Approximation results for PBVPs of nonlinear first order ordinary functional differential equations in a closed subset of the Banach space

JANHAVI B. DHAGE[1], **SHYAM B. DHAGE**[1], **BAPURAO C. DHAGE**[*1]

[1] *Kasubai, Gurukul Colony, Thodga Road, Ahmedpur - 413515, Dist. Latur, Maharashtra, India.*

This paper is dedicated to the occasion of Professor Gaston M. N'Guérékata's 70th birthday

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Abstract. In this paper we prove the approximation results for existence and uniqueness of the solution of PBVPs of nonlinear first order ordinary functional differential equations in a closed subset of the Banach space. We employ the Dhage monotone iteration method based on a recent hybrid fixed point theorem of Dhage (2022) and Dhage *et al.* (2022) for the main results of this paper. Finally an example is indicated to illustrate the abstract ideas involed in the approximation results.

Contents

1. Introduction

The study of periodic boundary value problems (in short PBVPs) and functional PBVPs of first order ordinary differential equations for existence and approximations using hybrid fixed point theory is initiated by Dhage and Dhage [9] and Dhage [5] respectively. Then after several results appeared in the literature for different types of hybrid PBVPs in the partially ordered Banach space. But to the knowledge of the present authors such results are not proved in the closed subsets of the Banach space. For details of functional differential equations and their importance, the readers are referred to Hale [15]. In this paper we prove the existence and approximation results for a PBVP more general than that studied in Dhage and Dhage [9] using the monotone iteration method of Dhage. This method relies on a recent hybrid fixed point theorem of Dhage *et al.* [12] in a partially ordered Banach space. Before stating the proposed PBVP, we give some preliminaries.

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*Corresponding author. Email address: **jbdhage@gmail.com** (Janhavi B. Dhage) **sbdhage4791@gmail.com** (Shyam B. Dhage) **bcdhage@gmail.com** (Bapurao C. Dhage)

Given the real numbers $r > 0$ and $T > 0$, consider the closed and bounded intervals $I_0 = [-r, 0]$ and $I = [0, T]$ in $\mathbb{R}$ and let $J = [-r, T]$. By $\mathcal{C} = C(I_0, \mathbb{R})$ we denote the space of continuous real-valued functions defined on I_0 with he norm $\| \cdot \|_\mathcal{C}$ defined by

$$\|x\|_\mathcal{C} = \sup_{-r \leq \theta \leq 0} |x(\theta)|. \tag{1.1}$$

The Banach space $\mathcal{C}$ with this supremum norm is called the history space of the functional differential equation in question. For any continuous function $x : J \to \mathbb{R}$ and for any $t \in I$, we denote by x_t the element of the space $\mathcal{C}$ defined by

$$x_t(\theta) = x(t + \theta), \quad -r \leq \theta \leq 0. \tag{1.2}$$

Now, given a history function $\phi \in \mathcal{C}$, we consider the PBVP of nonlinear first order ordinary functional differential equations (in short functional PBVP),

$$\left. \begin{array}{r} x'(t) + h(t)x(t) = f(t, x(t), x_t), \text{ a. e. } t \in I, \\ x(0) = \phi(0) = x(T), \\ x_0 = \phi, \end{array} \right\} \tag{1.3}$$

where $h : I \to \mathbb{R}$ and $f : I \times \mathbb{R} \times \mathcal{C}$ are continuous functions.

Definition 1.1. *A function $x \in AC(J, \mathbb{R})$ is said to be a solution of the functional PBVP* (1.3) *if*

(i) $x_0 = \phi$,

(ii) $x_t \in \mathcal{C}$ *for each $t \in I$, and*

(iii) x *satisfies the equations in* (1.3) *on J,*

where $AC(J, \mathbb{R})$ is the space of absolutely continuous real-valued functions defined on J.

In this paper we obtain the existence and approximation theorem for the functional PBVP (1.3) in a closed subset of the relevant function space. The rest of the paper is organized as follows. Below in Section 2, we give the auxiliary results needed later in the subsequent part of the paper. The main existence and uniqueness theorems are proved in Section 3 and a couple of illustrative examples are presented in Section 4.

2. Auxiliary Results

First we convert the functional PBVP (1.3) into an equivalent integral equation, because the integrals are easier to handle than differentials. We need the following result similar to Nieto [16, 17] and Dhage [2] which can proved by using the theory of calculus.

