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On Pillai's problem involving two linear recurrent sequences: Padovan and Fibonacci

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Abstract. In this paper, we find all integers c having at least two representations as a difference between linear recurrent sequences. This problem is a Pillai problem involving Padovan and Fibonacci sequence. The proof of our main theorem uses lower bounds for linear forms in logarithms, properties of continued fractions, and a version of the Baker-Davenport reduction method in Diophantine approximation.

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Contents

1. Introduction

It is well-known that the sequence $\{\mathcal{P}_k\}_{k\geq 1}$ of Padovan numbers is defined by

$$\mathcal{P}_0 = \mathcal{P}_1 = \mathcal{P}_2 = 1, \quad \mathcal{P}_{k+3} = \mathcal{P}_{k+1} + \mathcal{P}_k, \quad k \geq 0.$$

The first Padovan numbers are

$$1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, 114, 151, 200, 265 \ldots$$

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The sequence $\{F_k\}_{k\geq 1}$ of Fibonacci numbers is defined by

$$F_0 = 0, \quad F_1 = 1, \quad F_{k+2} = F_{k+1} + F_k, \quad k \geq 0.$$

The first Fibonacci numbers are

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, \ldots$$

In this paper, we are interested in the Diophantine equation

$$\mathcal{P}_m - F_n = c \tag{1.1}$$

for a fixed c and variable m and n. In particular, we are interested in those integers c admitting at least two representations as a difference between a Padovan number and Fibonacci number. This is a variation of the equation

$$a^x - b^y = c, \tag{1.2}$$

in non-negative integers (x, y) where a, b, c are given fixed positive integers. The history of equation (1.2) is very rich and goes back to 1935 when Herschfeld [8], [9] studied the particular case $(a, b) = (2, 3)$. Extending Herschfeld's work, Pillai [12], [13] proved that if a, b are coprime positive integers then there exists $c_0(a, b)$ such that if $c > c_0(a, b)$ is an integer, then equation (1.2) has at most one positive integer solution (x, y). Since then, variations of equation (1.2) has been intensively studied. Some recent results related to equation (1.1) are obtained by the third author and his collaborators in which they replaced Fibonacci numbers $\mathcal{P}_n$ by the Fibonacci numbers F_n (see [5]), Tribonacci numbers (see [2]), and k-generalized Fibonacci numbers (see [6]). The equation solved in this paper is an exponential Diophantine equation. The similar problem has been solved recently by the authors (see [14–16]). The aim of this paper is to prove the following result.

Theorem 1.1. *The only integers c having at least two representations of the form $\mathcal{P}_m - F_n$ with $m > 3, n > 1$ are*

$$c \in \{-226, -82, -52, -30, -27, -18, -9, -6, -5, -4, -3, -1, 0, 1, 2, 3, 4, 6, 7, 8, 10, 11, 13, 15, 16, 20, 25, 31, 32, 36, 44, 52, 62, 111, 262\}.$$

We organize this paper as follows. In Section 2, we recall some results useful for the proof of Theorem 1.1. The proof of Theorem 1.1 is done in the last section.

2. Auxiliary results

2.1. Some properties of Fibonacci and Padovan sequences

Here we recall a few properties of the Fibonacci sequence $\{P_k\}_{k\geq 0}$ and Padovan sequences $\{\mathcal{P}_k\}_{k\geq 0}$ which are useful to proof our theorem.
The characteristic equation of Padovan sequence is

$$x^3 - x - 1 = 0,$$

has roots $\alpha, \beta, \gamma = \overline{\beta}$, where

$$\alpha = \frac{r_1 + r_2}{6}, \quad \beta = \frac{-r_1 - r_2 + i\sqrt{3}(r_1 - r_2)}{12},$$

and

$$r_1 = \sqrt[3]{108 + 12\sqrt{69}} \text{ and } r_2 = \sqrt[3]{108 - 12\sqrt{69}}.$$

Further, Binet's formula is

$$P_k = a\alpha^k + b\beta^k + c\gamma^k, \quad \text{for all } k \geq 0, \tag{2.1}$$

where

$$\begin{aligned} a &= \frac{(1-\beta)(1-\gamma)}{(\alpha-\beta)(\alpha-\gamma)} = \frac{1+\alpha}{-\alpha^2+3\alpha+1}, \\ b &= \frac{(1-\alpha)(1-\gamma)}{(\beta-\alpha)(\beta-\gamma)} = \frac{1+\beta}{-\beta^2+3\beta+1}, \\ c &= \frac{(1-\alpha)(1-\beta)}{(\gamma-\alpha)(\gamma-\beta)} = \frac{1+\gamma}{-\gamma^2+3\gamma+1} = \overline{b}. \end{aligned} \tag{2.2}$$

Numerically, we have

$$\begin{aligned} &1.32 < \alpha < 1.33, \\ &0.86 < |\beta| = |\gamma| = \alpha^{-1/2} < 0.87, \\ &0.72 < a < 0.73, \\ &0.24 < |b| = |c| < 0.25. \end{aligned} \tag{2.3}$$

Using induction, we can prove that

$$\alpha^{k-2} \leq \mathcal{P}_k \leq \alpha^{k-1}, \tag{2.4}$$

for all $k \geq 4$.

