MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **12(01)**(2024), 43–56.
http://doi.org/10.26637/mjm1201/004

Certain operator algebras of star-like reducible $P\omega_n^*$ transformations

SULAIMAN AWWAL AKINWUNMI *[1], **GARBA RISQOT IBRAHIM** [2] **AND ADENIKE OLUSOLA ADENIJI** [3]

[1] *Faculty of Science, Department of Mathematics and Statistics, Federal University of Kashere, Gombe, Nigeria.*
[2] *Faculty of Science, Department of Mathematics and Statistics, Federal University of Kashere, Gombe, Nigeria.*
[3] *Faculty of Pure and Applied Sciences, Department of Mathematics and Statistics, kwara State University Malete, Kwara State, Nigeria and Department of Mathematics, University of Abuja, Abuja, Nigeria.*

***R**eceived 27 August 2023; **A**ccepted 21 October 2023*

Abstract. Let $Z_n = 1, 2, 3, \dots$ denote a distinct non-negative n-order collection of numbers, and $\alpha\omega_n^*$ denote a star-like transformation semigroup. The characterization of $P\omega_n^*$ star-like partial on the $\alpha\omega_n^*$ leads to the semigroup of linear operators. The research produced a completely new classical metamorphosis that was divided into inner product and norm parts. The study demonstrated that any specific star-like transformation $\lambda_i^*, \beta_j^* \in V^*$ is stable and uniformly continuous if there exists $T^{\vartheta^*} : (V^*, \langle v - \alpha^* u, u - \alpha^* v\rangle) \longrightarrow (V^*, \langle u - \alpha^* v, v - \alpha^* u\rangle)$ with a star-like polygon ϑ^* of $\vartheta^* V^*$ such that $T^{\vartheta^*}(v^*) = \vartheta^* V^*$. Every star-like composite vector space $V^* \in P\omega_n^*$ can be uniquely decomposed as the sum of subspaces $w_i^* \leq W_{i+1}^*$ and $s_j^* \leq S_{j+1}^*$ such that $W_{i+1}^* + S_{j+1}^* \subseteq V^* \in P\omega_n^*$. The study suggests that the research's findings be used to address issues in the mathematical disciplines of genetics, engineering, code theory, and telecommunications.

AMS Subject Classifications: 20M20, 20M20.

Keywords: Star-like transformation, semigroup, innerproduct, vector space.

Contents

1. Introduction and Background

The study of vector spaces and vector space functions is known as linear algebra. They form the fundamental objects of study in this paper. Once a star-like vector space is defined, its properties will be investigated. A non-empty star-like transformation $\alpha\omega_n^*$ on which a polygon

$$\vartheta^* : \alpha\omega_n^* \times \alpha\omega_n^* \longrightarrow \alpha\omega_n^*$$

is defined as a star-like groupoid $(\alpha\omega_n^*,\ \vartheta^*)$. Then, $(\alpha\omega_n^*,\ \vartheta^*)$ is a star-like semigroup if the operation ϑ^* disk associative.

---

*[**Corresponding author.** Email addresses: **molakinkanju@gmail.com, sakinwunmi@fukashere.edu.ng** (Sulaiman Awwal AKINWUNMI), **risqot.ibrahim@kwasu.edu.ng** (Garba Risqot IBRAHIM) **adeniji4love@yahoo.com** (Adenike Olusola ADENIJI)

Similar to how sets play a vital role in mathematics, mapping also aids in understanding the relationships between various algebraic structures. Instead of using the term mapping, which refers to the former, transformation is utilized. The publications of [1] and [2] provide additional details on semigroup mappings. According to the terminology employed by [3] the domain and image set of any given transformation $\lambda_i^* \in \alpha\omega_n^*$ were indicated by $D(\lambda_i^*)$ and $I(\lambda_i^*)$ respectively.

Let (-) signify the empty set and $\begin{pmatrix} u & v & l \\ r & s & t \end{pmatrix}$ be depicted as $\begin{pmatrix} r & s & t \end{pmatrix}$ not $\begin{pmatrix} r, & s, & t \end{pmatrix}$ not making the cycle notation more complex to the point that any transformation that contains an empty map is referred to as a star-like reducible transformation $P\omega_n^*$.

If a mapping $P\omega_n^*$ is a star-like linear vector in a semigroup such that any star-like vector can be metricized using the Hamming distance function method for every $i, j \in Z_n; i \leq j \Rightarrow ri \leq jr$, then it is said to be star-like order-preserving. One of the most potential transformation families for the current and upcoming generations of academics is created by the associative function composition [1]. Hence, the new classical finite $\alpha\omega_n*$ transformation semigroups.

Assuming that $\lambda_i^* \in P\omega_n^*$ is a star-like transformation, under the composition of mapping it generates another star-like transformation of its form with trace of any composed star-like matrix $\beta_j^* \in M^* \subseteq B\omega_n^*$ where $tr(\beta_j*)$ stands for the sum of its star-like diagonal points consisting of a finitely star-like polygon $\vartheta^* \in \vartheta^* V^*$. Then the star-like polygon (transformation) $\vartheta^* : R_0^* \longrightarrow Q$ is a rule $f : A \longrightarrow Q$ for some $A \neq \emptyset, A \subseteq R_0^*$, where R_0^* is a star-like disk operator and $P\omega_n^*(R_0^*)$ denotes the set of all star-like reducible transformations whose domain and rank are subsets of R_0^*, then $\beta^* \times \lambda^*$ of $\beta^*, \lambda^* \in P\omega_n^*(R_0^*)$ is the transformation with domain

$$Q = (I(\beta^*) \cap D(\lambda^*))\beta^{-1*}$$

so that for each $r_0^* \in R_0^*$,

$$r_0^*(\beta^* \times \lambda^*) = (r_0^*\beta^*)\lambda^*.$$

Given two associated star-like subspaces of V^*, W_{i+1}^* and S_{j+1}^* with the rule $\vartheta^* : W_{i+1}^* \longrightarrow S_{j+1}^*; \beta_j^* \in P\omega_n^*(V^*)$. The domain and rank of β_j^* are subspaces of V^* vector space V^*, and a subspace W_{i+1}^* of V^* whenever $P\omega_n^*(V^*, W_{i+1}^* \longrightarrow S_{j+1}^*) = \{\beta^* \in P\omega_n^*(V^*) : \alpha^* V^* \subseteq \alpha\omega_n^*$ if the following conditions are satisfy

(i) the range space $W_{i+1}^*(\beta^*)$ of β^*, which consists of all $\beta^* u$ with u in V^*

(ii) the null space $S_{j+1}^*(\beta^*)$ of β^*, which consists of all u in V^* such that $u\beta^* = 0$.