Lemma 2.1. *For any $h \in L^1(J, \mathbb{R}^+)$ and $\sigma \in L^1(J, \mathbb{R})$, x is a solution to the differential equation*

$$\left. \begin{array}{r} x' + h(t)x(t) = \sigma(t) \text{ a. e. } t \in I, \\ x(0) = \phi(0) = x(T), \\ x_0 = \phi, \end{array} \right\} \tag{2.1}$$

if and only if it is a solution of the integral equation

$$\left. \begin{array}{r} x(t) = \displaystyle\int_0^T G_h(t, s)\sigma(s)\, ds, \quad t \in I, \\ x_0 = \phi, \end{array} \right\} \tag{2.2}$$

where,

$$G_h(t,s) = \begin{cases} \dfrac{e^{H(s)-H(t)}}{1-e^{-H(T)}}, & 0 \le s \le t \le T, \\[2ex] \dfrac{e^{H(s)-H(t)-H(T)}}{1-e^{-H(T)}}, & 0 \le t < s \le T, \end{cases} \tag{2.3}$$

and $H(t) = \displaystyle\int_0^t h(s)\,ds.$

Notice that the Green's function G_h is nonnegative on $J \times J$ and the number

$$M_h := \max\{\,|G_h(t,s)| \,:\, t,s \in [0,T]\,\},$$

exists for all $L^1(J, \mathbb{R}^+)$. Note also that $H(t) > 0$ for all $t > 0$.

We need the following definition in the sequel.

Definition 2.2. *A mapping $\beta : I \times \mathbb{R} \times \mathcal{C} \to \mathbb{R}$ is said to be **Carathéodory** if*

(i) $t \mapsto \beta(t,x,y)$ is measurable for each $x \in \mathbb{R}$, $y \in \mathcal{C}$, and

(ii) $(x,y) \mapsto \beta(t,x,y)$ is jointly continuous almost everywhere for $t \in I$.

Again a Carathéodory function $\beta(t,x,y)$ is called L^1-Carathéodory if

(iii) for each real number $r > 0$ there exists a function $m_r \in L^1(J, \mathbb{R})$ such that

$$|\beta(t,x,y)| \le m_r(t) \quad a.e. \ \ t \in J,$$

for all $x \in \mathbb{R}$ and $y \in \mathcal{C}$ with $|x| \le r$ and $\|y\|_{\mathcal{C}} \le r$.

The following lemma is proved using the arguments similar to that given in Dhage and Dhage [9]. See also Dhage [2, 5] and references therein.

Lemma 2.3. *Suppose that there exists a function $u \in AC(J, \mathbb{R})$ such that*

$$\left.\begin{aligned} u'(t) + h(t)u(t) &\le f(t,u(t),u_t) \quad a.e. \ \ t \in I, \\ u(0) = \phi(0) &\le u(T), \\ u_0 &\le \phi. \end{aligned}\right\} \tag{2.4}$$

Then,

$$\left.\begin{aligned} u(t) &\le \int_0^T G_h(t,s)\sigma(s)\,ds, \ \ t \in I, \\ u_0 &\le \phi. \end{aligned}\right\} \tag{2.5}$$

Similarly, if there exists a function $v \in AC(J, \mathbb{R})$ such that the inequalities in (2.4) are satisfied with reverse sign, then the inequalities in (2.5) hold with reverse sign.

It is well-known that the fixed point theoretic technique is very much useful in the subject of nonlinear analysis for dealing with the nonlinear equations. See Granas and Dugundji [14], Zeidler [18] and the references therein. Here, we employ the Dhage monotone iteration method based on the following two hybrid fixed point theorems of Dhage [8] and Dhage *et al.* [12].

Theorem 2.4 (Dhage [8])**.** *Let S be a non-empty partially compact subset of a regular partially ordered Banach space $\left(E, \| \cdot \|, \preceq, \right)$ with every chain C in S is Janhavi set and let $\mathcal{T} \; : \; S \to S$ be a monotone nondecreasing, partially continuous mapping. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then the hybrid mapping equation $\mathcal{T}x = x$ has a solution ξ^* in S and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to ξ^*.*

Theorem 2.5 (Dhage [8])**.** *Let S be a non-empty partially closed subset of a regular partially ordered Banach space $\left(E, \|\cdot\|, \preceq \right)$ and let $\mathcal{T} \; : \; S \to S$ be a monotone nondecreasing nonlinear partial contraction. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then the hybrid mapping equation $\mathcal{T}x = x$ has a unique comparable solution ξ^* in S and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to ξ^*. Moreover, ξ^* is unique provided every pair of elements in E has a lower bound or an upper bound.*

Remark 2.6. *We note that every every pair of elements in a partially ordered set (**poset**) $(E, \preceq)$ has a lower or upper bound if $(E, \preceq)$ is a lattice, that is, $\preceq$ is a lattice order in E. In this case the poset $(E, \| \cdot \|, \preceq)$ is called a **partially lattice ordered Banach space**. There do exist several lattice partially ordered Banach spaces which are useful for applications in nonlinear analysis. For example, every Banach lattice is a partially lattice ordered Banach space. The details of the lattice structure of the Banach spaces appear in Birkhoff [1].*