On the other hand, let $(\delta, \eta) = \left(\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\right)$ be the roots of the characteristic equation $x^2 - x - 1 = 0$ of the Fibonacci sequence $\{F_k\}_{n\geq 0}$. The Binet formula for F_k

$$F_k = \frac{\delta^k - \eta^k}{\sqrt{5}} \quad \text{holds for all } k \geq 0. \tag{2.5}$$

This implies easily that the inequality

$$\delta^{k-2} \leq F_k \leq \delta^{k-1} \tag{2.6}$$

holds for all positive integers k.

2.2. A lower bound for linear forms in logarithms

The next tools are related to the transcendental approach to solve Diophantine equations. For a non-zero algebraic number γ of degree d over $\mathbb{Q}$, whose minimal polynomial over $\mathbb{Z}$ is $a\prod_{j=1}^{d}\left(X - \gamma^{(j)}\right)$, we denote by

$$h(\gamma) = \frac{1}{d}\left(\log|a| + \sum_{j=1}^{d} \log\max\left(1, \left|\gamma^{(j)}\right|\right)\right)$$

the usual absolute logarithmic height of γ.

Lemma 2.1. *Let $\gamma_1, \ldots, \gamma_s$ be a real algebraic numbers and let $b_1, \ldots, b_s$ be nonzero rational integer numbers. Let D be the degree of the number field $\mathbb{Q}(\gamma_1, \ldots, \gamma_s)$ over $\mathbb{Q}$ and let A_j be a positive real number satisfying*

$$A_j = \max\{Dh(\gamma_j), |\log\gamma_j|, 0.16\} \text{ for } j = 1, \ldots, s.$$

Assume that

$$B \geq \max\{|b_1|, \ldots, |b_s|\}.$$

If $\gamma_1^{b_1} \cdots \gamma_s^{b_s} \neq 1$, then

$$|\gamma_1^{b_1} \cdots \gamma_s^{b_s} - 1| \geq \exp(-C(s, D)(1 + \log B)A_1 \cdots A_s),$$

where $C(s, D) := 1.4 \cdot 30^{s+3} \cdot s^{4.5} \cdot D^2(1 + \log D)$.

2.3. A generalized result of Baker-Davenport

Lemma 2.2. *Assume that τ and μ are real numbers and M is a positive integer. Let p/q be the convergent of the continued fraction of the irrational τ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let $\varepsilon = ||\mu q|| - M \cdot ||\tau q||$, where $|| \cdot ||$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there is no solution of the inequality*

$$0 < m\tau - n + \mu < AB^{-k}$$

in positive integers m, n and k with

$$m \leq M \quad \text{and} \quad k \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

3. Proof of Theorem 1.1

Assume that there exist positive integers n, m, n_1, m_1 such that $(n, m) \neq (n_1, m_1)$, and

$$F_n - \mathcal{P}_m = F_{n_1} - \mathcal{P}_{m_1}.$$

Because of the symmetry, we can assume that $m \geq m_1$. If $m = m_1$, then $F_n = F_{n_1}$, so $(n, m) = (n_1, m_1)$, contradicting our assumption. Thus, $m > m_1$. Since

$$F_n - F_{n_1} = \mathcal{P}_m - \mathcal{P}_{m_1}, \tag{3.1}$$

and the right-hand side is positive, we get that the left-hand side is also positive and so $n > n_1$. Thus, $n \geq 2$ and $n_1 \geq 1$. Using the Binet's formulas (2.5) and (2.1), the equation (3.1) implies that

$$\delta^{n-4} \leq F_{n-2} \leq F_n - F_{n_1} = \mathcal{P}_m - \mathcal{P}_{m_1} < \alpha^{m-1}, \tag{3.2a}$$

$$\delta^{n-1} \geq F_n > F_n - F_{n_1} = \mathcal{P}_m - \mathcal{P}_{m_1} = \mathcal{P}_{m-5} \geq \alpha^{m-7}, \tag{3.2b}$$

therefore

$$1 + \left(\frac{\log \alpha}{\log \delta}\right)(m-1) < n < \left(\frac{\log \alpha}{\log \delta}\right)(m-7) + 4, \tag{3.3}$$

where $\dfrac{\log \alpha}{\log \delta} = 0.5843\ldots$. If $n < 300$, then $m \leq 190$. We ran a computer program for $2 \leq n_1 < n \leq 300$ and $1 \leq m_1 < m < 190$ and found only the solutions listed in the (3.2) at the end of the paper. From now, we assume that $n \geq 300$.

