}{w_5} \end{cases}$$

This puts an end to the proof. □

7.3. Numerical simulation

In this section, we use numerical simulation to illustrate the effect of control on the dynamics of the controlled compartments, in particular compartments B and T respectively. For reasons of clarity, the color red has been chosen for the curves of the classes with no control over the persistence parameters of brigandage and narcoterrorism, while the color blue has been chosen for the curves with control. Figure 6 shows that if u_1 control is very weak and u_2 control is effective, banditry persists and narcoterrorism stabilizes. Figure 7 shows that when u_1 control is effective and u_2 control is weak, banditry and narcoterrorism stabilize. Finally, there is a very quick stabilization in classes B and T when the u_1 and u_2 controls approach 1, as shown in Figure 8.

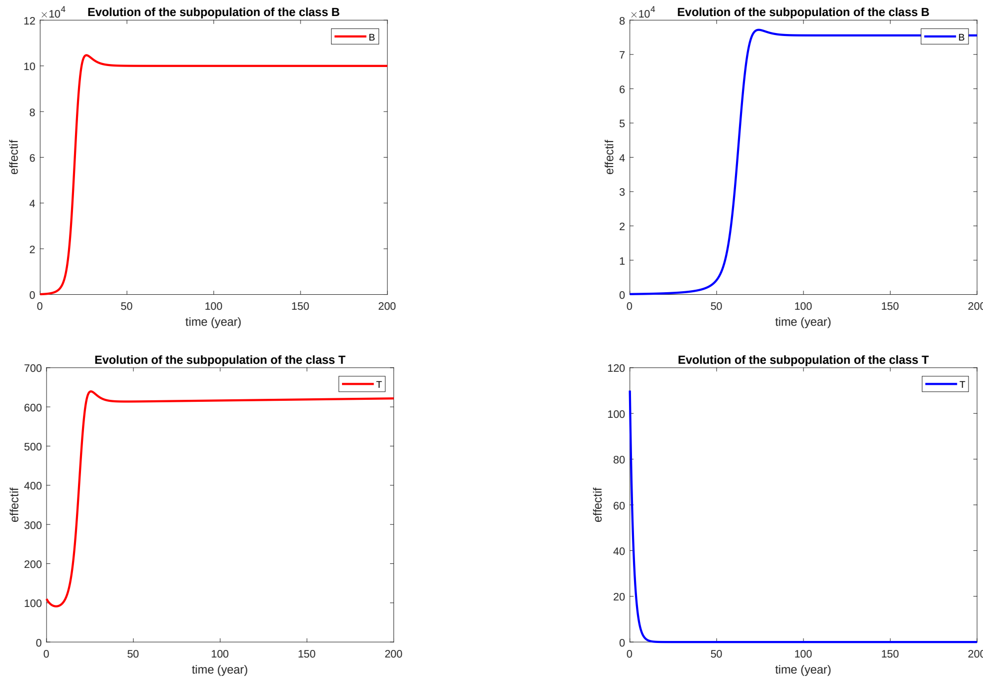

Figure 6: Dynamics of evolution of classes B and T illustrating the effect of control with $u_1 = 0.25$, and $u_2 = 0.75$