As a consequence of Remark 2.6, we obtain

Theorem 2.7 (Dhage [8])**.** *Let S be a non-empty partially closed subset of a regular partially lattice ordered Banach space $\left(E, \| \cdot \|, \preceq \right)$ and let $\mathcal{T} \; : \; S \to S$ be a monotone nondecreasing nonlinear partial contraction. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then the hybrid mapping equation $\mathcal{T}x = x$ has a unique solution ξ^* in S and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to ξ^*.*

If a Banach X is partially ordered by an order cone K in X, then in this case we simply say X is an **ordered Banach space** which we denote it by (X, K). Similarly, an ordered Banach space (X, K), where partial order $\preceq$ defined by the con K is a lattice order, then (X, K) is called the **lattice ordered Banach space**. Clearly, an ordered Banach space $\left(C(J, \mathbb{R}), K\right)$ of continuous real-valued functions defined on the closed and bounded interval J is lattice ordered Banach space, where the cone K is given by $K = \{x \in CJ, \mathbb{R}) \mid x \succeq 0\}$. The details of the cones and their properties appear in Guo and Lakshmikantham [13]. Then, we have the following useful results concerning the ordered Banach spaces proved in Dhage [7, 8].

Lemma 2.8 (Dhage [7, 8])**.** *Every ordered Banach space (X, K) is regular.*

Lemma 2.9 (Dhage [7, 8])**.** *Every partially compact subset S of an ordered Banach space (X, K) is a Janhavi set in X.*

As a consequence of Lemmas 2.8 and 2.9 we obtain the following hybrid fixed point theorem which we need in what follows.

Theorem 2.10 (Dhage [8] and Dhage et al. [12])**.** *Let S be a non-empty partially compact subset of an ordered Banach space (X, K) and let $\mathcal{T} \; : \; S \to S$ be a partially continuous and monotone nondecreasing operator. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then the hybrid operator equation $\mathcal{T}x = x$ has a solution ξ^* in S and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to ξ^*.*

Theorem 2.11 (Dhage [8] and Dhage et al. [12])**.** *Let S be a non-empty partially closed subset of a lattice ordered complete normed linear space (X, K) and let $\mathcal{T} \; : \; S \to S$ be a monotone nondecreasing partial contraction. If there exists an element $x_0 \in S$ such that $x_0 \preceq \mathcal{T}x_0$ or $x_0 \succeq \mathcal{T}x_0$, then the hybrid operator equation $\mathcal{T}x = x$ has a unique solution ξ^* in S and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^{\infty}$ of successive iterations converges monotonically to ξ^*.*

The details of the notions of partial order, Janhavi set, regularity, monotonicity, partial continuity, partial closure, partial compactness and nonlinear partial contraction along with their applications may be found in Guo and Lakshmikatham [13], Dhage [3, 6, 7], Dhage and Dhage [9, 10] and references therein.

3. Existence and Approximation Results

We place the nonlinear integral equation corresponding to the PBVP (1.3) in the Banach space $C(J, \mathbb{R})$ equipped with the norm $\| \cdot \|$ and the order relation $\preceq$ defined by

$$\|x\| = \sup_{t \in J} |x(t)|, \tag{3.1}$$

and

$$x \preceq y \iff y - x \in K, \tag{3.2}$$

where K is a cone in $C(J, \mathbb{R})$ given by

$$K = \{x \in C(J, \mathbb{R}) \mid x(t) \geq 0 \ \forall \, t \in J\}. \tag{3.3}$$

It is known that the partially ordered Banach space $C(J, \mathbb{R})$ is regular and lattice with respect to the *meet* and *join* lattice the operations $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$. Therefore, every pair of elements of $C(J, \mathbb{R})$ has a lower and an upper bound. See Dhage [6, 7] and the references therein. The following useful lemma concerning the partial compactness of the subsets of of $C(J, \mathbb{R})$ follows easily and is often times used in the theory of nonlinear differential and integral equations.