Note that the inequality (3.3) implies that $m < 2n$. So, to solve equation (3.1), we need an upper bound for n.

3.1. Bounding n

Note that using the numerical inequalities (2.3) we have

$$\frac{|\eta|^n}{\sqrt{5}} + \frac{|\eta|^{n_1}}{\sqrt{5}} + |b||\beta|^m + |c||\gamma|^m + |b||\beta|^{m_1} + |c||\gamma|^{m_1} < 1.9. \tag{3.4}$$

Using the Binet formulas in the Diophantine equation (3.1), we get

$$\begin{aligned}
\left|\frac{\delta^n}{\sqrt{5}} - a\alpha^m\right| &= \left|\frac{\eta^n}{\sqrt{5}} + \frac{\delta^{n_1} - \eta^{n_1}}{\sqrt{5}} + (b\beta^m + c\gamma^m) - (a\alpha^{m_1} + b\beta^{m_1} + c\gamma^{m_1})\right| \\
&\leq \frac{\delta^{n_1}}{\sqrt{5}} + a\alpha^{m_1} + \frac{|\eta|^n}{\sqrt{5}} + \frac{|\eta|^{n_1}}{\sqrt{5}} + |b||\beta|^m + |c||\gamma|^m + |b||\beta|^{m_1} + |c||\gamma|^{m_1} \\
&< \frac{\delta^{n_1}}{\sqrt{5}} + a\alpha^{m_1} + 1.9 \\
&< 3.08 \max\{\delta^{n_1}, \alpha^{m_1}\}.
\end{aligned}$$

Dividing through by $a\alpha^m$ and using the relation (3.2a), we obtain

$$\left|(\sqrt{5}a)^{-1}\delta^n\alpha^{-m}-1\right| < \max\left\{\frac{3.08}{a\alpha^m}\delta^{n_1}, \frac{3.08}{a}\alpha^{m_1-m}\right\} < \max\left\{3.24\frac{\delta^{n_1}}{\delta^{n-4}}, 4.28\alpha^{m_1-m}\right\}.$$

Hence, we get

$$\left|(\sqrt{5}a)^{-1}\delta^n\alpha^{-m}-1\right| < \max\{\delta^{n_1-n+6}, \alpha^{m_1-m+3}\}. \tag{3.5}$$

For the left-hand side, we apply Theorem 2.1 with the data

$$s=3, \quad \gamma_1=\sqrt{5}a, \quad \gamma_2=\delta, \quad \gamma_3=\alpha, \quad b_1=-1, \quad b_2=n, \quad b_3=-m.$$

Throughout we work with $\mathbb{K} := \mathbb{Q}(\sqrt{5}, \alpha)$ with $D=6$. Since $\max\{1,n,m\} \le 2n$ we take $B := 2n$. We have

$$h(\gamma_2)=\frac{\log\delta}{2} \quad \text{and} \quad h(\gamma_3)=\frac{\log\alpha}{3}.$$

Further, the minimal polynomial of γ_1 is $529x^6-1265x^4-250x^2-125$, then

$$h(\gamma_1)\approx 1.204.$$

Thus, we can take

$$A_1=7.23, \quad A_2=3\log\delta, \quad A_3=2\log\alpha.$$

Put

$$\Lambda=(\sqrt{5}a)^{-1}\delta^n\alpha^{-m}-1.$$

If $\Lambda=0$, then $\delta^n(\alpha^{-1})^m=\sqrt{5}a$, which is false, since $\delta^n(\alpha^{-1})^m \in \mathcal{O}_{\mathbb{K}}$ whereas $\sqrt{5}a$ does not, as can be observed immediately from its minimal polynomial. Thus, $\Lambda \neq 0$. Then, by Lemma 2.1, the left-hand side of (3.5) is bounded as

$$\log|\Lambda| > -1.4\cdot 30^6\cdot 3^{4.5}\cdot 6^2(1+\log 6)(1+\log 2n)(7.23)(3\log\delta)(2\log\alpha).$$

Comparing with (3.5), we get

$$\min\{(n-n_1-6)\log\delta, (m-m_1-3)\log\alpha\} < 8.45\times 10^{13}(1+\log 2n),$$

wich gives

$$\min\{(n-n_1)\log\delta, (m-m_1)\log\alpha\} < 8.45\times 10^{13}(1+\log 2n).$$

Now the argument splits into two cases.

