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Mild solutions for some nonautonomous evolution equations with state-dependent delay governed by equicontinuous evolution families

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This paper is dedicated to the occasion of Professor Gaston M. N'Guérékata's 70th birthday

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Abstract. In this work, we study the existence solutions and the dependence continuous with the initial data for some nondensely nonautonomous partial functional differential equations with state-dependent delay in Banach spaces. We assume that the linear part is not necessarily densely defined, satisfies the well-known hyperbolic conditions and generate a noncompact evolution family. Our existence results are based on Sadovskii fixed point Theorem. An application is provided to a reaction-diffusion equation with state-dependent delay.

AMS Subject Classifications: 35R10, 47J35.

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Contents

1. Introduction

Partial differential equations with delay are important for investigating some problems raised from natural phenomena. They have been successfully used to study a number of areas of biological, physical, engineering applications, and such equations have received much attention in recent years. It is generally known that taking into account the past states of the model, in addition to the present one, makes the model more realistic. This leads to the so called functional differential equations. In recent years, nonlinear evolution equations with

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state-dependent delay have been studied by several authors and some interesting results have been obtained, see [9–11, 14, 15, 17, 19].

In 1970, Kato in [12] initiated a study of the evolution family solution of hyperbolic linear evolution equations of the form

$$\begin{cases} x'(t) = A(t)x(t), & t \geq s, \\ x(s) = x_s \in X. \end{cases} \tag{1.1}$$

in a Banach space X. Some fundamental and basic results about the well posedness and dynamical behavior of equation (1.1) were established under the so called stability condition, ((B_2) in Section 2). The autors focus on the nonautonomous linear case.

In 2011, Belmekki et al investigated in [5] several results on the existence of solutions of the initial value problem for a new class of abstract evolution equations with state-dependent delay in Banach space X,

$$\begin{cases} x'(t) = Ax(t) + f(t, x(t - \rho(x(t)))), & t \in [0; a], \\ x(t) = \varphi(t), & t \in [-r; 0]. \end{cases} \tag{1.2}$$

where $f : [0; +\infty) \times X \longrightarrow X$ is a suitable nonlinear function, the initial data $\varphi : [-r; 0] \longrightarrow X$ is a continuous function, ρ is a positive bounded continuous function defined on X and r is the maximal delay given by $r = \sup\limits_{x \in X} \rho(x)$. The autors focus on the case where the differential operator in the main part is nondensely define and independent of time t in $[0, a]$. Here the equation is autonomous partial functional differential equations with state-dependent delay. Their approach is based on a nonlinear alternative of Leray-Schauder and integrated semigroup $(S(t))_{t\geq}$ which is considered to be compact for $t > 0$.

In 2019, Kpoumie et al. investigated in [15] several results on the existence of solutions of the following nonautomous equations:

$$\begin{cases} x'(t) = A(t)x(t) + f(t, x(t - \rho(x(t)))), & t \in [0; a], \\ x(t) = \varphi(t), & t \in [-r; 0]. \end{cases} \tag{1.3}$$

in a Banach space $(X, \|.\|)$, where the family of closed linear operator $(A(t))_{t\geq 0}$ on X is not necessarily densely, satisfying the hyperbolic conditions (B_1) through (B_3) and $\varphi : [-r; 0] \longrightarrow X$ the continuous function. Their approach is based on a nonlinear alternative of Leray-Schauder under the assumption of the compactness of evolution family generated by $(A(t))_{t\geq 0}$. They get the existence of mild solution under the Carathéodory condition on f.

In 2019, Chen and al. investigated in [7] several results on the existence of solutions of the nonautonomous parabolic evolution equations with non-instantaneous impulses in Banach space E:

$$\begin{cases} x'(t) - A(t)x(t) = f(t, x(t)), & t \in \bigcup_{k=0}^{m} (s_k, t_{k+1}], \\ x(t) = \gamma_k(t, x(t)), & t \in \bigcup_{k=1}^{m} (t_k, s_k], \\ x(0) = x_0). \end{cases} \tag{1.4}$$

by introducing the concepts of mild and classical solutions, where $A : D(A) \subset E \longrightarrow E$ is the generator of a $C_0 - semigroup$ of bounded linear operator $T(t)_{t\geq 0}$ defined on E, $u_0 \in E,\ 0 < t_1 < t_2 < \cdots < t_m < t_{m+1} := a,\ a > 0$ is a constant, $s_0 := 0$ and $s_k \in (t_k, t_{k+1})$ for each $k = 1, 2, \cdots, m,\ f : [0, a] \times E \longrightarrow E$ is a suitable nonlinear function, $\gamma_k : (t_k, s_k] \times E \longrightarrow E$ is continuous non-instantaneous impulsive function for all $k = 1, 2, \cdots, m$. Their results are based on Sadovskii fixed point

Theorem and they consider that evolution family is noncompact.

Therefore, it is for great significance and interesting to study the nonautonomous evolution equation where the family of closed linear operator $(A(t))_{t\geq 0}$ on X is not necessarily densely define and generates the noncompact evolution famillies. Driven by the above aspects, we will investigate the existence of mild solutions and the dependent continuous on the initial data of the following nonautonomous partial functional differential equations with state-dependent delay governed by noncompact evolution families of the form

$$\begin{cases} x'(t) = A(t)x(t) + f(t, x(t-\rho(x(t)))), & t \geq 0, \\ x(t) = \varphi(t), & t \in [-r; 0]. \end{cases} \tag{1.5}$$

in a Banach space $(X, \|.\|)$, where the family of closed linear operator $(A(t))_{t\geq 0}$ on X is not necessarily densely and satisfying the hyperbolic conditions (B_1) through (B_3) introduced by Kato in [12] that will be specified later. $f : [0; +\infty) \times X \longrightarrow X$ is a suitable nonlinear function satisfying some conditions which will be specified later. The initial data $\varphi : [-r; 0] \longrightarrow X$ is a continuous function and ρ is a positive bounded continuous function on X. The constant r is the maximal delay defined by $r = \sup\limits_{x\in X} \rho(x)$.