Lemma 3.1. *Let $\big(C(J, \mathbb{R}), K\big)$ be a partially ordered Banach space with the norm $\| \cdot \|$ and the order relation $\preceq$ defined by (3.1) and (3.2) respectively. Then every compact subset S of $C(J, \mathbb{R})$ is partially compact, but the converse may not be true.*

We introduce an order relation $\preceq_{\mathcal{C}}$ in $\mathcal{C}$ induced by the order relation $\preceq$ defined in $C(J, \mathbb{R})$. Thus, for any $x, y \in \mathcal{C}$, $x \preceq_{\mathcal{C}} y$ implies $x(\theta) \leq y(\theta)$ for all $\theta \in I_0$. Note that if $x, y \in C(J, \mathbb{R})$ and $x \preceq y$, then $x_t \preceq_{\mathcal{C}} y_t$ for all $t \in I$ (Cf. Dhage [4, 5]).

Let $C_{eq}(J, \mathbb{R})$ denote the subset of all equicontinuous functions in $C(J, \mathbb{R})$. Then for a constant $M > 0$, by $C_{eq}^M(J, \mathbb{R})$ we denote the class of equicontinuous functions in $C(J, \mathbb{R})$ defined by

$$C_{eq}^M(J, \mathbb{R}) = \{x \in C_{eq}(J, \mathbb{R}) \mid \|x\| \leq M\}.$$

Clearly, $C_{eq}^M(J, \mathbb{R})$ is a closed and uniformly bounded subset of the set of equicontinuous functions of the Banach space $C(J, \mathbb{R})$ which is compact in view of Arzelá-Ascoli theorem.

We need the following definition in what follows.

Definition 3.2. *A function $u \in C_{eq}^M(J, \mathbb{R})$ is said to be a* lower solution *of the PBVP (1.3) if the conditions (i) and (ii) of Definition 1.1 hold and u satisfies the inequalities*

$$\left.\begin{array}{r} u'(t) + h(t)u(t) \leq f(t, u(t), u_t) \ \ a.e. \ \ t \in I, \\ u(0) = \phi(0) \leq u(T), \\ u_0 \leq \phi. \end{array}\right\} \tag{3.4}$$

Similarly, a function $v \in C_{eq}^M(J, \mathbb{R})$ is called an upper solution of the functional PBVP (1.3) if the above inequality is satisfied with reverse sign. By a solution *of the PBVP (1.3) in a subset $C_{eq}^M(J, \mathbb{R})$ of the Banach space $C(J, \mathbb{R})$ we mean a function $x \in C_{eq}^M(J, \mathbb{R})$ which is both lower and upper solution of the functional PBVP (1.3) defined on J.*

We consider the following set of hypotheses in what follows:

(H_1) There exist constants $\ell_1 > 0$, $\ell_2 > 0$ such that

$$0 \le f(t, x_1, x_2) - f(t, y_1, y_2) \le \ell_1(x_1 - y_2) + \ell_2 \|x_2 - y_2\|_{\mathcal{C}},$$

for all $t \in J$, where $x_1, y_1 \in \mathbb{R}$ and $x_2, y_2 \in \mathcal{C}$ with $x_1 \ge y_1$, $x_2 \succeq_{\mathcal{C}} y_2$.

(H_2) The function f is L^1-Carathéodory on $I \times \mathbb{R} \times \mathcal{C} \to \mathbb{R}$.

(H_3) $f(t, x, y)$ is monotone nondecreasing in x and y for each $t \in I$.

(H_4) The functional PBVP (1.3) has a lower solution $u \in C_{eq}^M(J, \mathbb{R})$.

(H_5) The functional PBVP (1.3) has an upper solution $v \in C_{eq}^M(J, \mathbb{R})$.

Theorem 3.3. *Suppose that hypotheses (H_2) through (H_4) hold. Furthermore, if the inequality*

$$\|\phi\|_{\mathcal{C}} + M_h \|m_M\|_{L^1} \le M, \tag{3.5}$$

holds, then the PBVP (1.3) has a solution x^* defined on J and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by

$$
\left.
\begin{aligned}
x_0(t) &= u(t), \quad t \in J, \\
x_{n+1}(t) &=
\begin{cases}
\displaystyle\int_0^T G_h(t, s) f(s, x_n(s), x_s^n)\, ds, \quad t \in I, \\
\phi(t), \quad t \in I_0,
\end{cases}
\end{aligned}
\right\}
\tag{3.6}
$$

where $x_s^n(\theta) = x_n(s + \theta)$, $\theta \in I_0$, is monotone nondecreasing and converges to x^*.

Proof. Set $S = C_{eq}^M(J, \mathbb{R})$. Then, S is a uniformly bounded and equicontinuous subset of the ordered Banach space (X, K). Hence S is compact in view of Arzellá-Ascoli theorem. Consequently, S is partially compact subset of (X, K). Define an operator $\mathcal{T} : S \to C(J, \mathbb{R})$ by

$$
\mathcal{T}x(t) =
\begin{cases}
\displaystyle\int_0^T G_h(t, s) f(s, x(s), x_s)\, ds, \quad t \in I, \\
\phi(t), \quad t \in I_0.
\end{cases}
\tag{3.7}
$$

We shall show that the operator $\mathcal{T}$ satisfies all the conditions of Theorem 2.7 in a series of following steps.