Case 1. $\min\{(n-n_1)\log\delta, (m-m_1)\log\alpha\}=(n-n_1)\log\delta$.

In this case, we rewrite (3.1) as

$$\left|\left(\frac{\delta^{n-n_1}-1}{\sqrt{5}}\right)\delta^{n_1}-a\alpha^m\right| = \left|-a\alpha^{m_1}+\frac{\eta^n}{\sqrt{5}}-\frac{\eta^{n_1}}{\sqrt{5}}+(b\beta^m+c\gamma^m)-(b\beta^{m_1}+c\gamma^{m_1})\right|$$

by using (3.4) and dividing by α^m, we obtain

$$\left|\left(\frac{\delta^{n-n_1}-1}{\sqrt{5}a}\right)\delta^{n_1}\alpha^{-m}-1\right| < 3.65\alpha^{m_1-m}. \tag{3.6}$$

We put

$$\Lambda_1 = \left(\frac{\delta^{n-n_1}-1}{a\sqrt{5}}\right)\delta^{n_1}\alpha^{-m} - 1.$$

Clearly, $\Lambda_1 \neq 0$, for if $\Lambda_1 = 0$, then $\delta^n - \delta^{n_1} = \sqrt{5}a\alpha^m$. This is impossible if $\sqrt{5}a\alpha^m \in \mathbb{Q}(\sqrt{5},\alpha)$ but $\notin \mathbb{Q}(\sqrt{5})$. Therefore, let us assume that $\sqrt{5}a\alpha^m \in \mathbb{Q}(\sqrt{5})$. Since $a\alpha^m \in \mathbb{Q}(\alpha)$ and $\mathbb{Q}(\alpha)\cap\mathbb{Q}(\sqrt{5}) = \mathbb{Q}$, we deduce from $\sqrt{5}a\alpha^m \in \mathbb{Q}(\sqrt{5})$ that we have $\sqrt{5}a\alpha^m = y\sqrt{5}$ for some $y \in \mathbb{Q}$. Let $\sigma \neq id$ be the unique non trivial $\mathbb{Q}$-automorphism over $\mathbb{Q}(\sqrt{5})$. Then, we get

$$\delta^n - \delta^{n_1} = \sqrt{5}a\alpha^m = y\sqrt{5} = -\sigma(\sqrt{5}a\alpha^m) = -\sigma(\delta^n - \delta^{n_1}) = \eta^{n_1} - \eta^n.$$

However, the absolute value of the left-hand side is at least $\delta^n - \delta^{n_1} \geq \delta^{n-2} \geq \delta^{\cdots} > 2$, while the absolute value of right-hand side is at most $|\eta^{n_1} - \eta^n| \leq |\eta|^{n_1} + |\eta|^n < 2$. By this obvious contradiction we conclude that $\Lambda_1 \neq 0$.

We apply Lemma 2.1 by taking $s = 3$, and

$$\gamma_1 = \frac{\delta^{n-n_1}-1}{\sqrt{5}a}, \quad \gamma_2 = \delta, \quad \gamma_3 = \alpha, \quad b_1 = 1, \quad b_2 = n_1, \quad b_3 = -m.$$

On the other hand, the minimal polynomial of a is $23x^3 - 23x^2 + 6x - 1$ and has roots a, b, c. Since $|b| = |c| < 1$ and $a < 1$, then $h(a) = \dfrac{\log 23}{3}$.

Thus, we obtain

$$\begin{aligned} h(\gamma_1) &\leq h\left(\frac{\delta^{n-n_1}+1}{\sqrt{5}}\right) + h(a) \\ &\leq (n-n_1)h(\delta) + h(\sqrt{5}) + h(a) + \log(2) \\ &< \frac{1}{2}(n-n_1)\log\delta + \log(\sqrt{5}) + \frac{\log 23}{3} + \log(2) \\ &< 4.22 \times 10^{13} \cdot (1 + \log 2n). \end{aligned} \tag{3.7}$$

So, we can take $A_1 := 2.53 \times 10^{14}(1 + \log 2n)$. Further, as before, we can take $A_2 := 3\log\delta$ and $A_3 := 2\log\alpha$. Finally, since $\max\{1, n_1, m\} \leq 2n$, we can take $B := 2n$. We then get that

$$\log|\Lambda_1| > -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 6^2(1+\log 6)(1+\log 2n) \times (2.53 \times 10^{14}(1+\log 2n))(3\log\delta)(2\log\alpha).$$

Thus,

$$\log|\Lambda_1| > -2.96 \cdot 10^{27}(1 + \log 2n)^2.$$

Comparing this with (3.6), we get that

$$(m - m_1)\log\alpha < 2.96 \cdot 10^{27}(1 + \log 2n)^2.$$

Case 2. $\min\{(n-n_1)\log\delta, (m-m_1)\log\alpha\} = (m-m_1)\log\alpha$.