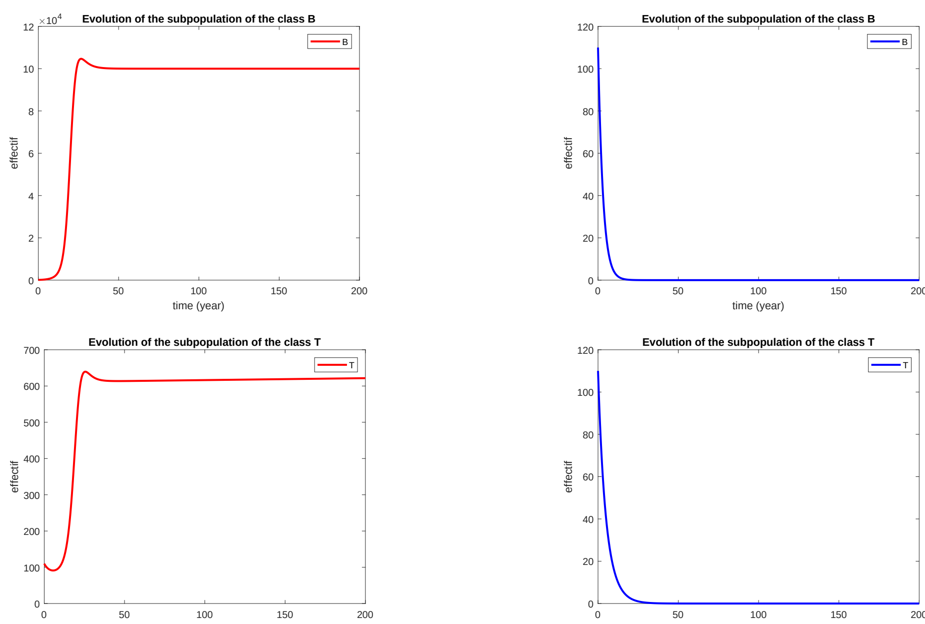

Figure 7: Dynamics of classes B and T illustrating the effect of control with $u_1 = 0.75$, and $u_2 = 0.25$

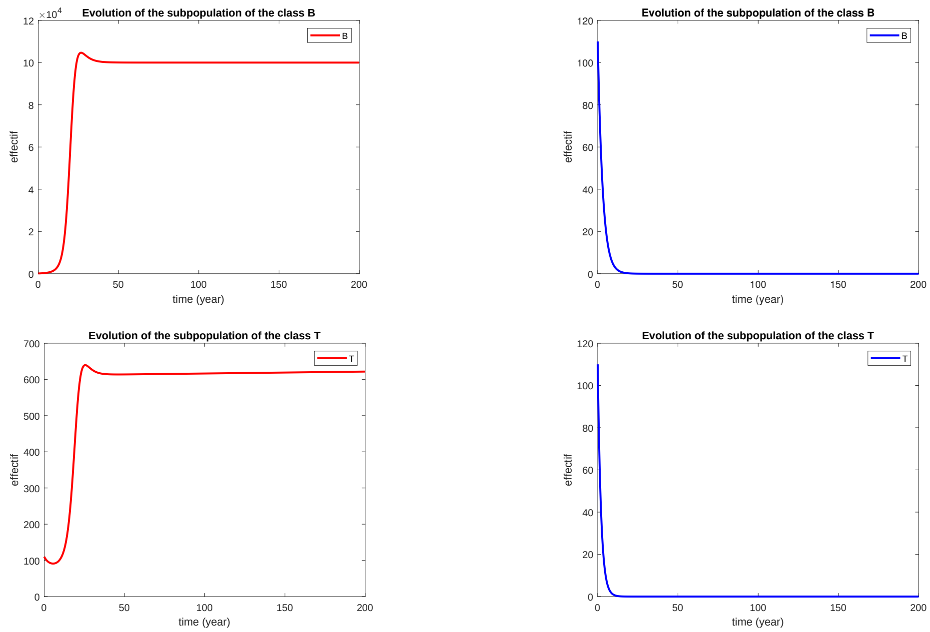

Figure 8: Dynamics of classes B and T illustrating the effect of control with $u_1 = 0.75$, and $u_2 = 0.75$

8. Conclusion

In this study, we first designed a mathematical model to illustrate the dynamics of narcoterrorism, based on the situation in certain Sahelian countries. The proposed mathematical model focused on the dynamics of recruitment into the narcoterrorist and brigand classes, showing the importance of contact and the deterrent presence of certain classes. We then carried out a rigorous mathematical analysis of the model. We then defined a first threshold $\mathcal{R}_0$ for this model, which designates the number of basic reproductions in the brigand or narcoterrorist class. In other words, the average number of people that a brigand or narcoterrorist manages to recruit into his class. From this threshold, we give asymptotic stability conditions for the equilibrium without brigands or terrorists. We also define two global thresholds, which are sufficient conditions for the eradication of narcoterrorism. Based on the results of the analysis, a strategy for combating narcoterrorism and banditry was proposed through a model check. The effectiveness of the strategy was then assessed using an optimality study based essentially on the Pontryagin maxima principle and Fleming's theorem. To make this study more readable, we carried out a numerical simulation of the analysis and control results. On the strength of some of the results of this study, we are convinced that to fight narcoterrorism and banditry more effectively, the Sahel and West African states must work to strengthen their systems of governance adapted to their realities. This strengthening of governance could be achieved through a better administrative and security network, as well as the development of local production activities and the promotion of local products. It is still time for the countries of the Sahel to take their destiny into their own hands. They will need to strengthen their cooperation on security, economic and social issues. There is still time for the Sahel countries to apply measures of good governance and virtuous governance adapted to their reality, all within a framework of faultless social cohesion and a local security system that is effective against violent extremism, narcoterrorism, and all forms of organized crime.