Step I: *$\mathcal{T}$ is well defined and $\mathcal{T} : S \to S$.*

Clearly, $\mathcal{T}$ is well defined in view of continuity of the functions k and f on $J \times J$ and $J \times \mathbb{R} \times \mathbb{R}$ receptively. We show that $\mathcal{T}(S) \subset S$. Let $x \in S$ be arbitrary. Now by hypothesis (H_2),

$$
\begin{aligned}
|\mathcal{T}x(t)| &\le
\begin{cases}
\displaystyle\int_0^T G_h(t, s)|f(s, x(s), x_s)|\, ds, \quad t \in I, \\
|\phi(t)|, \quad t \in I_0,
\end{cases} \\
&\le
\begin{cases}
\displaystyle\int_0^T M_h\, m_M(s)\, ds, \quad t \in I, \\
|\phi(t)|, \quad t \in I_0,
\end{cases} \\
&\le \|\phi\|_{\mathcal{C}} + M_h \|m_M\|_{L^1} \\
&= M,
\end{aligned}
$$

for all $t \in J$. Taking the supremum over t, we obtain $\|\mathcal{T}x\| \leq M$ for all $x \in C_{eq}^M(J, \mathbb{R})$.

Next, we prove that $\mathcal{T}(S) \subset S$. Let $y \in \mathcal{T}(S)$ be arbitrary. Then there is an $x \in S$ such that $y = \mathcal{T}x$. Now we consider the following three cases:

Case I : Suppose that $t_1, t_2 \in I$. Then, we have

$$|y(t_1) - y(t_2)| = \left|\mathcal{T}x(t_1) - \mathcal{T}x(t_2)\right|$$

$$\leq \int_0^T \left|G_h(t_1, s) - G_h(t_2, s)\right| |f(s, x(s), x_s)| \, ds$$

$$\leq \left[(\ell_1 + \ell_2)L + F_0\right] \int_0^T |G_h(t_1, s) - G_h(t_2, s)| \, ds. \tag{*}$$

Since k is continuous on compact $J \times J$, it is uniformly continuous there. Therefore, for each fixed $s \in J$, we have

$$|k(t_1, s) - k(t_2, s)| \to 0 \quad \text{as} \quad t_1 \to t_2$$

uniformly. This further in view of inequality (*) implies that

$$\left|\mathcal{T}x(t_1) - \mathcal{T}x(t_2)\right| \to 0 \quad \text{as} \quad t_1 \to t_2, \tag{i}$$

uniformly for all $x \in S$.

Case II : Suppose that $t_1, t_2 \in I_0$. Then, we have

$$|y(t_1) - y(t_2)| = \left|\mathcal{T}x(t_1) - \mathcal{T}x(t_2)\right| = |\phi(t_1) - \phi(t_2)| \to 0 \quad \text{as} \quad t_1 \to t_2,$$

uniformly for $x \in S$.

Case III : Let $t_1 \in I_)$ and $t_2 \in I$. Then we obtain

$$|y(t_1) - y(t_2)| = \left|\mathcal{T}x(t_1) - \mathcal{T}x(t_2)\right| \leq \left|\mathcal{T}x(t_1) - \mathcal{T}x(0)\right| + \left|\mathcal{T}x(0) - \mathcal{T}x(t_2)\right|.$$

If $t_1 \to t_2$, that is, $|t_1 - t_2| \to 0$, then $t_1 \to 0$ and $t_2 \to 0$ which in view of inequalities (i) and (ii) implies that

$$|y(t_1) - y(t_2)| \to 0 \quad \text{as} \quad t_1 \to t_2 \tag{iii}$$

uniformly for all $y \in \mathcal{T}(S)$. From above three cases (i)-(iii) it follows that $\mathcal{T}x \in S$ for all $x \in S$. As a result $\mathcal{T}(S) \subseteq S$.

Step II: $\mathcal{T}$ *is a monotone nondereasing operator on* S.

Let $x, y \in S$ be such that $x \succeq y$. Then, $x_t \succeq y_t$ for each $t \in I$. Therefore, by hypothesis (H$_2$), we get

$$\mathcal{T}x(t) = \begin{cases} \displaystyle\int_0^T G_h(t, s) f(s, x(s), x_s) \, ds, \, , & t \in I, \\ \phi(t), & t \in I_0, \end{cases}$$

$$\geq \begin{cases} \displaystyle\int_0^T G_h(t, s) f(s, y(s), y_s) \, ds, \, , & t \in I, \\ \phi(t), & t \in I_0, \end{cases}$$

$$= \mathcal{T}y(t),$$

for all $t \in J$. This shows that $\mathcal{T}x \succeq \mathcal{T}y$ and consequently the operator $\mathcal{T}$ is monotone nondecreasing on S.