In this case, we rewrite (3.1) as

$$\left|\frac{\delta^n}{\sqrt{5}} - a\alpha^m + a\alpha^{m_1}\right| = \left|\frac{\eta^n}{\sqrt{5}} + \frac{\delta^{n_1} - \eta^{n_1}}{\sqrt{5}} + (b\beta^m + c\gamma^m) - (b\beta^{m_1} + c\gamma^{m_1})\right|$$

so

$$\left|\frac{\delta^n\alpha^{-m_1}}{\sqrt{5}a(\alpha^{m-m_1}-1)} - 1\right| < \frac{2.35}{\sqrt{5}a(1-\alpha^{m_1-m})\alpha}\frac{\delta^{n_1}}{\alpha^{m-1}} < 17\delta^{n_1-n+4}. \tag{3.8}$$

Let

$$\Lambda_2 = (\sqrt{5}a(\alpha^{m-m_1}-1))^{-1}\delta^n\alpha^{-m_1} - 1.$$

Clearly, $\Lambda_2 \neq 0$, for if $\Lambda_2 = 0$ implies $\delta^{2n} = 5\alpha^{2m_1}a^2(\alpha^{m-m_1}-1)^2$.,However, $\delta^{2n} \in \mathbb{Q}(\sqrt{5})\backslash\mathbb{Q}$, whereas $5\alpha^{2m_1}a^2(\alpha^{m-m_1}-1)^2 \in \mathbb{Q}(\alpha)$, which is not possible.

We apply again Lemma 2.1. In this application, we take again $s = 3$, and

$$\gamma_1 = \sqrt{5}a(\alpha^{m-m_1}-1), \quad \gamma_2 = \delta, \quad \gamma_3 = \alpha, \quad b_1 = -1, \quad b_2 = n, \quad b_3 = -m_1.$$

We have

$$\begin{aligned} h(\alpha^{m-m_1}-1) &\leq h(\alpha^{m-m_1}) + h(-1) + \log 2 = (m-m_1)h(\alpha) + \log 2 \\ &= \frac{(m-m_1)\log\alpha}{3} + \log 2 < 9.51 \times 10^{13}(1+\log 2n). \end{aligned}$$

Thus, we obtain

$$\begin{aligned} h(\gamma_1) &< 2.82 \times 10^{13}(1+\log 2n) + \frac{\log 23}{3} + \log\sqrt{5} \\ &< 2.82 \times 10^{13}(1+\log 2n). \end{aligned}$$

So, we can take $A_1 := 1.69 \times 10^{14}(1+\log 2n)$. Further, as before, we can take $A_2 := 3\log\delta$ and $A_3 := 2\log\alpha$. Finally, since $\max\{1, n, m_1+1\} \leq 2n$, we can take $B := 2n$.

We then get that

$$\log|\Lambda_2| > -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 6^2(1+\log 6)(1+\log 2n) \times (1.69 \times 10^{14}(1+\log 2n))(3\log\delta)(2\log\alpha).$$

Thus,

$$\log|\Lambda_1| > -1.97 \cdot 10^{27}(1+\log 2n)^2.$$

Comparing this with (3.8), we get that

$$(n-n_1)\log\delta < 1.97 \cdot 10^{27}(1+\log 2n)^2.$$

Thus, in both Case 1 and Case 2, we have

$$\min\{(n-n_1)\log\delta, (m-m_1)\log\alpha\} < 8.45 \times 10^{13}(1+\log 2n) \tag{3.9a}$$

$$\max\{(n-n_1)\log\delta, (m-m_1)\log\alpha\} < 2.96 \cdot 10^{27}(1+\log 2n)^2. \tag{3.9b}$$

We now finally rewrite equation (3.1) as

$$\left|\frac{\delta^n}{\sqrt{5}} - \frac{\delta^{n_1}}{\sqrt{5}} - a\alpha^m + a\alpha^{m_1}\right| = \left|\frac{\delta^n}{\sqrt{5}} - \frac{\delta^{n_1}}{\sqrt{5}} + (b\beta^m + c\gamma^m) - (b\beta^{m_1} + c\gamma^{m_1})\right| < 1.9.$$

Dividing both sides by $a\alpha^{m_1}(\alpha^{m-m_1}-1)$, we get

$$\left|\left(\frac{\delta^{n-n_1}-1}{\sqrt{5}a(\alpha^{m-m_1}-1)}\right)\delta^{n_1}\alpha^{-m_1} - 1\right| < \frac{5.84}{a(1-\alpha^{m_1-m})\alpha}\frac{1}{\alpha^{m-1}} < 13.8\delta^{4-n}. \tag{3.10}$$