We point out that the work of this paper is the following of [5, 7, 12, 15]. But under appropriate circonstances, evolutionary families are not compact. Our work is organized as follows: First, we recall some preliminary results about the evolution family generated by $(A(t))_{t\geq 0}$ and recall also some preliminary results concern Kuratowski measure. Second, we use the alternative of Sadovskii fixed point Theorem to prove the existence of at least one mild solution and the dependent continuous on initial data. Third, we propose an application to illustrate the main result.

2. Preliminary results

Our notations in this section are the usual in the theory of evolution equations. In particular, we denote by $\mathcal{C}(E, F)$ the space of continuous functions from E into F and $\mathcal{C}^2(E, F)$ denotes the space of twice continuously differentiable functions from E into F.
We mention here some results on nonautonomous differential equations with nondense domaine. We cite [12, 13, 16, 18, 19]. We recall some properties and Theorems.

In the whole of this work, we assume the following hyperbolic assumptions:
(B_1) $D(A(t)) := D$ independent of t and not necessarily densely defined ($\overline{D} \subsetneq X$) .
(B_2) The family $(A(t)_{t\geq 0}$ is stable in the sense that there are constants $M \geq 1$ and $w \in \mathbb{R}$ such that $(\omega, +\infty) \subset \varrho(A(t))$ (resolvent set of A(t)) for $t \in [0, +\infty)$ and

$$\|\prod_{j=1}^{k} R(\lambda, A(t_j))\| \leq M(\lambda - \omega)^{-k}$$

for $\lambda > \omega$ and every finite sequence $\{t_j\}_{j=1}^k$ with $0 \leq t_1 \leq t_2 \leq \ \leq t_k$ and $k = 1, 2, ...$
(B_3) The mapping $t \longmapsto A(t)x$ is continuously differentiable in X for all $x \in D$.
We follow by recall the classical result which gives us the existence and explicit formula of the evolution family generated by $(A(t))_{t\geq 0}$ due to Kato [12]. Let $\lambda > 0$, $0 \leq s \leq t$ and $x \in \overline{D}$,

$$U_\lambda(t, s)x = \prod_{i=[\frac{s}{\lambda}]+1}^{[\frac{t}{\lambda}]} (I - \lambda A(i\lambda))^{-1} x.$$

Theorem 2.1. *[1, 12] Assuming the three conditions $(B_1) - (B_3)$. Then the limit*

$$U(t,s)x = \lim_{\lambda \longrightarrow 0^+} U_\lambda(t,s)x \tag{2.1}$$

exists for $x \in \overline{D}$ and $0 \leq s \leq t$, where the convergence is uniform on $\Gamma := \{(t,s) : 0 \leq s \leq t\}$. Moreover, the family $\{U(t,s) : (t,s) \in \Gamma\}$ satisfies the following properties:

(i) *$U(t,s)D(s) \subset D(t)$ for all $0 \leq s \leq t$, where $D(t)$ is defined by*

$$D(t) := \{x \in D : A(t)x \in D\}$$

,

(ii) *$U(t,s) : \overline{D} \longrightarrow \overline{D}$ for $(t,s) \in \Gamma$*

(iii) *$U_\lambda(t,t)x = x$ and $U_\lambda(t,s)x = U_\lambda(t,r)U_\lambda(r,s)x$ for $x \in \overline{D}$, $\lambda > 0$ and $0 \leq s \leq r \leq t$*

(iv) *$U(t,t)x = x$ and $U(t,s)x = U(t,r)U(r,s)x$ for $x \in \overline{D}$ and $0 \leq s \leq r \leq t$,*

(v) *the mapping $(t,s) \longmapsto U(t,s)x$ is continuous on Γ for any $x \in \overline{D}$,*

(vi) *for all $x \in D(s)$ and $t \geq s$, the function $t \longmapsto U(t,s)x$ is continuously differentiable with $\dfrac{\partial}{\partial t}U(t,s)x = A(t)U(t,s)x$, and $\frac{\partial^+}{\partial s}U(t,s)x = -U(t,s)A(s)x$.*

Corollary 2.2. *[1] Assume the condition (B_2). Then there exists $M \geq 1$ and $\omega \in \mathbb{R}$ such that*

$$\left\| U(t,s)x \right\| \leq Me^{\omega(t-s)}\|x\|, \qquad \text{for } x \in \overline{D} \text{ and } 0 \leq s \leq t.$$

Remark 2.3. *Since (B_2), $\lambda > \omega$ and hence for (2.1), we get that ω is non positive. And by using Corollary 2.2, we have $\|U(t,s)x\| \leq M\|x\|$ for each $x \in \overline{D}$ and $0 \leq s \leq t$.*

Definition 2.4. *[4, 7] An evolution family $\{U(t,s) : 0 \leq s \leq t \leq a\}$ is said to be equicontinuous if for any $s \geq 0$, the function $t \longmapsto U(t,s)$ is continuous by operator norm for $t \in (s; +\infty)$.*

In the following, we give some results on the existence of solutions for the following nondensely nonautonomous partial functional differential equation

$$\begin{cases} x'(t) = A(t)x(t) + f(t), & t \in [0;a], \\ x(0) = x_0. \end{cases} \tag{2.2}$$

where $f : [0,a] \longrightarrow X$ is a function. The following Theorem gives us the generalized variation of constants formula of equation (2.2).

Theorem 2.5. *[9] Let $x_0 \in \overline{D}$ and $f \in L^1([0,a]; X)$. Then the limit*

$$x(t) := U(t,0)x_0 + \lim_{\lambda \longrightarrow 0^+} \int_0^t U_\lambda(t,r)f(r)dr \tag{2.3}$$

exists uniformly for $t \in [0;a]$, x is a continuous function on $[0,a]$ and

$$\|x(t)\| \leq Me^{\omega t}\|x_0\| + \int_0^t Me^{\omega(t-s)}\|f(s)\|ds \leq M\|x_0\| + \int_0^t M\|f(s)\|ds \tag{2.4}$$

.

Definition 2.6. *[12] For $x_0 \in \overline{D}$, a continuous function $x : [0, a] \longrightarrow X$ is called a mild solution of equation (2.2) if it satisfies the equation (2.3).*

We introduce some basic definitions and properties of the Kuratowski noncompactness measure, this will be used to demonstrate our main result.