9. Acknowledgement

The authors express their gratitude to the World Bank and the Higher Education Support Project in Burkina Faso for their valuable support. They also thank all those whose suggestions have contributed to improving this document.

References

[1] A. BERMAN, R. J. PLEMMONS, *Nonnegative matrices in the mathematical science*, SIAM, (1994),

[2] J. BONY, PRINCIPE DU MAXIMUM, *inégalité de harnacket unicité du problème de cauchy pour les opérateurs elliptiques dégénérés*, Ann. Inst. Fourier (Grenoble), 19 (1969), pp. 277-304.

[3] R. BOYD AND P. RICHERSON, *Culture and the Evolutionary Process*, University of Chicago Press, Chicago, London, 1976.

[4] C. CASTILLO-CHAVEZ AND BAOJUN SONG, *Models for the transmission dynamics of fanatic behaviors*, in Bioterrorism: Mathematical Modeling Applications in Homeland Security" (eds. H. T. Banks and Carlos Castillo-Chavez), Frontiers in Applied Mathematics, Vol. 28, Society for Industrial and Applied Mathematics, Philadelphia, PA, (2003), 155.

[5] D. DENNETT, *Darwin's Dangerous Idea*, Simon and Schuster, New York, London, Toronto, Sydney, Tokyo, Singapore, 1995.

[6] M. W. FELDMAN AND L. L. CAVALLI-SFORZA, *On the theory of evolution under genetic and cultural transmission with application to the lactose absorption problem*, in Mathematical Evolutionary Theory, M. W. Feldman, ed., Princeton University Press, Princeton, NJ, 1989, pp. 145–173.

[7] W.H. FLEMING, R.W. RICHEL, *Deterministic and Stochastic Optimal Control*, Springer, New York, 1975

[8] K. P. HADELERAND, C.CASTILLO-CHAVEZ, *Acore group model for disease transmission*, J. Math. Biosci., 128 (1995), pp. 41–55.

[9] H. W. HETHCOTE AND J. A. YORKE, *Gonorrhea Transmission Dynamics and Control*, Lecture Notes in Biomath 56, Springer-Verlag, Berlin, 1984.

[10] D. L. LUKES, *Differential equations : classical to controlled / Dahlard L. Lukes,* Academic Press, New York: 1982.

[11] C. LUMSDEN AND E. O. WILSON, *Genes, Mind and Culture: The Co-evolutionary Process*, Havard University Press, Cambridge, MA, 1981.

[12] L.S. PONTRYAGIN, V.G. BOLTYANSKII, R.V. GAMKRELIDZE, E.F. MISHCHENKO, *The Mathematical Theory of Optimal Processes* (Wiley, New York, 1962)

[13] B. SONG, *Dynamical Epidemical Models and Their Applications*, Ph.D. thesis, Cornell University, Ithaca, NY, 2002.

[14] E.P. VALDIVIEZO, *Une brève étude du nombre de reproduction en épidémiologie et leurs applications*, Mémoire de Master de Mathématiques pour les Sciences du Vivant, Institut Polytechnique de Paris, 2 septembre 2020.

[15] R. VARGA, *Factorization and normalized iterative methods*, in Boundary problems in differential equations, R. Langer, ed., university of Wisconsin Press, 1960, pp. 121-142.

[16] R. VARGA, *Matrix iterative analysis*, Prentice-Hall, 1962.