To find a lower-bound on the left-hand side, we use again Lemma 2.1 with $s = 3$, and

$$\gamma_1 = \frac{\delta^{n-n_1}-1}{\sqrt{5}a(\alpha^{m-m_1}-1)}, \quad \gamma_2 = \delta, \quad \gamma_3 = \alpha, \quad b_1 = 1, \quad b_2 = n_1, \quad b_3 = -m_1.$$

Using $h(x/y) = h(x) + h(y)$ for any two nonzero algebraic numbers x and y, we have

$$\begin{aligned} h(\gamma_1) &\leq h\left(\frac{\delta^{n-n_1}-1}{\sqrt{5}a}\right) + h(\alpha^{m-m_1}-1) \\ &< \frac{1}{2}(n-n_1)\log\delta + \log\sqrt{5} + \frac{\log 23}{3} + \frac{(m-m_1)\log\alpha}{3} + \log 2 \\ &< 2.47 \cdot 10^{27}(1+\log 2n)^2, \end{aligned}$$

where in the above chain of inequalities, we used the argument from (3.7) as well as the bound (3.9b). So, we can take $A_1 := 1.78 \cdot 10^{28}(1+\log 2n)^2$ and certainly $A_2 := 3\log\delta$ and $A_3 := 2\log\alpha$. Using similar arguments as in the proof that $\Lambda_1 \neq 0$ we show that if we put

$$\Lambda_3 = \left(\frac{\delta^{n-n_1}-1}{\sqrt{5}a(\alpha^{m-m_1}-1)}\right)\delta^{n_1}\alpha^{-m_1} - 1,$$

then $\Lambda_3 \neq 0$. Lemma 2.1 gives

$$\log|\Lambda_3| > -1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 6^2(1+\log 6)(1+\log 2n) \times (1.78 \cdot 10^{28}(1+\log 2n)^2)(3\log\delta)(2\log\alpha),$$

which together with (3.10) gives

$$(n-4) < 2.08 \cdot 10^{41}(1+\log 2n)^3,$$

leading to $n < 2.83 \cdot 10^{47}$.

3.2. Reducing n

We now need to reduce the above bound for n and to do so we make use of Lemma 2.2 several times and each time $M := 2.83 \cdot 10^{47}$. To begin with, we return to (3.5) and put

$$\Gamma := n\log\delta - m\log\alpha - \log(\sqrt{5}a).$$

For technical reasons we assume that $\min\{n-n_1, m-m_1\} \geq 20$. We go back to the inequalities for $\Lambda,\ \Lambda_1,\ \Lambda_2$. Since we assume that $\min\{n-n_1, m-m_1\} \geq 20$ we get $|e^\Gamma - 1| = |\Lambda| < \frac{1}{4}$. Hence, $|\Lambda| < \frac{1}{2}$ and since the inequality $|x| < 2|e^x - 1|$ holds for all $x \in \left(-\frac{1}{2}, \frac{1}{2}\right)$, we get

$$|\Gamma| < 2\max\{\delta^{n_1-n+6}, \alpha^{m_1-m+3}\} \leq \max\{\delta^{n_1-n+8}, \alpha^{m_1-m+6}\}.$$

Assume $\Gamma > 0$. We then have the inequality

$$\begin{aligned} 0 < n\left(\frac{\log\delta}{\log\alpha}\right) - m - \frac{\log(1/(\sqrt{5}a))}{\log\alpha} &< \max\left\{\frac{\delta^8}{\log\alpha}\delta^{-(n-n_1)}, \frac{\alpha^6}{\log\alpha}\alpha^{-(m-m_1)}\right\} \\ &< \max\{170 \cdot \delta^{-(n-n_1)}, 20 \cdot \alpha^{-(m-m_1)}\}. \end{aligned}$$

We apply Lemma 2.2 with

$$\tau = \frac{\log\delta}{\log\alpha}, \quad \mu = \frac{\log(1/(\sqrt{5}a))}{\log\alpha}, \quad (A, B) = (170, \delta) \text{ or } (20, \alpha).$$

Let $\tau = [a_0, a_1, \ldots] = [1; 1, 2, 2, 6, 2, 1, 2, 1, 2, 1, 1, 11, 1, 2, 3, 1, 7, 37, 4, \ldots]$ be the continued fraction of τ. We choose consider the 98-th convergent

$$\frac{p}{q} = \frac{p_{98}}{q_{98}} = \frac{780930677042238317990327545345035018596353914355 17}{4563424307638745709704652808420849014759496830 8975}.$$