Definition 2.7. *[4, 7] The Kuratowski measure of noncoampactness $\mu(.)$ defined on bounded set V of Banach space E is*

$$\mu(V) := \inf\{\delta > 0 : V = \bigcup_{i=1}^{m} V_i \; and \; diam(V_i) \leq \delta \; for \; i = 1, 2, ..., m\}.$$

Definition 2.8. *[4, 7] Consider a Banach space X, and a nonempty subset E of X. A continuous operator $G : E \longrightarrow X$ is called to be $\lambda-$set-contractive if there exists a constant $\lambda \in [0; 1)$ such that, for every bounded set $B \subset E$,*

$$\mu\Big(G(B)\Big) \leq \lambda\mu(B).$$

Theorem 2.9. *[4, 7] Let E be a Banach space and $U, V \subseteq E$ be bounded. The following properties are satisfied:*

(a) $\mu(V) = 0$, *if and only if $\overline{V}$ is compact, where $\overline{V}$ means the closure hull of V;*

(b) $\mu(U) = \mu(\overline{U}) = \mu(convU)$, *where $convU$ means the convex hull of U;*

(c) $\mu(\lambda U) = |\lambda|\mu(U)$ *for any $\lambda \in \mathbb{R}$;*

(d) $U \subset V$ *implies* $\mu(U) \leq \mu(V)$*;*

(e) $\mu(U \bigcup V) = \max\{\mu(U), \mu(V)\}$*;*

(f) $\mu(U + V) \leq \mu(U) + \mu(V)$, *where* $U + V = \{x/x = u + v, u \in U, v \in V\}$*;*

(g) If $G : \mathcal{D}(G) \subset E \longrightarrow X$ is Lipschitz continuous with constant λ, then $\mu(G(V)) \leq \lambda\mu(V)$ for any bounded subset $V \subset D(G)$, where X is another Banach space.

For more details about properties of the Kuratowski measure of noncompactness, we refer to the monographs of Bana's and Goebel [4] and Deimling [7].

Theorem 2.10. *[4, 7] Consider a Banach space E, and $B \subset E$ bounded. Then, there exists a countable set $B_0 \subset B$, such that $\mu(B) < 2\mu(B_0)$.*

Theorem 2.11. *[4, 7] Let E be a Banach space and $B = \{u_n : n \in \mathbb{N}\} \subset \mathcal{C}([\alpha; \beta], E)$ be a bounded and countable set for constants $-\infty < \alpha < \beta < +\infty$. Then, $t \longmapsto \mu(B(t))$ is Lebesgue integral on $[\alpha; \beta]$, and*

$$\mu\Big(\Big\{ \int_{\alpha}^{\beta} u_n(t)dt \mid n \in \mathbb{N}\Big\}\Big) \leq 2\int_{\alpha}^{\beta} \mu(B(t))dt.$$

Theorem 2.12. *[4, 7] Consider E a Banach space, and $B \subset \mathcal{C}([\alpha; \beta], E)$ a bounded and equicontinuous. Then, the mapping $t \longmapsto \mu(B(t))$ is continuous on $[\alpha; \beta]$, and $\mu(B) = \max_{t \in [\alpha;\beta]} \mu(B(t))$.*

The following Sadovskii fixed point theorem plays a key role in the proof of our main results.

Theorem 2.13. *[4, 7] Consider a Banach space E and suppose that, $\Omega \subset E$ is bounded, closed and convex. If the operator $G : \Omega \longrightarrow \Omega$ is condensing, which means that $\mu(G(\Omega)) < \mu(\Omega)$, then G has at least one fixed point in Ω.*

3. Existence of mild solution

In this section, we try our self to prove the existence of global mild solutions for equation (1.5) using the equicontinuity of $\{U(t,s) : 0 \leq s \leq t < +\infty\}$. We begin by define the mild solution that correspond to the definition in (1.5) and denote $\mathcal{C}_r := \mathcal{C}[-r, 0]$ with $r > 0$.

Definition 3.1. *Let $\varphi \in \mathcal{C}_r$ such that $\varphi(0) \in \overline{D}$. We say that a continuous function $x : (-r; +\infty) \longrightarrow X$ is a mild solution of the equation (1.5), if it satisfies the following equation*

$$x(t) = \begin{cases} U(t,0)\varphi(0) + \lim\limits_{\lambda \to 0^+} \displaystyle\int_0^t U_\lambda(t,s) f(s, x(t - \rho(s, x(s))))ds, & for \quad t \geq 0, \\ \varphi(t) \qquad -r < t \leq 0. \end{cases} \tag{3.1}$$

Firstly we study the local mild solution of equation (1.5).
To obtain our result, we consider the following assumptions :

($\mathbf{H}_1$) The nonlinear function $f : [0; \infty) \times X \longrightarrow X$ is continuous; and for some $r > 0$ there exist a constant $\delta_1 > 0$ and $\phi_r \in L^1([0,a], \mathbb{R}^+)$ such that for all $t \in [0,a]$ and $u \in \mathcal{C}([-r,a], X)$ satisfying $\|u\| \leq r$, $\|f(t,u)\| \leq \phi_r(t)$ and $\limsup\limits_{r \to +\infty} \dfrac{\|\phi_r\|_{L^1([0,a],\mathbb{R}^+)}}{r} = \delta_1 < +\infty$.

($\mathbf{H}_2$) There exists positive constant L_1 such that for any countable set $D \subset X$,

$$\mu(f(t,D)) \leq L_1 \mu(D), \quad t \in [0; a].$$

($\mathbf{H}_3$) We assume that the evolution family $\Big(U(t,s)\Big)_{t \geq s \geq 0}$ is equicontinuous i.e for any $s \geq 0$, the function $t \longmapsto U(t,s)$ is continuous by operator norm for $t \in (s; +\infty)$.