If $\alpha\omega_n^*$ is considered to be star-like, then

$$|v - \alpha^* u| \leq |u - \alpha^* v| \tag{1.1}$$

for all $u, v \in D(\lambda_i^*, \beta_j^*)$ and $\alpha^* u, \alpha^* v \in I(\beta_j^*,\ \lambda_i^*)$ where

$$V^* = \begin{pmatrix} u & v & \dots & uv_{i,j} \\ \alpha^* u & \alpha^* v & \dots & \alpha^* uv_{i,j} \end{pmatrix} = (u, v, uv_{i,j},\ \alpha^* u, \alpha^* v, \alpha^* uv_{i,j}). \tag{1.2}$$

The inner product was characterized by star-like transformation semigroups $\alpha\omega_n^*$ such that for all $\beta^*, \lambda^* \in P\omega_n^*$

$$\lambda_i^* = \left\langle \begin{pmatrix} u & v & u \dots & uv_{i,j} \\ q_1 & q_2 & q_3 \dots & q_{i,j} \end{pmatrix} \right\rangle,\ and\ \beta_j^* = \left\langle \begin{pmatrix} v & u & v \dots & vu_{j,i} \\ k_1 & k_2 & k_3 \dots & k_{j,i} \end{pmatrix} \right\rangle$$

equals to

$$\langle \beta^* \rangle = \langle (\beta^* k_1, \beta^* k_2, \dots \beta^* k_{i,j}) \rangle\,,$$

$$\langle \lambda^* \rangle = \langle (\lambda^* q_1, \lambda^* q_2, \dots \lambda^* q_{j,i}) \rangle\,.$$

Then $U_j = \beta_j^*$ and $U_i = \lambda_i$ such that

$$U_i^* = (q+1, q+1-i),\ U_j^* = (k+1, k+1-j)$$

which implies

$$\langle U_i, U_j \rangle = \left\{ \begin{pmatrix} u_1 & u_2 & u_3 \ldots & u_{i+1} \\ k_1 & k_2 & k_3 \ldots & k_{i+1} \end{pmatrix}, \begin{pmatrix} v_1 & v_2 & v_3 \ldots & v_{j+1} \\ q_1 & q_2 & q_3 \ldots & q_{j+1} \end{pmatrix} \right\} \tag{1.3}$$

with the star-like disk operator $R_0^* \geq 0$ on the inner product of a star-like vector space $V^* : [0, \infty) \to [0, \infty)$ such that $0\alpha^* = 0$, and $\beta^*(R_0^*) \leq R_0^*$, in which

$$\| \langle R_0^*(k), (u) \rangle \| \leq \| \langle R_0^*(v), (q) \rangle \|, \tag{1.4}$$

such that $k_{i,j}$, and $q_{i,j}$ are lower diagonal elements and upper diagonal elements of $\lambda_i^*, \beta_j^* \in P\omega_n^*$ star-like reducible semigroup respectively.

Let $M\omega_n^*$ denote a star-like monoid semigroup with unique identity $1 \in M\omega_n^* : 1\lambda^* = \lambda^* = \lambda^* 1$ for all $\lambda^* \in M\omega_n^*$. Putting $\lambda^{0*} = 1$ (index law) holds for all b, d in $\mathbb{N} \cup \emptyset$ then $\alpha\omega_n^*$ contains a unique element 0 (zero):

$$\begin{aligned} \beta^* \lambda^* &= \lambda^* \beta^* \\ 0\beta^* &= 0 \\ \lambda^* 0 &= 0 \end{aligned}$$

For all β^*, λ^* in $P\omega_n^*$ is disk associative, by equation (1.1) $P\omega_n^* \cup \emptyset$ is a semigroup obtained from $P\omega_n^*$ by adjoining zero where necessary. If a semigroup $P\omega_n^*$ has the property that for all $\beta^*, \lambda^* \in P\omega_n^*$:
$\lambda^{0*} = 1$, for all $b, d \in \mathbb{R}$. Then

$$\lambda^{b*} \lambda^{d*} = \lambda^{b+d*} \Longrightarrow (\lambda^b)^{d*} = \lambda^{bd*}.$$

Thus, equations (1.3) and equation give useful characterizations of inner product normed space on the star-like vector of a star-like mapping such that the star-like vector of order n in equations (1.1) and (1.2) for any given $\beta_i^* \in P\omega_n^*$ is given by

$$\beta_{i,\,j}^* = \begin{cases} \beta_{i,n}^* - D(\lambda_{i,n}^*) \\ V_{i,j}^* - n \end{cases} \tag{1.5}$$

such that

$$V_{i,j}^* = \begin{pmatrix} \beta_i^* & \beta_j^* & \ldots & \beta_n^* \\ \lambda_i^* & \lambda_i^* & \ldots & \lambda_n^* \end{pmatrix}$$

The star-like order reversing of $V_{i,\,j}^*$ in (1.5) generates elements of $P\omega_n^*$. Hence, a star-like vector space is a triple $(V^*,\ +, \times)$ over $P\omega_n^*(n,\ F)$ comprised of a set V^* and F^n along with the operation $'+'$ and $'\times'$ by real integers such that the operations most produce vectors in the space and the following statements must be true;

(i) if β^*, λ^* are vectors in V^* then $\beta^* + \lambda^*$ is a vector in V^*

(ii) if β^* is a vector in V^* and b is a star-like scalar in $P\omega_n^*(n,\ F) \in \mathbb{R}$ then $b\beta^*$ is a vector in V^*.

As a result, a star-like vector space is a triple $(V^*,\ +, \times)$ over $P\omega_n^*(n,\ F)$ consisting of a set V^* and F^n as well as the operations $'+'$ and $'\times'$ by real integer and star-like disk operator such that the operations most produce star-like vectors in the space and the following must be true:

(i) if β^*, λ^* are vectors in V^* then $\beta^* + \lambda^*$ is a vector in V^*

(ii) if β^* is a vector in V^* and b is a star-like scalar in $P\omega_n^*(n,\ F) \in \mathbb{R}$ then β^* is a vector in V^*.