If satisfied $q = q_{98} > 6M$. Further, it yields $\varepsilon > 0.35$, and therefore either

$$n - n_1 \leq \frac{\log(170q/\varepsilon)}{\log\delta} < 250, \text{ or } m - m_1 \leq \frac{\log(20q/\varepsilon)}{\log\alpha} < 420.$$

In the case of $\Gamma < 0$, we consider the following inequality instead:

$$\begin{aligned} m\left(\frac{\log\alpha}{\log\delta}\right) - n + \frac{\log(\sqrt{5}a)}{\log\delta} &< \max\left\{\frac{\delta^9}{\log\delta}\alpha^{-(n-n_1)}, \frac{\alpha^{12}}{\log\delta}\alpha^{-(m-m_1)}\right\} \\ &< \max\{98 \cdot \delta^{-(n-n_1)}, 12 \cdot \alpha^{-(m-m_1)}\}, \end{aligned}$$

instead and apply Lemma 2.2 with

$$\tau = \frac{\log \alpha}{\log \delta}, \quad \mu = \frac{\log(\sqrt{5}a)}{\log \delta}, \quad (A, B) = (98, \delta) \text{ or } (12, \alpha).$$

Let $\tau = [a_0, a_1, \ldots] = [0; 1, 1, 2, 2, 6, 2, 1, 2, 1, 2, 1, 1, 11, 1, 2, 3, 1, 7, 37, \ldots]$ be the continued fraction of τ (note that the current τ is just the reciprocal of the previous τ). We consider the 98-th convergent

$$\frac{p}{q} = \frac{p_{98}}{q_{98}} = \frac{1000540334879242934726141761162813294034885977722}{1712206861451396832387596141129961335575127483549}.$$

which satisfies $q = q_{98} > 6M$. This yields again $\varepsilon > 0.47$, and therefore either

$$n - n_1 \leq \frac{\log(98q/\varepsilon)}{\log \delta} < 242, \text{ or } m - m_1 \leq \frac{\log(12q/\varepsilon)}{\log \alpha} < 406.$$

These bounds agree with the bounds obtained in the case that $\Gamma > 0$. As a conclusion, we have either $n - n_1 \leq 250$ or $m - m_1 \leq 420$ whenever $\Gamma \neq 0$.

Now, we have to distinguish between the cases $n - n_1 \leq 250$ and $m - m_1 \leq 420$. First, let assume that $n - n_1 \leq 250$. In this case, we consider inequality (3.6) and assume that $m - m_1 \geq 20$. We put

$$\Gamma_1 = n_1 \log \delta - m \log \alpha + \log\left(\frac{\delta^{n-n_1} - 1}{\sqrt{5}a}\right).$$

Then inequality (3.6) implies that

$$|\Gamma_1| < 7.3\alpha^{m_1 - m}.$$

If we further assume that $\Gamma_1 > 0$, we then get

$$0 < n_1 \left(\frac{\log \delta}{\log \alpha}\right) - m + \frac{\log((\delta^{n-n_1} - 1)/(\sqrt{5}a)}{\log \alpha} < 26 \cdot \alpha^{-(m-m_1)}.$$

Again we apply Lemma 2.2 with the same τ as in the case when $\Gamma > 0$. We use the 98-th convergent $p/q = p_{98}/q_{98}$ of τ as before. But in this case we choose $(A, B) := (26, \alpha)$ and use

$$\mu_k = \frac{\log((\delta^k - 1)/(\sqrt{5}a))}{\log \alpha},$$

instead of μ for each possible value of $k := n - n_1 \in [1, 2, \ldots 250]$. For the remaining values of k, we get $\varepsilon > 0.0004$. Hence, by Lemma 2.2, we get

$$m - m_1 < \frac{\log(26q/0.0004)}{\log \alpha} < 446.$$

Thus, $n - n_1 \leq 250$ implies $m - m_1 \leq 446$.

In the case that $\Gamma_1 < 0$ we follow the ideas from the case that $\Gamma_1 > 0$. We use the same τ as in the case that $\Gamma < 0$ but instead of μ we take

$$\mu_k = \frac{\log((\sqrt{5}a)/(\delta^k - 1))}{\log \delta},$$

for each possible value of $n - n_1 = k = 1, 2, \ldots, 250$. Using Lemma 2.2 with this setting we also obtain in this case that $n - n_1 \leq 250$ implies $m - m_1 \leq 429$.