Theorem 3.2. *Let $a > 0$ and assume that the family of linear operators $\Big(A(t)\Big)_{t \geq 0}$ satisfies the hyperbolic conditions ($\boldsymbol{B}_1$)-($\boldsymbol{B}_3$), the assumptions ($\boldsymbol{H}_1$) - ($\boldsymbol{H}_3$) and $\varphi(0) \in \overline{D}$. Then the problem (1.5) has at least one local mild solution defined on $[-r, a]$. Moreover, the mild solution depends continuously on the initial data.*

Proof. Our proof is based on Sadoskii's fixed Point Theorem.
Let $(G_1 u)(t) = U(t,0)\varphi(0)$ and $(G_2 u)(t) = \lim\limits_{\lambda \longrightarrow 0^+} \displaystyle\int_0^t U_\lambda(t,s) f(s, u(s - \rho(u(s))))ds$ for each $u \in \mathcal{C}([-r; a]; X)$ and $0 \leq s \leq t \leq a$.
We claim that $G = G_1 + G_2$ is well defined on $\mathcal{C}([0; a]; X)$ to itself. Let $u \in \mathcal{C}([0; a]; X)$, we show that $Gu \in \mathcal{C}([0; a]; X)$. By the strongly continuity of the evolution family $\{U(t,s) : 0 \leq s \leq t \leq a\}$, we get for $0 \leq s \leq t \leq a$ that:

$$\begin{aligned} \|(Gu)(t) - (Gu)(s)\| &\leq \|U(t,0)\varphi(0) - U(s,0)\varphi(0)\| \\ &+ \|\lim_{\lambda \longrightarrow 0^+} \int_0^t U_\lambda(t,r) f(r, u(r - \rho(u(r))))dr - \lim_{\lambda \longrightarrow 0^+} \int_0^s U_\lambda(s,r) f(r, u(r - \rho(u(r))))dr\| \\ &\leq \|U(t,0)\varphi(0) - U(s,0)\varphi(0)\| + \|\lim_{\lambda \longrightarrow 0^+} \int_0^s \Big(U_\lambda(t,r) - U_\lambda(s,r)\Big)(f(r, u(r - \rho(u(r))))dr\| \\ &+ \|\lim_{\lambda \longrightarrow 0^+} \int_s^t U_\lambda(t,r) f(r, u(r - \rho(u(r))))dr\|. \end{aligned}$$

By $(\mathbf{H}_3)$, we get that: $\lim_{s\to t}\|U(t,0)\varphi(0)-U(s,0)\varphi(0)\| = 0$ and using $(\mathbf{H}_1)$ we deduce from the Lebesgue dominated convergence theorem that:

$$\lim_{s\to t}\| \lim_{\lambda\longrightarrow 0^+}\int_s^t U_\lambda(t,r)f(r,u(r-\rho(u(r))))dr\| = 0$$

and

$$\lim_{s\to t}\| \lim_{\lambda\longrightarrow 0^+}\int_0^s \Big(U_\lambda(t,r)-U_\lambda(s,r)\Big)(f(r,u(r-\rho(u(r))))dr\| = 0.$$

Thus

$$\lim_{s\to t}\|(Gu)(t)-(Gu)(s)\| = 0.$$

Therefore, our operator $Gu \in \mathcal{C}([0;a];X)$ for any $u \in \mathcal{C}([-r;a];X)$.

Case 1: Assume that

$$Ma\max\{\delta_1, 4L_1\} < 1 \tag{3.2}$$

We claim that there exists a constant $R > 0$ such that $G(B_R) \subset B_R$ where

$$B_R = \{u \in \mathcal{C}([0,a],X), \|u\| \leq R\}$$

By virtue of (3.2), we choose R such that $R \geq \frac{M\|\varphi(0)\|}{1-M\delta_1 a}$. Let $u \in B_R$ and $(\mathbf{H}_1)$ hypothesis , we get that

$$\begin{aligned}
\|(Gu)(t)\| &\leq \|U(t,0)\varphi(0)\| + \| \lim_{\lambda\longrightarrow 0^+}\int_0^t U_\lambda(t,s)f(s,u(s-\rho(u(s))))ds\| \\
&\leq M\|\varphi(0)\| + \int_0^t M\|f(s,u(s-\rho(u(s))))\|ds \\
&\leq M\|\varphi(0)\| + \int_0^t M\phi_r(s)ds \\
&\leq M\|\varphi(0)\| + \int_0^t MR\delta_1 ds \\
&\leq M\|\varphi(0)\| + MR\delta_1 t
\end{aligned}$$

Then, $\max_{s\in[0;t]}\|(Gu)(s)\| \leq M\|\varphi(0)\| + MR\delta_1 a \leq R.$

Therefore $G(B_R) \subset B_R$.

We claim that $G : B_R \longrightarrow B_R$ is continuous.

Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in B_R such that $u_n \to u \in B_R$ as $n \to +\infty$. From $(\mathbf{H}_1)$, we consider the definition of the operator G, the continuity of ρ and $(\mathbf{H}_2)$ hypothesis. We get for any $t \in [0;a]$ that:

$$\begin{aligned}
\|(Gu_n)(t)-(Gu)(t)\| &\leq \int_0^t M\|f(s,u_n(s-\rho(u_n(s)))) - f(s,u(s-\rho(u(s))))\|ds \\
&\leq \int_0^t M\|f(s,u_n(s-\rho(u_n(s)))) - f(s,u_n(s-\rho(u(s))))\|ds \\
&+ \int_0^t M\|f(s,u_n(s-\rho(u(s)))) - f(s,u(s-\rho(u(s))))\|ds
\end{aligned}$$

Since $(u_n)_{n\in\mathbb{N}} \subset B_R$, then for each $n \in \mathbb{N}$, u_n is continuous on $[-r,a]$. And by using $(\mathbf{H}_1)$ we have:

$$\|f\Big(s, u_n(s-\rho(u_n(s)))\Big) - f\Big(s, u_n(s-\rho(u(s)))\Big)\| \to 0 \; as \; n \to +\infty$$

and

$$\|f\Big(s, u_n(s-\rho(u(s)))\Big) - f\Big(s, u(s-\rho(u(s)))\Big)\| \to 0 \; as \; n \to +\infty$$

By using the Lebesgue dominated convergence theorem, we get that:

$$\lim_{n\to+\infty} \|(Gu_n)(t) - (Gu)(t)\| = 0 \;\; for \;\; each \;\; t \in [0, a].$$

Consequently, $Gu_n \to Gu$ as $n \to +\infty$. So the operator G is continuous in B_R.