That is, given two vectors β^*, λ^* in $V^*_{i,\ j}$ of equation (1.5), it associates a new vector in $V^*_{i,j}$ denoted by $\beta^* + \lambda^*$:

$$(+) : V^*_{i,\ j} \times V^*_{i,\ j} \longrightarrow V^*_{i,\ j}$$
$$(\beta^*, \lambda^*) \longrightarrow \beta^* + \lambda^*$$

and given a vector β^* in $V^*_{i,\ j}$ and a star-like disk operator $r_0^* \in R_0^*$, it associate a new vector in $r_0^*\beta^* \in V^*_{i,\ j}$ such that

$$(\times) : \mathbb{R} \times V^*_{i,\ j} \longrightarrow V^*_{i,\ j}$$
$$(r_0^*, v^*) \longrightarrow r_0^* v^*.$$

Let $V^* \in P\omega_n^*(n, F)$ represent a star-like vector space. A mapping $T^{\vartheta^*} : V^* \longrightarrow V^*$ is a star-like mapping if there exists a star-like disk operator (constant) $r_0^* \subseteq V^*$ with $0 \leq b \leq 1$ such that

$$V^*(T^{\vartheta^*}(\beta^*), \lambda^*) \leq r_0^* V^*(T^{\vartheta^*}(\lambda^*), \beta^*) \tag{1.6}$$

As a result, a star-like map points closer diagonally together. For every $\beta^*, \lambda^* \in V^*$ and $r \leq 0$, all points λ^* in the ball $B_r(\beta^*)$ are mapped diagonally into a ball $B_s(T^*(\beta^*))$ with $s \leq r$. This is depicted in 1. It also follows from equation (1.3) and (1.5) that a star-like mapping is uniformly continuous. If $T^{\vartheta^*} : V^* \longrightarrow V^*$ then a point $v^* \in V^*$ such that

$$\left|\left|T^{\vartheta^*}(v^*)\right|\right| = ||r_0^* V^*|| \tag{1.7}$$

is called a star-like fixed point of T^{ϑ^*}

The following is a partial list of papers and books:[4], [5], [6], [7], and [10] for basic and standard notions in transformation semigroup theory. [8] factorized assertions about the relationships between metric spaces, normed linear spaces, and inner product spaces. Refer to [9] for an introduction to functional analysis with algebraic applications. The characterization relations of algebraic structure to linear operators have not been investigated on $P\omega_n^*$, hence the need for this research.

That exists between metric spaces, normed linear spaces, and inner product spaces.

2. Preliminary

There is a need to demonstrate the application of algebraic theory to other relevant pure mathematical topics. The research study created mathematical relationships to connect some operator algebras with transformation semigroups. Some fundamental concepts and preliminary information that would be required in the following part were defined:

Definition 2.1. *Star-like Mapping ($\longrightarrow^*$): Consider the star-like sets of disk operators R_n^* and Q_n^* to be non-empty. A star-like rule $\vartheta^* : R_n^* \longrightarrow^* Q_n^*$ is a function ϑ^* that transforms Q_n^* into R_n^*.*

(i) $D(\vartheta^*) = R_n^*$

(ii) for every $r, n, k', l' \in \rho^* \Longrightarrow r = k',\ n = l'$.

Definition 2.2. *Star-like fixed point: A fixed point element $m^* \in I(\beta^*)$ of $P\omega_n^*$ is a function $\vartheta^* : \beta^* \longrightarrow \beta^*$ such that $f(\beta^*) = |m^*(\beta^*)|$. It is read that β^* fixes m^*.*

Definition 2.3. *Star-like vector space: A triple $(V^*, +, \times)$ is a star-like space $V^* \subseteq P\omega_n^*$ containing a set of mapping (vectors) and star-like operator $+$ and $\times$ by real integer as follows:*

(i) Given two vectors $\beta^*, \lambda^* \in V^*$, a new star-like vector in V^* denoted by $\beta^* + \lambda^*$ is obtained

$$\begin{aligned}(+) : V^* \times V^* &\longrightarrow V^*\\ (\beta^* \lambda^*) &\longrightarrow \beta^* + \lambda^*\end{aligned}$$

(ii) Given a star-like vector $\beta^* \in V^*$ and a real star-like disk operator $r_0^* \in R_0^*$, associates a new star-like vector in V^* denoted by $r_0^* V^*$ and a real number $b \in \mathbb{R}$ so that

$$\begin{aligned}(\times) : \mathbb{R} \times V^* &\longrightarrow V^*\\ (r_0^*,\ V^*) &\longrightarrow r_0^* V^*,\end{aligned}$$

Then, $(V^*,\ +, \times)$ is a (real) star-like vector space V^* if

(i) $(\beta^* + \lambda^*) + \gamma^* = \beta^* + (\lambda^* + \gamma^*)$

(ii) $0 \in V^*$ *such that* $\beta^* + 0 = 0 + \beta^* = \beta^*$

(iii) If $\beta^* \in V^*$ there exists $-\beta^* \subseteq V^*$ which satisfies $\beta^* + (-\beta^*) = 0$

(iv) $\beta^* + \lambda^* = \lambda^* + \beta^*$

(v) $(r_0^*\ b) \times \lambda^* = r_0^* \times (b \times \lambda^*)$*, for all* $r_0^*,\ b \in \mathbb{R}$

(vi) $r_0^* \times (\beta^* + \lambda^*) = r_0^* \beta^* + r_0^* \lambda^*$*, for every* $\beta^*, \lambda^* \subseteq V^*$

(vii) $(r_0^*\ b) \times \beta^* = r_0^* \times (b\beta^*)$

(viii) $1_{\beta^*} = \beta^*$ *for every* $\beta^*, \lambda^*, \gamma^* \subseteq V^*$

Definition 2.4. *Supplementary subspaces: Let $W_{i+1}^*, S_{j+1}^* \in V^*$ be two subspaces of a star-like vector space V^*, Then W_{i+1}^* and S_{j+1}^* are said to be supplementary subspaces if*

$$W_{i+1}^* + S_{j+1}^* = V^* and\ W_{i+1}^* \bigcap S_{j+1}^* = \langle 0 \rangle .$$

Definition 2.5. *Star-like inner product space: Let $(V*, +, \times)$ represent a star-like vector space over the field $P\omega_n(n,\ F)$. A star-like inner product is a space function $\langle \times, \times \rangle : V^* \times V^* \longrightarrow \mathbb{R}$ that assigns to each ordered pair (γ^*, λ^*) in V^* and a scalar (real number) given that the following propositions are true.*

(i) $\langle \gamma^*,\ \gamma^* \rangle \geq 0$ *and* $\langle \gamma^*,\ \gamma^* \rangle = 0$ *if and only if* $\gamma^* = 0$ *for all* $\gamma^* \in V^*$

(ii) $\langle r_0^* \gamma^*,\ \lambda^* \rangle = r_0^* \langle \gamma^*,\ \lambda^* \rangle$ *for* $r_0^* \in \mathbb{R}$

(iii) $\langle \gamma^*,\ \lambda^* \rangle = \langle \lambda^*,\ \gamma^* \rangle$ *for all* $\gamma^*,\ \lambda^* \in V^*$

(iv) $\langle \gamma^*,\ r_0^* \lambda^* + b\beta^* \rangle = r_0^* \langle \gamma^*,\ \lambda^* \rangle + b \langle \gamma^*,\ \beta^* \rangle$ *for every* $\beta^*,\ \lambda^*, \gamma^* \in V^*$.