Now let us turn to the case that $m - m_1 \leq 420$ and let us consider inequality (3.8). We put

$$\Gamma_2 = n \log \delta - m_1 \log \alpha + \log(1/(\sqrt{5}a(\alpha^{m-m_1} - 1))),$$

and we assume that $n-n_1 \geq 20$. We then have

$$|\Gamma_2| < \frac{34\delta^4}{\alpha^{n-n_1}}.$$

Assuming $\Gamma_2 > 0$, we get

$$0 < n\left(\frac{\log\delta}{\log\alpha}\right) - m_1 + \frac{\log((1/(\sqrt{5}a(\alpha^{m-m_1}-1)))}{\log\alpha} < \frac{34\delta^4}{(\log\alpha)\alpha^{n-n_1}} < 830\delta^{-(n-n_1)}.$$

We apply again Lemma 2.2 with the same τ, q, M, $(A,B) := (830, \delta)$ and

$$\mu_k = \frac{\log((1/(\sqrt{5}a(\alpha^k-1)))}{\log\alpha} \quad \text{for } k = 1, 2, \ldots 420.$$

We get $\varepsilon > 0.00077$, therefore

$$n - n_1 < \frac{\log(830q_{98}/0.00077)}{\log\delta} < 263.$$

A similar conclusion is reached when $\Gamma_2 < 0$. To conclude, we first get that either $n-n_1 \leq 250$ or $m-m_1 \leq 446$. If $n-n_1 \leq 250$, then $m-m_1 \leq 446$, and if $m-m_1 \leq 420$ then $n-n_1 \leq 263$. In conclusion, we always have $n-n_1 < 263$ and $m-m_1 < 446$.

Finally we go to (3.10). We put

$$\Gamma_3 = n_1\log\delta - m_1\log\alpha + \log\left(\frac{\delta^{n-n_1}-1}{\sqrt{5}a(\alpha^{m-m_1}-1)}\right).$$

Since $n \geq 300$, inequality (3.10) implies that

$$|\Gamma_3| < \frac{17}{\delta^{n-4}}.$$

Assume that $\Gamma_3 > 0$. Then

$$0 < n_1\left(\frac{\log\delta}{\log\alpha}\right) - m_1 + \frac{\log((\delta^k-1)/(\sqrt{5}a(\alpha^l-1)))}{\log\alpha} < 390\delta^n,$$

where $(k,l) := (n-n_1, m-m_1)$. We apply again Lemma 2.2 with the same τ, M, q, $(A,B) := (390, \delta)$ and

$$\mu_{k,l} = \frac{\log((\delta^k-1)/(\sqrt{5}a(\alpha^l-1)))}{\log\alpha} \quad \text{for } 1 \leq k \leq 264,\ 1 \leq l \leq 446.$$

We consider the 99th convergent $\frac{p_{99}}{q_{99}}$. For all pairs (k,l) we get that $\varepsilon > 2\times10^{-5}$. Thus, Lemma 2.2 yields that

$$n < \frac{\log(390\times q_{99}/\varepsilon)}{\log\delta} < 274.$$

Theorem 1.1 is therefore proved.

On the next page is presented the table that gives the couples for which we obtain the different representations of c on the form $\mathcal{P}_m - F_n = c$.

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c	(m,n)
-226	$(8,13),(19,14)$
-82	$(8,11),(19,13)$
-52	$(5,10),(14,11)$
-30	$(6,9),(18,12)$
-27	$(8,9),(13,10)$
-18	$(5,8),(11,9),(14,10)$
-9	$(6,7),(10,8)$
-6	$(4,6),(8,7),(13,9),(15,10)$
-5	$(5,6),(11,8)$
-4	$(6,6),(9,7)$
-3	$(4,5),(7,6),(17,11)$
-1	$(4,4),(6,5),(8,6),(10,7)$
0	$(4,3),(5,4),(7,5),(12,8)$
1	$(4,2),(5,3),(6,4),(9,6)$
2	$(5,2),(6,3),(7,4),(8,5)$
3	$(6,2),(7,3),(11,7),(14,9)$
4	$(7,2),(8,4),(9,5),(10,6)$
6	$(8,2),(9,4),(24,15)$
7	$(9,3),(10,5),(13,8),(19,12)$
8	$(9,2),(11,6),(12,7)$
10	$(10,3),(16,10)$
11	$(10,2),(11,5)$
13	$(11,4),(12,6)$
15	$(11,2),(13,7),(15,9)$
16	$(12,5),(14,8)$
20	$(12,2),(13,6)$
25	$(13,4),(18,11)$
31	$(16,9),(17,10)$
32	$(14,5),(21,13)$
36	$(14,2),(15,7)$
44	$(15,5),(16,8)$
52	$(16,7),(17,9)$
62	$(16,4),(19,11)$
111	$(18,4),(20,11)$
262	$(21,4),(22,11)$

Table 1: Representations

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