We claim that the operator $G : B_R \longrightarrow B_R$ is equicontinuous. For all $u \in B_R$, $0 < t_1 < t_2 \leq a$ and $\varepsilon > 0$ small enough, by using $(\mathbf{H}_3)$, we get that:

$$\begin{aligned}
\|(Gu)(t_2) \; - \; (Gu)(t_1)\| &\leq \|U(t_2,0)\varphi(0) - U(t_1,0)\varphi(0)\| \\
&+ \|\lim_{\lambda\longrightarrow 0^+}\Big[\int_0^{t_2} U_\lambda(t_2,s)f(s,u(s-\rho(u(s))))ds - \int_0^{t_1} U_\lambda(t_1,s)f(s,u(s-\rho(u(s))))ds\Big]\| \\
&\leq \|[U(t_2,0) - U(t_1,0)]\varphi(0)\| + \|\lim_{\lambda\longrightarrow 0^+}\int_{t_1}^{t_2} U_\lambda(t_2,s)f(s,u(s-\rho(u(s))))ds\| \\
&+ \|\lim_{\lambda\longrightarrow 0^+}\int_0^{t_1-\varepsilon} [U_\lambda(t_2,s) - U_\lambda(t_1,s)]f(s,u(s-\rho(u(s))))ds\| \\
&+ \|\lim_{\lambda\longrightarrow 0^+}\int_{t_1-\varepsilon}^{t_1} [U_\lambda(t_2,s) - U_\lambda(t_1,s)]f(s,u(s-\rho(u(s))))ds\| \\
&\leq \|U(t_2,0) - U(t_1,0)\|_{\mathcal{L}(X)}\|\varphi(0)\| + M\int_{t_1}^{t_2} \|f(s,u(s-\rho(u(s))))\|ds \\
&+ \sup_{s\in[0,t_1-\varepsilon]}\|\lim_{\lambda\longrightarrow 0^+}[U_\lambda(t_2,s) - U_\lambda(t_1,s)]\|_{\mathcal{L}(X)}\int_0^{t_1-\varepsilon} \|f(s,u(s-\rho(u(s))))\|ds \\
&+ \sup_{s\in[t_1-\varepsilon,t_1]}\|\lim_{\lambda\longrightarrow 0^+}[U_\lambda(t_2,s) - U_\lambda(t_1,s)]\|_{\mathcal{L}(X)}\int_{t_1-\varepsilon}^{t_1} \|f(s,u(s-\rho(u(s))))\|ds \\
&\to 0 \; as \; t_2 \to t_1 \;\; and \;\; \varepsilon \to 0.
\end{aligned}$$

We claim that the operator $G : B_R \longrightarrow B_R$ is condensing.
For any $B \subset B_R$, B is bounded. By using Theorem 2.10, there exists a countable set $A = \{v_n : n \in \mathbb{N}\} \subset B$ such that

$$\mu(G(B)) \leq 2\mu(G(A)). \tag{3.3}$$

Because $A \subset B \subset B_R$, we get that $G(A) \subset G(B_R)$ then $G(A)$ is bounded. And since the operator $G : B_R \longrightarrow B_R$ is equicontinuous, by the Theorem 2.12 we get that

$$\mu(G(A)) = \max_{t\in[0;a]} \mu(G(A)(t)). \tag{3.4}$$

By using the definition of the operator G_1, we get that $(G_1u)(t) = U(t,0)\varphi(0)$ for all $u \in B$ and $0 \leq t \leq a$. Therefore $G_1(B)(t) = \{U(t,0)\varphi(0)\}$ for $t \in [0;a]$. From the definition of μ, we have

$\mu(G_1(B)(t)) = 0$ for all $t \in [0;a]$ and according to the Theorem 2.12, we get $\mu(G_1(B)) = 0$.
By using Theorem 2.9, Theorem 2.11, the assumptions $(\mathbf{H}_1)$ and the definition of G_2, we have

$$\begin{aligned}\mu(G_2(A)(t)) &= \mu(\{\lim_{\lambda \longrightarrow 0^+} \int_0^t U_\lambda(t,s) f(s, u_n(s - \rho(u_n(s))))ds \mid n \in \mathbb{N}\}) \\ &\leq M\mu(\{\int_0^t f(s, u_n(s - \rho(u_n(s))))ds \mid n \in \mathbb{N}\}) \\ &\leq M\mu(\{\int_0^t f(s, u_n(s))ds \mid n \in \mathbb{N}\}) \\ &\leq 2M \int_0^t \mu(f(s, A(s))ds \\ &\leq 2ML_1 \int_0^t \mu(A(s))ds \\ &\leq 2ML_1 \int_0^t \mu(A)ds \\ &\leq 2ML_1 t\mu(A) \\ &\leq 2ML_1 a\mu(A). \end{aligned} \tag{3.5}$$

We know that $A \subset B$, and using Theorem 2.9,

$$\mu(A) \leq \mu(B). \tag{3.6}$$

$$\mu(G(A)) = \mu(G_1(A) + G_2(A)) \leq \mu(G_1(A)) + \mu(G_2(A)) = \mu(G_2(A)). \tag{3.7}$$

By using (3.3)-(3.7), we have

$$\mu(G(B)) \leq 4ML_1 a\mu(B). \tag{3.8}$$

Since (3.2) and (3.8), we have

$$\mu(G(B)) < \mu(B). \tag{3.9}$$

The inequality (3.9) proves that the operator $G : B_R \longrightarrow B_R$ is condensing. From the Theorem 2.13, the problem (1.5) has at least one local mild solution defined on $[-r, a]$.

Case 2: We assume that $Ma\max\{\delta_1, 4L_1\} \geq 1$.