Definition 2.6. *In the case of any particular star-like transformation $\lambda_i^* \in P\omega_n^*$, there exists a unique identity star-like rule $e_{\lambda_i^*}^* : \lambda_i^* \longrightarrow \lambda_i^*$ defined by $e_{\lambda_i^*}{}^*(u) = u$ for all $u \in \lambda_i^*$.*

3. Main Results

The following results explain how particular operator algebras affect star-like $P\omega_n^*$ reducible transformation semigroups.

Lemma 3.1. *A $R_0^* \subseteq V^*$ in $P\omega_n^*$ element is a star-like disk operator if and only if $R_0^* f(\lambda_i^*) \leq I(R_0^*)$.*

Proof. Suppose $R_0^* f(\lambda_i^*) \leq I(R_0^*)$, there exist $\lambda_i^* s_i^* \in I(R_0^*)$ such that $\lambda_i^* s_i^*(R_0^*) = \lambda_i^* s_i^*$.
If

$$I(R_0^*) = \{\lambda_i^* s_i^* R_0^* : \lambda_i^* s_i^* \in P\omega_n^*\},$$

Then

$$R_0^* \in P\omega_n^* \iff |R_0^*(v) - R_0^*(\lambda_i^* \alpha^* u)| \leq |R_0^*(\lambda_i^* u) - R_0^*(\alpha^* v)| \leq R_0^*$$

Implies

$$\lambda_i^* v \{|R_0^*(v) - R_0^*(\lambda_i^* \alpha^* u)| \leq |R_0^*(\lambda_i^* u) - R_0^*(\alpha^* v)|\} \leq \lambda_i^* s_i^* R_0^*.$$

By (1.1)

$$|R_0^*(u) - R_0^*(\lambda_i^* \alpha^* u)| \leq |R_0^*(\lambda_i^* v) - R_0^*(\alpha^* v)| \leq K_{R_0^*}$$

shows that

$$(\lambda_i^* \alpha^* v K_{R_0^*}) R_0^* \leq \alpha^* u R_0^*$$

and

$$\lambda_i^* s_i^* R_0^* \leq \lambda_i^* s_i^*.$$

Thus, $R_0^* f(\lambda_i^*) \leq I(R_0^*)$, for every $\lambda_i^* s_i^* \in I(R_0^*)$. ■

Theorem 3.2. *Assume $\vartheta^* V^*$ is a star-like disknorm of $V^* \in \alpha\omega_n^*(n,\ F)$ such that $P\omega_n^* \subseteq \alpha\omega_n^*$ then the following are true:*

i Every element $\lambda_n^* \in P\omega_n^*$ is star-like reducible

ii $P\omega_n^*$ contains $w^+(\vartheta^* V^*) \leq w^-(\vartheta^* V^*)$

iii There exists a unique $r_0^* \in P\omega_n^*(V^*) : \{r_0^* = \langle Max(n, w^+ V^*) \times Min(n, w^- V^*)\rangle\}$ and $\langle (n, w^+(V^*), w^-(V^*))\rangle = \sum_{\vartheta_i^*=1}^{n} \begin{pmatrix} 2^{\vartheta^*-1} \\ \vartheta^* + n - 1 \end{pmatrix}$ such that $r_0^* \subseteq R_0^*$ is the star-like disk operator degree of $P\omega_n^*$.

Proof. $(i) \implies (ii)$
If $r_0^* \in \vartheta^* V^*$ is a star-like reducible degree order, then $b \in Z_n$ is in the range set $d \in Z_n : \lambda_n^*(b)d$. Because $\vartheta^* V^*$ is a star-like vector

$$(\lambda_n^*(b)\vartheta^* V^*) = \lambda_n^*(b)\vartheta^* V^* \tag{3.1}$$

Implies

$$\lambda_n^*(d\vartheta^* V^*) = d\vartheta^* V^*. \tag{3.2}$$

Then

$$\langle Max(n, w^+ \vartheta^* V^*)\rangle \leq \langle Min(n, w^- \vartheta^* V^*)\rangle$$

for some $b,\ d \in D(\lambda_n^*)$ with a star-like disknorm

$$\lambda_n^*(bV^*) = dv^* : r_0^*(\lambda_n^*) \leqslant dV^*. \tag{3.3}$$

$(ii) \longrightarrow (iii)$
Let $\vartheta^* \in V_n^* : \vartheta^* = w^+(\lambda_n^*) \times w^-(\lambda_n^*)$.
By star-like folding principle and composition of star-like reducible transformation.

$$D\left\langle Max(n, w^+(V^*))\right\rangle$$

gives the star-like order-reversing of

$$I\left\langle Min(n, w^-((V^*))\right\rangle.$$

The theorem follows from 2

Since $\vartheta^* V^*$ possesses a reducible order of $F(r_0^*)$ in $V^* \subseteq P\omega_n^*$, then $F(n; w^+(\lambda_n^*), w^-(\lambda_n^*)$ generates a finitely reducible recurrence star-like disknorm:

$$\left\langle (n, w^+(V^*), w^-(V^*))\right\rangle = \sum_{\vartheta_i^*=1}^{n} \binom{2^{\vartheta^*-1}}{\vartheta^* + n - 1} \tag{3.4}$$

$(iii) \longrightarrow (i)$
Assuming $\lambda_n^* \in P\omega_n^*$ is star-like reducible such that $\vartheta^* V^*$ is a star-like vector space with a star-like disknorm, then for any given star-like transformation

$$\lambda^* \in D(\vartheta^* V_n^*) : r_0^*(\lambda_n^*) \leq Z_n$$

such that $b_{i+1} - b_i$ is the domain and $d_{j+1} - d_j$ is the image order of $\lambda_n^* \subseteq \vartheta^* V^*$

$$\vartheta^* V_n^* \mid b_{i+1} - b_i \mid \leq \vartheta^* V_n^* \mid d_{j+1} - d_j \mid \tag{3.5}$$

then $r_0^* \in R_n^* : \vartheta^* V_n^* \times \vartheta^* V_n^* = \vartheta^* V_n^*$ which completes the proof. ∎

Proposition 3.3. *Given a star-like vector space $(V^*, +, \times)$, the following statements are true:*

(i) $0 \times \lambda^* = 0$ *for any* $\lambda^* \in V^*$

(ii) $(-r_0^*) \times \lambda^* = r_0^* \times (\lambda^*)$ *for any* $r_0^* \in \mathbb{R}$

(iii) $r_0^* \times 0 = 0$ *for any* $r_0^* \in \mathbb{R}$

(iv) If $r_0^* \times \lambda^* = 0$ *then either* $r_0^* = 0$ *or* $\lambda^* = 0$.