Step III: $\mathcal{T}$ *is partially continuous on* S.

Let C be a chain in the closed and bounded subset $C_{eq}^M(J, \mathbb{R})$ of the ordered Banach space $\big(C(J, \mathbb{R}), K\big)$ and let $\{x_n\}$ be a sequence of points in C such that $x_n \to x$ as $n \to \infty$. Then, by definition of the operator $\mathcal{T}$, we obtain

$$\lim_{n \to \infty} \mathcal{T} x_n = \lim_{n \to \infty} \begin{cases} \displaystyle\int_0^T G_h(t,s) f(s, x_n(s), x_s^n) \, ds, \, , \ t \in I, \\ \phi(t), \ \ t \in I_0, \end{cases}$$

$$= \begin{cases} \displaystyle\int_0^T G_h(t,s) \Big[\lim_{n \to \infty} f(s, x_n(s), x_s^n) \Big] \, ds, \, , \ t \in I, \\ \phi(t), \ \ t \in I_0, \end{cases}$$

$$= \begin{cases} \displaystyle\int_0^T G_h(t,s) f(s, x(s), x_s) \, ds, \, , \ t \in I, \\ \phi(t), \ \ t \in I_0, \end{cases}$$

$$= \mathcal{T} x(t),$$

for all $t \in J$. This shows that $\mathcal{T} x_n \to \mathcal{T} x$ pointwise on J. Next, by following the arguments asin Step II, it is proved that $\{\mathcal{T} x_n\}$ is an equicontinuous sequence of points in S. This shows that $\mathcal{T} x_n \to \mathcal{T} x$ uniformly on J. Consequently $\mathcal{T}$ is a partially continuous operator on S into itself.

Thus $\mathcal{T}$ satisfies all the conditions of Theorem 2.7 on a partially compact subset S of the Banach space $C(J, \mathbb{R})$. Hence $\mathcal{T}$ has a fixed point $x^* \in S$ and the sequence $\{\mathcal{T}^n x_0\}_{n=0}^\infty$ of successive iterations converges monotone nondecreasingly to x^*. This further implies that the PBVP (1.3) has a solution x^* on J and the sequence $\{x_n\}_{n=0}^\infty$ successive approximations defined by (3.6) converges monotone nondecreasingly to x^*. This completes the proof. $\qquad \square$

Theorem 3.4. *Suppose that the hypotheses (H_1) and (H_4) hold. Furthermore, if $M_h T(\ell_1 + \ell_2) < 1$, then the PBVP (1.3) has a unique solution x^* defined on J and the sequence $\{x_n\}_{n=0}^\infty$ of successive approximations defined by (3.6) is monotone nondecreasing and converges to x^*.*

Proof. Set $S = C_{eq}^M(J, \mathbb{R})$. Then $S = C_{eq}^M(J, \mathbb{R})$ is a closed subset of an ordered Banach space (X, K) and soo it is partially closed set in (X, K). Define an operator $\mathcal{T}$ on S by (3.7). Then $\mathcal{T}$ is well defined. We shall show that $\mathcal{T}$ is a partial contraction on S.

Let $x, y \in S$ be such that $x \succeq y$. Then, by hypothesis (H_1), we have

$$\big| \mathcal{T} x(t) - \mathcal{T} y(t) \big| = \left| \int_0^T G_h(t,s) \big[f(s, x(s), x_s) - f(s, y(s), y_s) \big] \, ds \right|$$

$$\leq \int_0^T G_h(t,s) \big| f(s, x(s), x_s) - f(s, y(s), y_s) \big| \, ds$$

$$\leq \int_0^T G_h(t,s) \big[f(s, x(s), x_s) - f(s, y(s), y_s) \big] \, ds$$

$$\leq \int_0^T G_h(t,s) \big[\ell_1 | x(s) - y(s)) | + \ell_2 \| x_s - y_s \|_{\mathcal{C}} \big] \, ds$$