We know that $\frac{4ML_1 a}{k} \to 0 \ \ as \ \ k \to +\infty$ and $\frac{M\delta_1 a}{k} \to 0 \ \ as \ \ k \to +\infty$. Then there exists a constance $n \in \mathbb{N}\backslash\{0,1\}$ such that $\frac{4ML_1 a}{n} < 1$ and $\frac{M\delta_1 a}{n} < 1$. Let $b = \frac{a}{n}$, hence $nb = a$ and $4ML_1 b < 1$ and $M\delta_1 b < 1$. We deduce from **Case 1** that there exists at least one local mild solution $x_1 : [-r; b] \longrightarrow X$ of the problem (1.5).

We denote $\varphi_1 \in \mathcal{C}([-r;0], X)$ such that $\varphi_1(t) = x_1(t+b)$ for any $t \in [-r-b; 0]$ and $\mathcal{C}_{\varphi_1}([b, 2b], X) := \{y \in \mathcal{C}([b, 2b], X) : y(b) = \varphi_1(0)\}$. We consider the following problem

$$\begin{cases} x'(t) = A(t)x(t) + f(t, x(t - \rho(x(t)))), & t \in [b; 2b], \\ x(t) = x_1(t), & t \in [-r; b]. \end{cases} \tag{3.10}$$

The problem (3.10) is equivalent to the following problem

$$\begin{cases} y'(s) = B(s)y(s) + f_1(s, y(s - \rho(y(s)))), & s \in [0; b], \\ y(s) = \varphi_1(s), & s \in [-r; 0]. \end{cases} \tag{3.11}$$

where $s = t - b$, $y(s) = x(s+b)$, $B(s) = A(b+s)$ and $f_1(s,.) = f(b+s,.)$.
In this case, f_1 satisfies (H_1) and (H_2). And the family of linear operator $\{B(t) : 0 \leq t \leq b\}$ satisfies $(B_1)-(B_3)$ and its evolution family satisfies all condition that the evolution family generated by $\{A(t) : 0 \leq t \leq a\}$ does. It follows from **Case 1** that there exists at least one local mild solution $y : [-r; b] \longrightarrow X$ of the problem (3.11). Then the problem (1.5) has at least one local solution in $[b, 2b]$ defined by $x_2(t) = y(t)$ for $t \in [b; 2b]$.
By use the inductive reasoning, we get that the problem (1.5) has at least one local solution x_k in $[(k-1)b, kb]$, $k = 1, 2, \cdots, n$. Hence, the problem (1.5) has at least one local solution defined by:

$$x(t) = x_k(t) \quad for \;\; t \in [(k-1)b; kb], \;\; k = 1, 2, \cdots, n.$$

Therefore, the problem (1.5) has at least one local mild solution on $[-r, a]$.

Let $y = y(., \varphi)$ and $z = z(., \psi)$ be two solutions of equation (1.5) corresponding respectively to initial data $\varphi, \psi \in \mathcal{B}$ with $\varphi(0), \psi(0) \in \overline{D}$. Then

$$\begin{aligned}
\|y(t) - z(t)\| \leq & \|U(t,0)\Big[\varphi(0) - \psi(0)\Big]\| \\
& + \|\lim_{\lambda \longrightarrow 0^+} \int_0^t U_\lambda(t,s)\Big[f(s, y(s - \rho(y(s)))) - f(s, z(s - \rho(z(s))))\Big]ds\| \\
& \leq M\|\varphi(0) - \psi(0)\| + M\int_0^t \|f(s, y(s - \rho(y(s)))) - f(s, z(s - \rho(z(s))))\|ds \\
& \leq M\|\varphi - \psi\|_\infty + M\int_0^t \|f(s, y(s - \rho(y(s)))) - f(s, y(s - \rho(z(s))))\|ds \\
& + M\int_0^t \|f(s, y(s - \rho(z(s)))) - f(s, z(s - \rho(z(s))))\|ds.
\end{aligned}$$

For all $\epsilon > 0$, we find $\delta > 0$ such that $\|\varphi - \psi\|_\infty < \delta \Rightarrow \max\limits_{0 \leq s \leq a} \|y(s) - z(s)\| < \epsilon$.
ρ is continuous function, then there exists $\delta_1 > 0$ such that:

$$\|y(s) - z(s)\| < \epsilon \Rightarrow \|\rho(y(s)) - \rho(z(s))\| < \delta_1, \;\; s \in [0, a].$$

y is continuous function, then there exists $\delta_2 > 0$ such that:

$$\|\rho(y(s)) - \rho(z(s))\| < \delta_1 \Rightarrow \|y(s - \rho(y(s))) - y(s - \rho(z(s)))\| < \delta_2, \;\; s \in [0, a].$$

f is continuous function, then there exists $\delta_3 > 0$ such that

$$\|y(s - \rho(y(s))) - y(s - \rho(z(s)))\| < \delta_2 \Rightarrow$$

$$\|f(s, y(s - \rho(y(s)))) - f(s, y(s - \rho(z(s))))\| < \frac{\delta_3}{2aM}$$

and

$$\|f(s, y(s - \rho(z(s)))) - f(s, z(s - \rho(z(s))))\| < \frac{\delta_3}{2aM}, \;\; s \in [0, a].$$

Consequently,

$$\|\varphi - \psi\|_\infty < \delta \Rightarrow \max_{0\le s\le a} \|f(s, y(s-\rho(y(s)))) - f(s, y(s-\rho(z(s))))\| < \frac{\delta_3}{2aM}$$

and

$$\max_{0\le s\le a} \|f(s, y(s-\rho(z(s)))) - f(s, z(s-\rho(z(s))))\| < \frac{\delta_3}{2aM}.$$

Therefore

$$\begin{aligned} M\|\varphi-\psi\|_\infty &+ M\int_0^t \|f(s, y(s-\rho(y(s)))) - f(s, y(s-\rho(z(s))))\|ds \\ &+ M\int_0^t \|f(s, y(s-\rho(z(s)))) - f(s, z(s-\rho(z(s))))\|ds < \epsilon. \end{aligned} \tag{3.12}$$

Relation (3.12) implies that:

$$M\delta + \delta_3 < \epsilon. \quad \text{Then,} \quad \delta < \frac{\epsilon-\delta_3}{M}.$$

Choose ϵ *and* δ_3 such that $\epsilon > \delta_3$ and take $\delta = \frac{\epsilon-\delta_3}{2M}$.
Therefore, the mild solution of (1.5) depends continuously on the initial data.