Proof. (i) Suppose $\lambda^* \in V^*$ such that $V^* \in \lambda\omega_n^*(n,\ F)$, by definition 2.5 using (viii), (v) and (ii) gives

$$\begin{aligned} \lambda^* + 0 \times \lambda^* &= 1 \times \lambda^* + 0 \times \lambda^* \\ &= (1 + 0) \times \lambda^* = \lambda^* + 0. \end{aligned}$$

By adding $(-\lambda^*)$ to both sides of the equality: $-\lambda^* + \lambda^* + 0 \times \lambda^* = -\lambda^* + \lambda^* + 0$.
Thus, $0 \times \lambda^* = 0$.
(ii) Let $-(r_0^* \times \lambda^*)$ be an element in V^* that satisfies property (iii) in definition 2.5, replace λ^* by $r_0^* \times \lambda^*$.
Now $r_0^* \times 0 = 0$ for any $r_0^* \in R_0^*$ holds if $-(r_0^* \times \lambda^*) = (-r_0^*) \times \lambda^*$.
Using $(-r_0^*) \times \lambda^* = r_0^* \times (-\lambda^*)$ and $r_0^* \times \lambda^* = 0$
where $(-r_0^*) \times \lambda^* + r_0^* \times \lambda^* = (-r_0^* + r_0^*) \times \lambda^* = 0 \times \lambda^* = 0$.
Then, $(-r_0^*) \times \lambda^* = r_0^* \times (-\lambda^*) = 0$.
(iii) In general, if $\lambda_j^* \in V^*$ we see that:

$$\begin{aligned} r_0^* \times 0 &= r_0^* \times (\lambda_j^* - \lambda_j^*) = r_0^* \times \lambda_j^* + r_0^* \times (-\lambda_j^*) \\ &= r_0^* \times \lambda_j^* + r_0^* \times \left\{(-1) \times \lambda_j^*\right\} = r_0^* \times \lambda_j^* + (-r_0^*) \times \lambda_j^* \\ &= r_0^* \times \lambda_j^* - r_0^* \lambda_j^* = 0 \end{aligned}$$

Therefore, $r_0^* \times 0 = 0$ for any $\lambda_j^* \in \alpha\omega_n^*(n,\ F)$.
(iv) Suppose $r_0^* \neq 0$, and $r_0^*\lambda^* = 0$.
Consider $\frac{1}{r_0^*}$, then,

$$\begin{aligned}\lambda^* &= 1\lambda^* = (\frac{1}{r_0^*}\, r_0^*)\lambda^* \\ &= \frac{1}{r_0^*}\,(r_0^*\lambda^*) = \frac{1}{r_0^*}\ 0\ = 0\end{aligned}$$

Thus, if $r_0^* = 0$, then $\lambda^* = 0$ or r_0^*. A reducible star-like transformation of $P\omega_n^*$ is the subset $K^* \subset \alpha\omega_n^*$, which is closed by the same operation on $P\omega_n^*$. If $K^* \subset \alpha\omega_n^*$, then it is a legitimate star-like sub-vector of V_n^* that is not equivalent to $|P\omega_n^*|$. Then

$$\cap_{j\in J} K_j^* \neq \emptyset \Longrightarrow \cap_{j\in J} K_j^* \subseteq P\omega_n^*.$$

Similarly, $S_{j+1}^* \subseteq V^* \in \alpha\omega_n^*(n,\ F)$ is a star-like vector addition and scalar multiplication closed subspace of V^*.
Therefore a star-like subspace $S_{j+1}^* \subseteq V^*$ in any given star-like triple $(V^*,\ +,\ \times)$ is a vector subspace of V^* if it is a vector space with the induced operation and still satisfies properties (i) - (vii) of definition 2.5. ■

Proposition 3.4. *Any star-like subset S_{i+j}^* of V^* is a star-like subspace if and only if the following requirements are met:*

(i) $w_i^* + s_j^* \in V_{i,j}^*$, *for any* $w_i^*, s_j^* \in V^*$

(ii) $b\ s_{i+j}^* \in P\omega^*$ *for any* $b \in \mathbb{R}$.

Proof. (i) Given any $S_{i+j} \in V^*$ such that $w_i^* + s_j^* \subseteq S_{i+j}^*$ and any star-like real number $b \in \mathbb{R}$ such that $w_i^* + s_j^* \leq S_{i+j}^*$. As a result of the limited vectors being well defined on S_{i+j}^*, the outcome vector is still in S_{i+j}^*.
Furthermore, $w_i^* + s_j^* \in V^* : 0 = 0(w_i^* + s_j^*) \in S_{i+j}^*$ and $-S_{i+j}^* = (-1)w_i^* + s_j^* V^*$.

(ii) Suppose $V^* \in \alpha\omega_n^*(n,\ F)$ and $S_{i+j}^* \in V^*$ with $i, j \in \mathbb{Z}_i \cup \emptyset$; $Z_i(i = \{0, 1, 2, \times\}) : \emptyset \in \mathbb{R}$, by the properties of $V^* \in P\omega_n^*$ it is obvious that S_{i+j}^* is star-like subspace.
Therefore, by properties (i) - (iv) of definition 2.6, the proof is complete. ■

Remark 3.5. *A star-like subset containing only zero vector, $z^* \in V^* = \emptyset$, and the whole space V^* are trivial subspaces, in which z^* is the smallest possible star-like subspace and $V^* \subseteq P\omega^*$ is the largest one.*

Proposition 3.6. *Let $(V^*,\ +,\ \times)$ represent a star-like vector space and $W_{i+1}^*,\ S_{j+1}^*$ represent two star-like subspaces. The following are interchangeable:*

(i) $W_{i+1}^* \bigcap S_{j+1}^* = \langle 1 \rangle$

(ii) There exists a unique couple $(w_i^*,\ s_j^*) \in W_{i+1}^* \times S_{j+1}^*$ *for each* $r_0^* \in W_{i+1}^* + S_{j+1}^*$ *such that* $r_0^* = w_i^* + s_j^*$.

Proof. (i) $\Longrightarrow$ (ii) Assume a star-like vector operator $r_0^* \in W_{i+1}^* + S_{j+1}^*$ n be expressed in two paths: $r_0^* = u_{i,j}^* + v_{j,i}^*$ and $r_0^* = v_{i,j}^* + u_{j,i}^*$ with $uv_{i,j}^* \in W_{i+1}^*$, and $vu_{j,i}^* \in S_{j+1}^*$.
Take note that

$$u_{i,j}^* - v_{i,j}^* \leq v_{j,i}^* - u_{j,i}^* \in W_{i+1}^* \bigcap S_{j+1}^* = \langle 1 \rangle\,.$$

As a result, $r_0^* v \leq \alpha^* u$ and $r_0^* u \leq \alpha^* v$ are equal
(ii) $\Longrightarrow$ (i) suppose by contradiction, there exists $0 \neq r_0^* \in W_{i+1}^* \bigcap S_{j+1}^*$.
Therefore,

$$r_0^* = 0 + r_0^* \leq r_0^* + 0 \in W_{i+1}^* \bigcap S_{j+1}^*.$$

This contradict the initial proposed statement, because any star-like vector in the same transformation can be expressed uniquely as the combination of vectors in W_{i+1}^* and S_{j+1}^*, this means that $r_0^* V^*$ can be decomposed in two different ways as a vector of $W_{i+1}^* + S_{j+1}^* \in V^* \subseteq P\omega_n^*$. ■

Remark 3.7. *Every vector of equation* (1.5) *can be uniquely decomposed in proposition* **??** *as the combination of a star-like vector in* W_{i+1}^* *and* S_{j+1}^*.