$$\leq \int_0^T G_h(t,s) (\ell_1 + \ell_2) \| x - y \| \, ds$$

$$\leq M_h T(\ell_1 + \ell_2) \| x - y \|,$$

for all $t \in J$. Taking the supremum over t, we obtain

$$\| \mathcal{T} x - \mathcal{T} y \| \leq M_h T(\ell_1 + \ell_2) \| x - y \|$$

for all comparable elements $x, y \in S$. This shows that $\mathcal{T}$ is a partial contraction on S. We know that every partially Lipschitz operator is partially continuous, so $\mathcal{T}$ is a partially continuous operator on S. Now, we apply Theorem **??** to the operator $\mathcal{T}$ and conclude that $\mathcal{T}$ has a unique fixed point $x^* \in S$ and the sequence $\{\mathcal{T}^n u\}_{n=0}^{\infty}$ of successive iterations converges to x^*. This further implies that functional PBVP (1.3) has a unique solution x^* and the sequence $\{x_n\}_{n=0}^{\infty}$ of successive approximations defined by (3.6) is monotone nondecreasing and converges to x^*. □

Remark 3.5. *The conclusion of existence and uniqueness theorems, Theorems 3.3 and 3.4 for the problem* (1.3) *also remains true if we replace the hypothesis (H_4) by (H_5). In this case the sequence $\{y_n\}_{n=0}^{\infty}$ defined similar to* (3.6) *converges monotone nonincreasingly to the solution x^* of the functional PBVP* (1.3) *defined on J.*

4. An Example

Example 4.1. *Let $I_0 = \left[-\dfrac{\pi}{2}, 0\right]$ and $I == [0, 1]$ be two closed and bounded intervals in $\mathbb{R}$, the set of real number and let $J = \left[-\dfrac{\pi}{2}, 0\right] \bigcup [0, 1] = \left[-\dfrac{\pi}{2}, 1\right]$. Given a history function $\phi(t) = \sin t$, $t \in \left[-\dfrac{\pi}{2}, 0\right]$, consider the nonlinear two point functional BVP*

$$
\left.
\begin{aligned}
x'(t) + x(t) &= f_1(t, x(t), x_t), \quad t \in [0, 1], \\
x(0) &= \phi(0) = 0 = x(1), \\
x_0 &= \phi,
\end{aligned}
\right\}
\tag{4.1}
$$

for all $t \in [0, 1]$, where $x_t(\theta) = x(t + \theta)$, $\theta \in \left[-\frac{\pi}{2}, 0\right]$ and the function f_2 is given by

$$
f_1(t, x, y) =
\begin{cases}
0, & \text{if } x \leq 0, \ y \preceq_{\mathcal{C}} 0, \\[2mm]
\dfrac{1}{4} \dfrac{x}{1 + x}, & \text{if } x > 0, \ y \preceq_{\mathcal{C}} 0, \\[3mm]
\dfrac{1}{4} \dfrac{\|y\|_{\mathcal{C}}}{1 + \|y\|_{\mathcal{C}}}, & \text{if } x \leq 0, \ y \succeq_{\mathcal{C}} 0, \ y \neq 0, \\[3mm]
\dfrac{1}{4} \left[\dfrac{x}{1 + x} + \dfrac{\|y\|_{\mathcal{C}}}{1 + \|y\|_{\mathcal{C}}} \right], & \text{if } x > 0, \ y \succeq_{\mathcal{C}} 0, \ y \neq 0,
\end{cases}
$$

for all $t \in [0, 1]$.

Here, $h = 1 = T$ and f_1 defines a continuous function $f : [0, 1] \times \mathbb{R} \times \mathcal{C} \to \mathbb{R}$. We shall show that f_1 satisfies all the conditions of Theorem 3.4. Now, let $x_1, y_1 \in \mathbb{R}$ and $x_2, y_2 \in \mathcal{C}$ be such that $x_1 \geq y_1 \geq 0$ and $x_2 \succeq_{\mathcal{C}} y_2 \succeq 0$. Therefore, we have

$$
\begin{aligned}
0 \leq{}& f(t, x_1, x_2) - f(t, y_1, y_2) \\
\leq{}& \frac{1}{4} \cdot \frac{x_1}{1 + x_1} - \frac{y_1}{1 + y_1} + \frac{\|x_2\|_{\mathcal{C}}}{1 + \|x_2\|_{\mathcal{C}}} - \frac{\|y_2\|_{\mathcal{C}}}{1 + \|y_2\|_{\mathcal{C}}} \\
\leq{}& \frac{1}{4} \cdot \frac{x_1 - y_1}{1 + x_1 - y_1} + \frac{1}{4} \cdot \frac{\|x_2\|_{\mathcal{C}} - \|y_2\|_{\mathcal{C}}}{1 + \|x_2\|_{\mathcal{C}} - \|y_2\|_{\mathcal{C}}} \qquad (\because |x_1 - y_1| \leq |x_1| + |y_1|) \\
\leq{}& \frac{1}{4} \cdot \frac{|x_1 - y_1|}{1 + |x_1 - y_1|} + \frac{1}{4} \cdot \frac{\big| \|x_2\|_{\mathcal{C}} - \|y_2\|_{\mathcal{C}} \big|}{1 + \big| \|x_2\|_{\mathcal{C}} - \|y_2\|_{\mathcal{C}} \big|} \\
\leq{}& \frac{1}{4} \cdot \frac{|x_1 - y_1|}{1 + |x_1 - y_1|} + \frac{1}{4} \cdot \frac{\|x_2 - y_2\|_{\mathcal{C}}}{1 + \|x_2 - y_2\|_{\mathcal{C}}} \\
\leq{}& \frac{1}{4} \cdot |x_1 - y_1| + \frac{1}{4} \cdot \|x_2 - y_2\|_{\mathcal{C}},
\end{aligned}
$$