■

Our subsequent objective is to establish the global mild solution of problem (1.5).

($\mathbf{H}_4$) The nonlinear function $f : [0;\infty) \times X \longrightarrow X$ is continuous; and for some $R > 0$ there exist a constant $\delta_0 > 0$ and $\phi_r \in L^1([0,+\infty), \mathbb{R}^+)$ such that for all $t \ge 0$ and $u \in \mathcal{C}([-r,+\infty), X)$ satisfying $\|u\| \le R$, $\|f(t,u)\| \le \phi_R(t)$ and $\limsup_{R\to+\infty} \frac{\|\phi_R\|_{L^1([0,+\infty),\mathbb{R}^+)}}{R} = \delta_0 < +\infty$.

($\mathbf{H}_5$) There exists positive constant L_0 such that for any countable set $D \subset X$,

$$\mu(f(t,D)) \le L_0\mu(D), \quad t \ge 0.$$

Theorem 3.3. *Assume that the family of linear operators* $\Big(A(t)\Big)_{t\ge0}$ *satisfies the hyperbolic conditions* ($\boldsymbol{B}_1$)*-*($\boldsymbol{B}_3$)*, the evolution family* $\Big(U(t,s)\Big)_{t\ge s\ge0}$ *is equicontinuous,* ($\boldsymbol{H}_3$) $-$ ($\boldsymbol{H}_5$) *hold. Then problem (1.5) has at least one global mild solution on* $[-r;+\infty)$.

Proof. Using Theorem 3.2, We deduce that there exists an unique local mild solution x^n of problem (1.5) defined on $[-r;n]$ for each $n \in \mathbb{N}$. It is clear that $x^{n+1}|_{[-r;n]} = x^n$ for each $n \in \mathbb{N}$. Hence the problem (1.5) has at least one global mild solution $x(.)$ on $[-r;+\infty)$ and it is defined by $x(t) = x^n(t)$ *for each* $-r \le t \le n$ and for all $n \in \mathbb{N}$. ■

4. Application

In this section, we apply our results to the following non-autonomous partial differential equation of evolution.

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \theta(t)\dfrac{\partial^2}{\partial x^2}u(t,x) + \dfrac{t}{6(t+1)^3 + \mid u(t-\psi(u(t,x),x) \mid}, \ for\ (t,x) \in [0;+\infty) \times \Omega, \\ u(t,x) = 0, \ for\ (t,x) \in [0;+\infty) \times \partial\Omega, \\ u(t,x) = \phi(t,x), \ for\ (t,x) \in [-T;0] \times \Omega. \end{cases} \tag{4.1}$$

where $\Omega \subset \mathbb{R}$ is a bounded and closed domain with smooth boundary $\partial\Omega$ and the function $\theta \in \mathcal{C}^1([0;+\infty),\mathbb{R})$. The delay function ψ is a bounded positive continuous function in $\mathbb{R}$ and let T be its upper bound element in $\mathbb{R}$ and the function $\phi \in \mathcal{C}^2([-T;0]\times\Omega;\mathbb{R})$.

Theorem 4.1. *The problem (4.1) has at least one mild solution.*

Proof. We consider X, the Banach space defined by $X = \mathcal{C}(\Omega;\mathbb{R})$ and the operator $A : D \subset X \longrightarrow X$ defined by

$$\begin{cases} D = D(A) = \{z \in \mathcal{C}^2(\Omega;\mathbb{R}) : z(x) = 0, x \in \partial\Omega\}, \\ Az(t,x) = \dfrac{\partial^2}{\partial x^2} z(t,x), \quad (t,x) \in [0;\infty) \times \Omega. \end{cases}$$

We have $\overline{D} = \{z \in \mathcal{C}(\Omega;\mathbb{R}) : z(x) = 0, x \in \partial\Omega\} \neq X$. We know from [**?**] that

$$(0,+\infty) \subset \varrho(A) \;\; and \;\; \|R(\lambda,A)\| \leq \frac{1}{\lambda} \;\; for \;\; \lambda > 0. \tag{4.2}$$

Let $(A(t))_{t\geq 0}$ be a family of operators defined by $A(t) = \theta(t)\dfrac{\partial^2}{\partial x^2}$.
For any $t \geq 0$, we have $D(A(t)) = D$ independent of t.
Then it is well know that for every $t \geq 0$

$$R(\lambda, A(t)) = \frac{1}{\theta(t)} R(\frac{\lambda}{\theta(t)}, A) \tag{4.3}$$

Since (4.2), we have $\|R(\lambda,A)\| \leq \dfrac{1}{\lambda}$ for each $\lambda \in (0,+\infty) \cap \varrho(A)$.
Then, by adding the (4.3) and for $0 \leq t_1 \leq t_2 \leq ... \leq t_k < +\infty$ we get

$$\|\prod_{j=1}^{k} R(\lambda, A(t_j))\| \leq \frac{1}{\lambda^k}$$

Using the definition of the function θ and Banach's space X, the mapping $t \longmapsto A(t)x$ is continuously differentiable in X for all $x \in D$.
Hence, the family of linear operator $(A(t))_{t\geq 0}$ on X satisfies the assumptions $(B_1) - (B_3)$.
Since [2], the operator $A_0(.)$ of $A(.)$ in $\overline{D(A)}$ generates an evolution family $\Big(U(s,t)\Big)_{t\geq s\geq 0}$ given by

$$U(s,t) = T_0\Big(\int_s^t \theta(r)dr\Big)$$

which is equicontinuous for each $t \geq s \geq 0$. where the operator $\triangle_0$ of $\triangle$ in $\overline{D(\triangle)}$ give by

$$\begin{cases} D(\triangle_0) = \{x \in D(\triangle) : \triangle x \in \overline{D(\triangle)}\}, \\ \triangle_0 x = \triangle x, \end{cases}$$