Theorem 3.8. *Let* $\|V^*\|$ *denote a star-like norm vector space with the disk operator* r_0^* *and let* u^* *and* v^* *be any two star-like vectors in* V^* *then* $2d(u^*, v^*) = \|E^*(u^*) - F^*(v^*)\| + 2\phi r_0^*$.

Proof. By a star-like operator

$$r_0^*(u^*) = r_0^*(u^* - v^* + v^*) \leq r_0^*(u^* - u^* + v^*) + r_0^*(V^*)$$

which is equivalent to

$$\frac{E^* - F^*}{2} + \phi r_0^*(u^*) - r_0^*(v^*) \leq \frac{1}{2} V^* r_0^*(v^* - u^*)$$

Then,

$$r_0^*(v^*) - r_0^*(u^*) \leq r_0^*(u^* - v^*) = \frac{1}{2} V^* r_0^*(v^* u^*) \tag{3.6}$$

It was deduced in equation (1.6) that r_0^* is a continuous star-like disk when using it as a norm on the star-like vector space $r_0^* V^*$, using the absolute value as a norm on the real star-like space,

$$-2r_0^*(v^* - u^*) \leq E^* r_0^*(v^*) - F^* r_0^*(u^*)$$

Gives

$$|r_0^*(u^*) - v^*| \leq \frac{1}{2} V^* r_0^*(u^* - v^*).$$

Given a star-like mapping

$$T^{\vartheta^*} : (r_0^* V^*, \|\times\|) \longrightarrow (r_0^* V^*, \ \frac{1}{2} V^* \ \|\times\|)$$

such that a star-like operator $r_0^* < b \leq 1$. Then

$$\frac{1}{2} V^* \left\| E^*(T^{\vartheta^*}(v^*), F^*(u^*)) \right\| \leq \frac{1}{2} V^* b \left\| (T^{\vartheta^*}(u^*), T^{\vartheta^*}(v^*)) \right\| \tag{3.7}$$

As a result, the diagonal distance between two star-like vectors in a star-like disknorm space $\vartheta^* V^*$ is provided by

$$2d(u^*, v^*) = \|E^*(u^*) - F^*(v^*)\| + 2\phi r_0^*.$$

■

Example 3.9. *Consider a star-like 3- dimensional real space* $\mathbb{R}^3 \in V_{i,j}^*$ *such that*

$$V_{i,j}^* = \begin{pmatrix} \beta_i^* & \beta_j^* & \dots & \beta_n^* \\ \lambda_i^* & \lambda_i^* & \dots & \lambda_n^* \end{pmatrix}$$

Then

$$\mathbb{R}^3 = \left\{ \begin{pmatrix} \beta_1^* \\ \beta_2^* \\ \beta_3^* \end{pmatrix} : \beta_1^*, \beta_2^*, \beta_3^* \in \mathbb{R} \right\}$$

with the usual operation $+$ *and* $\times$ *where*

$$\beta^* = \begin{pmatrix} \beta_1^* \\ \beta_2^* \\ \beta_3^* \end{pmatrix},$$

$$\lambda^* = \begin{pmatrix} \lambda_1^* \\ \lambda_2^* \\ \lambda_3^* \end{pmatrix}.$$

Then $\begin{pmatrix} \beta_1^* \\ \beta_2^* \\ \beta_3^* \end{pmatrix} + \begin{pmatrix} \lambda_1^* \\ \lambda_2^* \\ \lambda_3^* \end{pmatrix} = \begin{pmatrix} \beta_1^* + \lambda_1^* \\ \beta_2^* + \lambda_2^* \\ \beta_3^* + \lambda_3^* \end{pmatrix}$ *such that*

$$b \begin{pmatrix} \beta_1^* \\ \beta_2^* \\ \alpha_3^* \end{pmatrix} = \begin{pmatrix} b\beta_1^* \\ b\beta_2^* \\ b\beta_3^* \end{pmatrix}$$

for every $\beta^*, \lambda^* \in V_{i,j}^*$ *and* $b \in \mathbb{R}$.

Example 3.10. *: Let* $V_{i,j}^* = \mathbb{R}$ *be the star-like space of real number with usual star-like norm:* $T^{\vartheta^*} : \mathbb{R} \longrightarrow \mathbb{R}$ *be defined by*

$$T^{\vartheta^*}(\beta^*) = 2\beta^*$$

given that $\alpha^*, \beta^* \in V_{i,j}^*$

$$\left\| T^{\vartheta^*}(\beta^*) \leq V_{i,j}^* - \lambda^* \right\| = \|2\beta^* - 2\lambda^*\| = 2\left\|V_{i,j}^* - 1\right\|$$

So T^{ϑ^*} *is a star-like mapping shown in 1.*

Lemma 3.11. *Suppose* $\vartheta^* \subseteq R_0^*$ *is a star-like polygon with star-like inner angles of* $Area(R_0^*) = (\sum_{n=1}^{\infty} \lambda_n) - (n-2)\pi$. *Then the star-like norm* $\|\vartheta^*\| : \vartheta^* V^* \longrightarrow \mathbb{R}$ *is then continuous.*

Proof. Let ξ and δ be any star-like elements such that $\xi > 0$ and $\delta = \xi$. The star-like convex polygon in ϑ^* of R_0^* can be strictly accomplished by arranging ϑ^* so that the origin is in the interior of R_0^* and projecting the boundary of ϑ^* on T^{*2} using

$$\vartheta^*(i,j,k) = \frac{(i,j,k)}{\sqrt{i^2+j^2+k^2}}. \tag{3.8}$$

The vertices of ϑ_n^* correspond to a portion of T^{*2}, the edges correspond to a portion of great circles of (ϑ_n^*), and the faces correspond to the star-like polygon. The union of $\vartheta_0^*, + \cdots \vartheta_n^*$ forms a star-like polygon on T^{*2}.