for all $t \in [0, 1]$. Similarly, we get the same estimate for other values of the function f_1. So the hypothesis (H_1) holds with $\ell_1 = \frac{1}{4}$ and $\ell_2 = \frac{1}{4}$. Again, the Green's function G is continuous and nonnegative on $[0, 1] \times [0, 1]$ with bound $M_1 \approx 1.6$, so that the hypothesis (H_3) holds. Moreover, here we have

$$M_h \, T(\ell_1 + \ell_2) \cong 1.6 \left(\frac{1}{4} + \frac{1}{4} \right) = 0.8 < 1,$$

and so all the conditions of Theorem 3.4 are satisfied for $M = \frac{9}{5}$. Finally, the functions u and v defined by

$$u(t) = \begin{cases} -\displaystyle\int_0^1 G_1(t,s)\,ds, & t \in [0,1], \\[2mm] \sin t, & t \in \left[-\frac{\pi}{2}, 0 \right], \end{cases}$$

and

$$v(t) = \begin{cases} \dfrac{1}{2} \displaystyle\int_0^1 G_1(t,s)\,ds, & t \in [0,1], \\[2mm] \sin t, & t \in \left[-\frac{\pi}{2}, 0 \right], \end{cases}$$

satisfy respectively the inequalities of the lower solution and upper solution of the functional PBVP (4.1) with $u \preceq v$ on J. Hence the functional PBVP (4.1) has a unique solution $x^* \in C_{eq}^{9/5}(J, \mathbb{R})$ defined on $J = \left[-\frac{\pi}{2}, 1 \right]$. Moreover, the sequence $\{x_n\}_{n=0}^\infty$ defined by

$$x_0(t) = u(t), \quad t \in \left[-\frac{\pi}{2}, 1 \right],$$

$$x_{n+1}(t) = \begin{cases} \displaystyle\int_0^1 G_1(t,s) f(s, x_n(s), x_s^n))\,ds, & t \in [0,1], \\[2mm] \sin t, & t \in \left[-\frac{\pi}{2}, 0 \right], \end{cases}$$

is monotone nondecreasing and converges to x^*. Similarly, the sequence $\{y_n\}_{n=0}^\infty$ defined by

$$y_0(t) = v(t), \quad t \in \left[-\frac{\pi}{2}, 1 \right],$$

$$y_{n+1}(t) = \begin{cases} \displaystyle\int_0^1 G_1(t,s) f(s, y_n(s), y_s^n))\,ds, & t \in [0,1], \\[2mm] \sin t, & t \in \left[-\frac{\pi}{2}, 0 \right], \end{cases}$$

is monotone nonincreasing and converges to the unique dolution $x^* \in C_{eq}^{9/5}(J, \mathbb{R})$. Thus, we have

$$x_0 \preceq x_1 \preceq \cdots \preceq x_n \preceq x^* \preceq y_n \preceq \cdots \preceq y_1 \preceq y_0.$$

5. Remarks and Conclusion

We observe that the existence of solutions of the PBVP (1.3) can also be obtained by an application of topological Schauder fixed point principle under the hypothesis (H_2), but in that case we do not get any sequence of successive approximations that converges to the solution. Again, we can not apply analytical or geometric Banach contraction mapping principle to the problem (3.1) under the considered hypotheses (H_1) and (H_3) in order to get the desired conclusion, because here the nonlinear function f does not satisfy the usual Lipschitz condition on the domain $I \times \mathbb{R} \times \mathcal{C}$. Similarly, we can not apply algebraic Knaster-Tarski fixed point theorem to PBVP (1.3) for proving the existence of solution, because $C(J, \mathbb{R})$ is not a complete lattice. Therefore, all these arguments show that our hybrid fixed point principle, Theorem 2.4 has more advantages than classical fixed point theorems to get more information about the solution of nonlinear equations in the subject of nonlinear analysis.

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