generates the semigroup $\Big(T_0(t)\Big)_{t\geq 0}$ such that

$$\|T_0(t)\| \leq e^{-t} \;\; for \;\; each \;\; t \geq 0.$$

Hence,

$$\|U(t,s)\| \leq 1 \;\; for \;\; each \;\; t \geq s \geq 0.$$

We get that $M = 1$.
Let $f : [0; +\infty) \times X \longrightarrow X$ defined by $f(t,z)x = \dfrac{t}{6(t+1)^3 + \mid z(x) \mid}$ for $x \in \Omega$ and $t \geq 0$. The initial data φ is defined by $\varphi(t)x = \phi(t,x)$ for $x \in \Omega$ and $t \geq 0$.
and $z(t)x = u(t,x)$.
Therefore (4.1) becomes

$$\begin{cases} z'(t) = A(t)z(t) + f(t, z(t - \psi(z(t)))),\ for\ t \geq 0, \\ z(t) = \varphi(t),\ for\ t \in [-T; 0]. \end{cases} \tag{4.4}$$

For every $t \geq 0\ \ and\ \ z, y \in X$,

$$\begin{aligned} |f(t,z) - f(t,y)| &= \frac{t}{[6(t+1)^3 + |z|][6(t+1)^3 + |y|]}||y| - |z|| \\ &\leq \frac{t}{[6(t+1)^3 + |z|][6(t+1)^3 + |y|]}|z - y| \\ &\leq \frac{t}{36(t+1)^6}|z - y| \\ &\leq \frac{1}{36}|z - y| \end{aligned}$$

we get that $L_0 = \dfrac{1}{36}$. Thus (H_5) is verified.
Let $r > 0$, for every $t \geq 0$, $u \in B_r$, we get that:

$$\|f(t,u)\| \leq \frac{t}{6(t+1)^3} \leq \frac{(t+1)}{6(t+1)^3} \leq \frac{r+1}{(t+1)^2} = \phi_r(t), \quad \phi_r \in L^1([0,+\infty), \mathbb{R}^+)$$

$$\|\phi_r\|_{L^1([0,+\infty),\mathbb{R}^+)} = r + 1$$

and

$$\lim_{r \to +\infty} \frac{\|\phi_r\|_{L^1([0,+\infty),\mathbb{R}^+)}}{r} = 1.$$

Then (H_4) is verified and we take $\delta_0 = 1$. Therefore, by using Theorem 3.3, we get that the problem (4.4) has at least one global mild solution $u : [-T; +\infty) \longrightarrow X$. ■

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References

[1] Akrid, T., Maniar, L., Ouhinou, A., Periodic solutions of non densely nonautonomous differential equations with delay, *Afr. Diaspora J. Math.,* 15(1)(2013), 1–18.

[2] Arendt, W., Grabosch, A., Greiner, G., Groh, U., Lotz, H.P., Moustakas, U., Nagel, R., Neubrander, B., Schlotterbeck, U., *One-parameter Semigroup of Positive Operators,* Springer, Berlin (1984).

[3] Arino, O., Hbib, M.L.,Ait Dads, E., Delay Differential Equations and Applications, *Mathematics, Physics and Chemistry*, 205(2002).

[4] BANA'S, J., *On measures of non compactness in Banach spaces*, *Comment. Math. Univ. Carolin.,* 21(1)(1980), 131–143.

[5] BELMEKKI, M., BENCHOHRA, M., EZZINBI, K., Existence results for some partial functional differential equations with state-dependent delay, *Appl. Math. Lett.,* 24(2011), 1810–1816.

[6] CHEN, P., ZHANG, X., LI, Y., Existence of mild solutions to partial differential equations with non-instantaneous impulses, *Electron. J. Differ. Equ.,* 2016(241)(2016), 1–11.

[7] CHEN, P., ZHANG, X., LI, Y., Non-autonomous parabolic evolution equations with non-instantaneous impulses governed by noncompact evolution families, *Journal of Fixed Point Theory and Applications*, 21(84)(2019), 1–17.

[8] DA PRATO, G., SINESTRARI, E., Differential operators with nondense domain, *Annali della Scuola Normale Superiore di Pisa, Classe di Scienze*, 14(2)(1987), 285–344.

[9] EZZINBI, K., BEKOLLE, D., KPOUMIE, M.E.K., Periodic solutions for some nondensely nonautonomous partial functional differential equations in fading memory spaces, *Differ. Equ. Dyn. Syst.,* 26(1-3)(2018), 177–197.

[10] HALE, J.K, KATO, J., Phase space for retarded equations with infinite delay, *Funkcialaj Ekvacioj*, 21(1978), 11–41.

[11] HINO, Y., MURAKAMI, S., NAITO T., *Functional Differential Equations with Infinite Delay*, Lectures Notes in Mathematics, vol. 1473. Springer 2006

[12] KATO, T., Linear evolution equations of hyperbolic type, *J. Fac. Sci. Univ. Tokyo*, 17(1970), 241–258.

[13] KATO, T., Linear evolution equations of hyperbolic type II, *J. Math. Soc. Japan*, 25(4)(1973), 648–666.

[14] KPOUMIE, M.EL-K, EZZINBI, K., BEKOLLE, D., Nonautonomous partial functional differential equations; existence and regularity, *Nonautonomous Dynamical Systems*, 4(1)(2017), 108–127.

[15] KPOUMIE, M. E. K., NSANGOU, A. H. G., NDAMBOMVE, P., ZABSONRE, I., MBOUTNGAM, S., Existence of solutions for some nonautonomous partial functional differential equations with state-dependent delay, *SeMA*, 77(2020), 107–118.

[16] OKA, H., TANAKA, N., Evolution operators generated by non densely defined operators, *Math. Nachr.,* 278(11)(2005), 1285–1296.

[17] PAZY, A., *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer, Berlin (1983).

[18] TANAKA, N., Semilinear equations in the hyperbolic case, *Nonlinear Anal. Theory Methods Appl.,* 24(5)(1995), 773–788.

[19] TANAKA, N., Quasilinear evolution equations with non-densely defined operators, *Differ. Integral Equ.,* 5(1996), 1067–1106.