$$U(R_0^*) + U(R_1^*) + \cdots + U(R_n^*) = Area(T^{*2}) \tag{3.9}$$

For each β^*, and λ^* in $\vartheta^* V^*$

$$\vartheta^*(\lambda^*,\ \beta^*) = \vartheta^*|\lambda^* - \beta^*|.$$

generates

$$\frac{1}{2}V^* = \frac{E^* - F^*}{2} + \vartheta^*$$

Such that every star-like edge is shared by two star-like polygons and $\||v - \alpha^* u| \leq |u - \alpha^* v|\| : V^* \longrightarrow \mathbb{R}$ gives

$$\bigcup_{i=1}^{n} \bigcup_{j=1}^{n} v - \alpha^* u_{ij} - \bigcup_{i=1}^{n} u - \alpha^* v + \bigcup_{i=1}^{n} 2T^{*2} = 4\pi \tag{3.10}$$

which shows that

$$\vartheta^* T^{*2}(\|\lambda^*\|,\ \|\beta^*\|) = \vartheta^* T^{*2} |\ \|\lambda^*\| - \vartheta^* T^{*2}\ \|\beta^*\|\ |. \\ \leq \vartheta^* T^{*2}\ \|\lambda^* - \beta^*\| < \xi = \delta$$

and since the sum of star-like polygons at each vertex is 2π we obtain

$$2\pi V^* - 2\pi E^* + 2\pi F^* = 4\pi$$

As a result, $\|\vartheta^*\|$ is continuous on $\vartheta^* V^*$. ■

Theorem 3.12. *Let $V_{i,j}^* \in P\omega_n^*$ be a star-like symmetric reducible vector space, then for any $\vartheta^* \in V_{i,j}^*$ then $\langle ab\rangle \parallel \langle cd\rangle$ such that $\langle abc\rangle \leq \langle bcd\rangle$ for any a, b, c, d of $\vartheta^* \in V_{i,j}^*$.*

Proof. Given that a and d are on the star-like opposite side of the line bc in a star-like symmetric reducible vector space shown in 3 below,

Then, by using the folding principle of a star-like reducible transformation

$$\langle abc\rangle \leq |< abc\ | \longrightarrow \langle bcd\rangle \leq |< bcd\ |\ . \tag{3.11}$$

Then

$$\langle ab\rangle \leq \langle cd\rangle = \langle v - \alpha^* u\rangle \leq \langle u - \alpha^* v\rangle \tag{3.12}$$

Therefore, for any given reducible star-like vector space, the transverse of each $V_{i,j}^* \in P\omega_n^*$ makes an equal alternative angle on two sides because the lines of any reducible star-like vector space $V_{i,j}^*$ are always reducible. ■

Theorem 3.13. *Assume V^* is a vector norm space with a star-like disknorm. Then, on V^*, every star-like mapping T^{ϑ^*} is uniformly continuous.*

Proof. Given the fact that $(V^*, \|\times\|)$ denotes a star-like vector normed space and $T^{\vartheta^*} : (V^*, \|\times\|) \longrightarrow (V^*, \|\times\|)$ represents a star-like map, so, by equation (1.3) a star-like disknorm $\vartheta^* \in \mathbb{R}$ is defined, with $0 < \vartheta^* < 2$. Where $\xi > 0$ denotes an arbitrary element and $\delta = \frac{\xi}{\beta^*} > 0, then$ Then $\left\|T^{\vartheta^*}(\beta^*), (\lambda^*)\right\| < \delta$ such that

$$\left\|T^{\vartheta^*}(\beta^*), (\lambda^*)\right\| < \vartheta^* \times \frac{\xi}{\alpha^*} = \xi$$

Then, according to equations (1.5) and (1.7), every star-like mapping is continuous, implying that T^{ϑ^*} is uniformly continuous on $\vartheta^* V^*$. As a result, a star-like inner product space $(V^*, \langle\vartheta^*, \vartheta^*\rangle)$ is a normed vector space with the disknorm $||\vartheta^*|| = \sqrt{\langle v - \alpha^* u, u - \alpha^* v\rangle}$. ■

Theorem 3.14. *Let $\beta^*, \lambda^* \in V^*$ then* $\langle\beta^*, \lambda^*\rangle = \left\langle \begin{pmatrix} q-k \\ k-1 \end{pmatrix} = \begin{pmatrix} q-(k-1) \\ q-k \end{pmatrix} \right\rangle$.

Proof. ;
Suppose $D(\beta^*, \lambda^*) \subseteq Z_n$.If

$$F(q, k) = \ \langle\beta^*, \lambda^* \in V^* \subseteq P\omega_n^* : r(\beta^*, \lambda^*)\rangle = \langle I(\beta^*, \lambda^*)\rangle = k$$

Consider $u_{ij} v_{ji} \in D(\beta^*, \lambda^*)$ such that

$$u_{ij}\ \langle\beta^*, \lambda^*\rangle \leq \langle\lambda^*, \beta^*\rangle\ v_{ji} \tag{3.13}$$

Implies

$$\langle u_{ij}v_{ji}\rangle = 0$$

So, $u_{ij}v_{ji} \subseteq V^*$ has a $q-0+1$ disknorm degree of freedom with star-like order

$$\left\langle \begin{pmatrix} q-k \\ q-1 \end{pmatrix} = \begin{pmatrix} q-(k-1) \\ q-k \end{pmatrix} \right\rangle = 1.$$

Therefore, since for star-like reducible transformation, $\vartheta^* V^*$ is a star-like subspace of all star-like vector space and that if $\langle u_{ij}v_{ji}\rangle \in V^* : r(\beta^*, \lambda^*) = k$, irrespective of the value of $q \geq 2$ whenever $q = (q-1)$, there are exactly two star-like disknorm of rank such that

$$\langle \beta^*, \lambda^* \rangle = \left\langle \begin{pmatrix} q-(k-1) \\ q-k \end{pmatrix} \right\rangle$$

. ■

Theorem 3.15. *If* $T^{\vartheta^*} : (V^*, \langle v-\alpha^*u, u-\alpha^*v\rangle) \longrightarrow (V^*, \langle u-\alpha^*v, v-\alpha^*u\rangle)$ *is a star-like map, then for each positive integer* $n \in Z_n$ *,* $T^{\vartheta^*}n* : (V^*, \langle \vartheta^*, \vartheta^*\rangle) \longrightarrow (V^*, \langle \vartheta^*, \vartheta^*\rangle)$ *is also a star-like map.*

Proof. Assume $T^{\vartheta^*} : (V^*, \langle v-\alpha^*u, u-\alpha^*v\rangle) \longrightarrow (V^*, \langle u-\alpha^*v, v-\alpha^*u\rangle)$. Because T^{ϑ^*} is a star-like map, there exists a positive real integer $b \in \mathbb{R}$ that satisfies $\left\langle T^{\vartheta^*}\vartheta^*(u_{ij}v_{ji}), \vartheta^*(v_{ji}u_{ij})\right\rangle \leq b\left\langle T^{\vartheta^*}\vartheta^*(u_{ij}), (v_{ji})\right\rangle$. Then

$$\begin{aligned} \left\langle T^2\vartheta^*(u_{ij}), (v_{ji})\right\rangle &= \left\langle T^{\vartheta^*}(T^{\vartheta^*}(\vartheta^*(u_{ij}))), T^{\vartheta^*}(\vartheta^*(v_{ji}))\right\rangle \\ &\leq b\left\langle T^{\vartheta^*}(T^{\vartheta^*}(\vartheta^*(v_{ji}))), T^{\vartheta^*}(\vartheta^*(u_{ij}))\right\rangle \\ &\leq b^2\left\langle T^{\vartheta^*}(\vartheta^*(u_{ij})), (\vartheta^*(v_{ji}))\right\rangle \\ &= d\left\langle T^{\vartheta^*}(\lambda^*), (\beta^*)\right\rangle. \end{aligned}$$

Where $d = b^2 \leq 2$. So, for $n = 2$, see that

$$T^2\vartheta^* : (V^*, \langle v-\alpha^*u, u-\alpha^*v\rangle) \longrightarrow (V^*, \langle u-\alpha^*v, v-\alpha^*u\rangle)$$

is a star-like map. Now, for $n = \vartheta^*$

$$T^n : (V^*, \langle v-\alpha^*u, u-\alpha^*v\rangle) \longrightarrow (V^*, \langle u-\alpha^*v, v-\alpha^*u\rangle)$$

is a star-like map:

$$\langle T^n(v_{ji}), (u_{ij})\rangle \leq b^n \langle T^n((u_{ij}), (v_{ji}))\rangle$$

for every $\beta^*, \lambda^* \in V^*$. Then,

$$\begin{aligned} \left\langle T^{\vartheta^*+1}(u_{ij}), (v_{ji})\right\rangle &= \left\langle T^{\vartheta^*+1}(v_{ji}), T^{\vartheta^*}(T^{\vartheta^*}(u_{ij}))\right\rangle \\ &\leq b\left\langle T^{\vartheta^*+1}(u_{ij}), T^{\vartheta^*}(v_{ji})\right\rangle \\ &\leq b^{\vartheta^*+1}\left\langle T^{\vartheta^*}(v_{ji}), (u_{ij})\right\rangle. \end{aligned}$$

Hence, by mathematical induction, we deduced that

$$T^{\vartheta^*} : (V^*, \langle v-\alpha^*u, u-\alpha^*v\rangle) \longrightarrow (V^*, \langle u-\alpha^*v, v-\alpha^*u\rangle)$$

is a star-like map for all positive integers $Z_n = 1, 2, 3, \cdots$. ■

4. Acknowledgement

The author thanks the referee for his helpful recommendations that improved the paper's presentation.

References

[1] J.M. Howie, Fundamentals of semigroup theory, *The Claredon Press , Oxford University press, New York, Oxford Science Publication*, **12**(1995), London Mathematical society, monographs. New series.

[2] P.B. Bhattacharya, S.K. Jaini and S.K. Nagpaul, First Course in Linear Algebra, *New Age International Publishers*. London, New-Delhi, Nairobi.

[3] S.A. Akinwunmi, M.M. Mogbonju and G.R. Ibrahim, Some characterizations of equivalence relation on contraction mappings, *Scientific African*, **10**(2020)e00643, https://doi.org/10.1016/j.sciaf.2020.e00643.

[4] A.H. Clifford and G.B. Preston, The algebraic theory of semigroups, *American Mathematical Society, Providence, R.I.*., Vol. 1. Mathematical surveys, No. 7 (1998).

[5] G.M.S.Gomes and J.M. Howie, On the ranks of certain semigroups of order-preserving transformations, *Semigroup Forum,* **45**(1992), 272–282.

[6] O. Ganyushkin and V. Mazorchuk, Classical Finite Transformation Semigroups; An Introduction, *Springer-Verlag*, (2009), London Limited.

[7] G.U. Garba, On the idempotents ranks of certain semigroups of order-preserving transformation, *Portugal. Math.*, **51**(1994), 185–204.

[8] J. Klipfel, A brief introduction to hilbert space and quantum logic, *retrieved in www.google.com*, (2008).

[9] E. Kreyszig, Introductory functional analysis with applications, *John Wiley and Sons, New York*, (1978) Santa Barbara, London Sydney Toronto.

[10] A. Laradji and A. Umar, On certain finite semigroups of order-decreasing transformations, *International Semigroup Forum*, **6**(2004), 184–200.

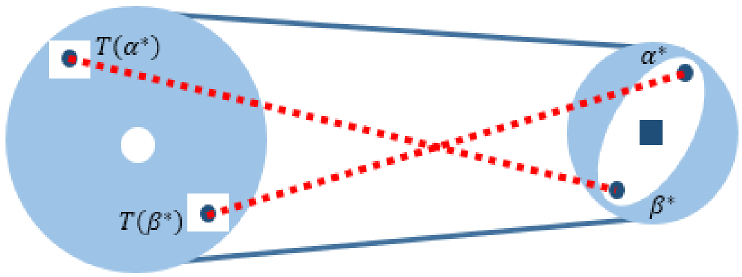

Figure 1: A star-like map $T^{\vartheta}*: T^{\vartheta}*(v^*) = \vartheta^* V^*$

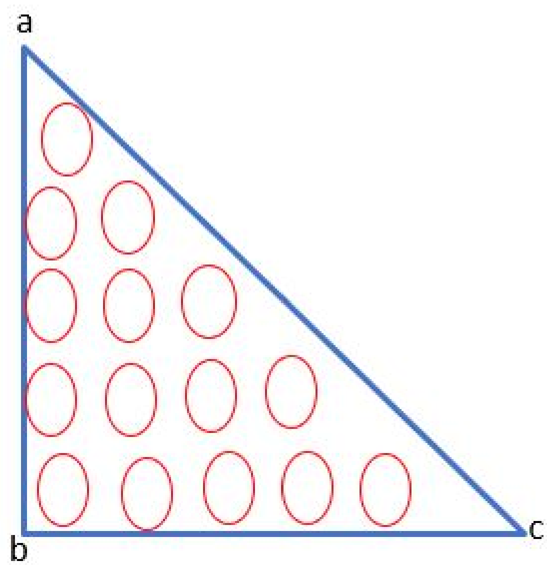

Figure 2:

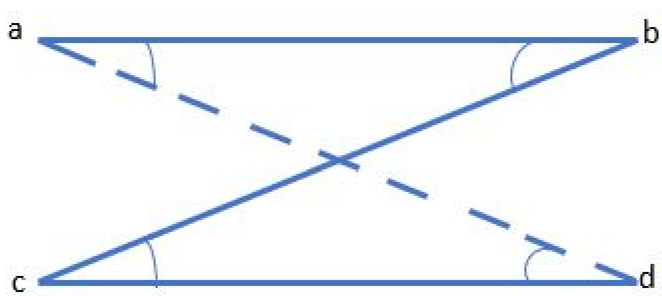

Figure 3: