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On a conformable fractional differential equations with maxima

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Abstract. This study deals with the existence and uniqueness of solutions for a class of first order conformable fractional differential equations with maxima. We also provide some examples to illustrate the application of the results.

Keywords: Conformable, maxima, monotone, uniqueness, upper solutions, lower solutions.

Contents

1. Introduction

The purpose of this study the following problem

$$\begin{cases} \left(\mathfrak{D}_\alpha \mathfrak{u}\right)(\tau) = \mathfrak{f}(\tau, \mathfrak{u}(\tau), \max\limits_{s\in[\tau-r,\tau]} \mathfrak{u}(s)),\ \tau \in J = [0, \mathfrak{T}], \\ \mathfrak{u}(\tau) = \varphi(\tau),\ \tau \in [-r, 0], \end{cases} \tag{1.1}$$

where $\mathfrak{D}_\alpha$ represents the conformable fractional derivative of order α, $0 < \alpha \leq 1$, $\mathfrak{T} > 0$ and $r > 0$, $\mathfrak{f} : J \times \mathbb{R} \times C \to \mathbb{R}$ is continuous with $C = C([-r, \mathfrak{T}], \mathbb{R})$ and $\varphi : [-r, 0] \to \mathbb{R}$ continuous.

Conformable fractional derivative was first introduced in [23], later developed in [1] and it appears in many fields (see [2], [3], [11], [17], [25], [35] along with the cited references therein).

However differential equations with maxima and differential inequalities with maxima were initially used in automatic control and in the study stability of equations with retarded argument (see [30] and [19, Chapter 4 Section 5]). Nevertheless, a variety of fields, including there are a wide range of areas such as psychology (e.g., dynamic model for happiness), optimal control, theory of lateral inhibition, chemostat models and economy (see [5], [6], [8], [15], [18], [20], [21], [28] and [33]) use differential equations with maxima.

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Some authors have studied conformable fractional differential equations with deviating arguments using fixed point theorems, numerical methods, monotone iterative technique, and upper and lower solutions method see [14], [16], [22], [24] and [31]). Let us recall some of them.

In [14], the authors studied the problem

$$\begin{cases} (\mathfrak{D}_\alpha \mathfrak{u})(\tau) = f(\tau, \mathfrak{u}(\tau), \mathfrak{u}(\theta(\tau))), \ \tau \in J = [0, \mathfrak{T}], \\ \mathfrak{u}(0) = \mathfrak{u}(T), \end{cases} \tag{1.2}$$

where $0 < \alpha \leq 1$, $\mathfrak{T} > 0$, $f : J \times \mathbb{R} \times C(J, \mathbb{R}) \to \mathbb{R}$ and $\theta : J \to J$ are continuous with $\theta(J) \subseteq J$.

The authors used the monotone iterative technique to establish some sufficient conditions for the existence of extremal solutions for periodic boundary value problem (1.2).

In [16], the author studied the following problem

$$\begin{cases} (\mathfrak{D}_\alpha \mathfrak{v})(\tau) = f(\tau, \mathfrak{v}(\tau), \mathfrak{v}(\theta(\tau))), \ \tau \in J = [0, \mathfrak{T}], \\ \mathfrak{v}(0) = g(\mathfrak{v}), \end{cases} \tag{1.3}$$

where $0 < \alpha \leq 1$, $\mathfrak{T} > 0$, $f : J \times \mathbb{R} \times C(J, \mathbb{R}) \to \mathbb{R}$ and $\theta : J \to J$ continuous with $\theta(J) \subseteq J$, and $g : C(J, \mathbb{R}) \to \mathbb{R}$ continuous increasing.

The author established the existence of minimal and maximal solutions for the problem (1.3) by combining the upper and lower solutions method with the monotone iterative technique.

In [22], the authors studied the following problem

$$\begin{cases} (\mathfrak{D}_\alpha y)(\tau) + y(\tau) = \mu y(\mu\tau), \ \tau > 0, \\ y(0) = \lambda, \end{cases} \tag{1.4}$$

where $0 < \alpha \leq 1$, λ and μ are real numbers with $\mu < 1$.

The approximate solution for problem (1.4) was provided by the authors using the homotopy perturbation method.

One well know that the existence of solutions for first order differential equations with maxima is proved using the monotone iterative technique (see [4], [7], [8, Chapter 6] and the references cited therein). The aim of this work is to demonstrate its successful application to problems of type (1.1).

This work is structured to the following plan. We provide some definitions and preliminaries results in Section 2. Section 3 presents and demonstrates the main results and finally Section 4 offers how our results are applied.

2. Definitions and Preliminary Results

Definition 2.1. *[23]Let $h : J \to \mathbb{R}$ continuous $0 < \alpha \leq 1$. The conformable fractional integral of order α of h is defined by*

$$(I_\alpha h)(\tau) = \int_0^\tau s^{\alpha-1} h(s)\, ds, \text{ for } \tau > 0.$$

Definition 2.2. *[23]Let $h : J \to \mathbb{R}$ and $0 < \alpha \leq 1$. The Conformable fractional derivative of order α of h is defined by*

$$\begin{cases} (\mathfrak{D}_\alpha h)(\tau) = \lim\limits_{\rho \to 0} \frac{h\left(\tau + \rho\tau^{1-\alpha}\right) - h(\tau)}{\rho}, \text{ for } \tau > 0, \\ (\mathfrak{D}_\alpha h)(0) = \lim\limits_{\tau \to 0^+} (D_\alpha h)(\tau). \end{cases} \tag{2.1}$$

Example 2.3. *We have*

(i) $(\mathfrak{D}_\alpha c)(\tau) = 0$, *where* $c \in \mathbb{R}$.

(ii) $(\mathfrak{D}_\alpha \tau^\lambda)(\tau) = \begin{cases} \lambda\tau^{\lambda-\alpha} \text{ if } \tau > 0, \\ \lambda \text{ if } \lambda = \alpha \text{ and } \tau = 0, \\ 0 \text{ if } \lambda > \alpha \text{ and } \tau = 0. \end{cases}$

(iii) $(\mathfrak{D}_\alpha e^{\tau^\alpha})(\tau) = \alpha e^{\tau^\alpha}$.

(iv) $(\mathfrak{D}_\alpha \sin\left(\frac{t^\alpha}{\alpha}\right))(\tau) = \cos\left(\frac{\tau^\alpha}{\alpha}\right)$.

(v) $(\mathfrak{D}_\alpha \cos\left(\frac{t^\alpha}{\alpha}\right))(\tau) = -\sin\left(\frac{\tau^\alpha}{\alpha}\right)$.

Theorem 2.4. *[23, Theorem 2.1]If $\mathtt{h} : J \to \mathbb{R}$ is α-differentiable at $\tau_0 > 0$, then $\mathtt{h}$ is continuous at τ_0.*

Lemma 2.5. *[23, Theorem 3.1]Let $\mathtt{h} : J \to \mathbb{R}$ be a continuous function and $0 < \alpha \leq 1$, then we have $(\mathfrak{D}_\alpha \circ I_\alpha)\mathtt{h} = \mathtt{h}$.*

Lemma 2.6. *[23, Theorem 2.4]Let $\mathtt{h} : [a, b] \to \mathbb{R}$ continuous with $0 \leq a < b$ and $0 < \alpha \leq 1$. If $\mathtt{h}$ is α-differentiable in (a, b), then*

$$\mathtt{h}(b) - \mathtt{h}(a) = \left(\frac{b^\alpha - a^\alpha}{\alpha}\right)(\mathfrak{D}_\alpha \mathtt{h})(c),$$

with c in (a, b).

Notation 2.7. *For $0 < \alpha \leq 1$, we define $C^{\alpha,0}(J, \mathbb{R})$ as follows*

$$C^{\alpha,0}(J, \mathbb{R}) = \{\mathtt{h} \in C(J, \mathbb{R}) : \mathfrak{D}_\alpha \mathtt{h} \in C(J, \mathbb{R})\}.$$

Lemma 2.8. *Let $\mathtt{h} \in C^{\alpha,0}([a, b], \mathbb{R})$ with $0 \leq a < b$. Then $\mathfrak{D}_\alpha \mathtt{h} \equiv 0$ in $[a, b]$ if and only if $\mathtt{h} \equiv c$ in $[a, b]$, where c is a real constant.*

Proof. Assume that $\mathtt{h} \in C^{\alpha,0}([a, b], \mathbb{R})$ with $0 \leq a < b$.

Suppose that $\mathfrak{D}_\alpha \mathtt{h} \equiv 0$ in $[a, b]$ and we put by definition

$$\mathtt{h}(\tau_0) = \min_{\tau \in [a,b]} \mathtt{h}(\tau) \text{ and } \mathtt{h}(\tau_1) = \max_{t \in [a,b]} \mathtt{h}(t).$$

From Lemma 2.6, one has

$$\mathtt{h}(\tau_0) = \mathtt{h}(\tau_1),$$

which means that

$$\mathtt{h} \equiv c \text{ in } [a, b], \text{ with } c \in \mathbb{R}.$$

Conversely if $\mathtt{h} \equiv c$ in $[a, b]$ with $c \in \mathbb{R}$, then by using the definition of Conformable fractional derivative, we obtain $\mathtt{h} \in C^{\alpha,0}([a, b], \mathbb{R})$. ■

Lemma 2.9. *Assume that $\mathtt{h} \in C^{\alpha,0}(J, \mathbb{R})$, then we have*

$$(I_\alpha \circ \mathfrak{D}_\alpha)(\mathtt{h}(\tau)) = \mathtt{h}(\tau) - h(0), \text{ for } \tau \in J.$$

Proof. We put by definition

$$\mathtt{g}(\tau) = (I_\alpha \circ \mathfrak{D}_\alpha)(\mathtt{h}(\tau)), \text{ for } t \in J.$$

From Lemma 2.5, we obtain

$$(\mathfrak{D}_\alpha \mathtt{g})(\tau) = (\mathfrak{D}_\alpha \mathtt{h})(\tau), \text{ for } \tau \in J,$$

which means that

$$(\mathfrak{D}_\alpha)(g - h)(t) = 0, \text{ for } t \in J,$$

and consequently since $g(0)=0$ and from the preceding Lemma, we deduce that

$$\mathrm{g}(\tau)=\mathrm{h}(\tau)-\mathrm{h}(0)\,,\text{ for }\tau\in J.$$

That is

$$\left(I_\alpha\circ\mathfrak{D}_\alpha\right)(\mathrm{h}(\tau))=\mathrm{h}(\tau)-h(0)\,,\ \text{ for }\tau\in J.$$

■

Lemma 2.10. *[32, Theorem 1 page 44] If the functions* $\mathrm{u}:[\mathfrak{c},\mathfrak{d}]\to\mathbb{R}$ *and* $\mathrm{v}:[\mathfrak{c},\mathfrak{d}]\to\mathbb{R}$ *are continuous on the segment* $[\mathfrak{c},\mathfrak{d}]$*, then*

$$\max_{\tau\in[\mathfrak{c},\mathfrak{d}]}|\mathrm{u}(\tau)-\mathrm{v}(\tau)|\geq\left|\max_{\tau\in[\mathfrak{c},\mathfrak{d}]}\mathrm{u}(\tau)-\max_{\tau\in[\mathfrak{c},\mathfrak{d}]}\mathrm{v}(\tau)\right|.$$

Lemma 2.11. *If the functions* $\mathrm{u}:[\mathfrak{c},\mathfrak{d}]\to\mathbb{R}$ *and* $\mathrm{v}:[\mathfrak{c},\mathfrak{d}]\to\mathbb{R}$ *are continuous on the segment* $[\mathfrak{c},\mathfrak{d}]$*, then*

$$\max_{\tau\in[\mathfrak{c},\mathfrak{d}]}\mathrm{u}(\tau)-\max_{\tau\in[\mathfrak{c},\mathfrak{d}]}\mathrm{v}(\tau)\geq\min_{\tau\in[\mathfrak{c},\mathfrak{d}]}\left(\mathrm{u}(\tau)-\mathrm{v}(\tau)\right).$$

Proof. We have

$$\max_{\tau\in[\mathfrak{c},\mathfrak{d}]}\mathrm{u}(\tau)-\max_{\tau\in[\mathfrak{c},\mathfrak{d}]}\mathrm{v}(\tau)=\max_{\tau\in[\mathfrak{c},\mathfrak{d}]}\mathrm{u}(\tau)-\mathrm{v}(\varsigma)\,,$$

where $\varsigma\in[\mathfrak{c},\mathfrak{d}]$.

Which implies that

$$\begin{aligned}\max_{\tau\in[\mathfrak{c},\mathfrak{d}]}\mathrm{u}(\tau)-\max_{\tau\in[\mathfrak{c},\mathfrak{d}]}\mathrm{v}(\tau)&\geq\mathrm{u}(\varsigma)-\mathrm{v}(\varsigma)\\&\geq\min_{\tau\in[\mathfrak{c},\mathfrak{d}]}\left(\mathrm{u}(\tau)-\mathrm{v}(\tau)\right).\end{aligned}$$

That is

$$\max_{\tau\in[\mathfrak{c},\mathfrak{d}]}\mathrm{u}(\tau)-\max_{\tau\in[\mathfrak{c},\mathfrak{d}]}\mathrm{v}(\tau)\geq\min_{\tau\in[\mathfrak{c},\mathfrak{d}]}\left(\mathrm{u}(\tau)-\mathrm{v}(\tau)\right).$$

■

Now consider the problem

$$\begin{cases}\left(\mathfrak{D}_\alpha u\right)(\tau)=\widetilde{g}(\tau,\mathfrak{u}(\tau)\,,\max\limits_{s\in[\tau-r,\tau]}\mathfrak{u}(\tau)),\ \tau\in J,\\ \mathfrak{u}(\tau)=\psi(t)\,,\ \tau\in[-r,0]\,,\end{cases}\tag{2.2}$$

where $0<\alpha\leq1$, $\widetilde{g}:J\times\mathbb{R}\times C\left([-r,\mathfrak{T}]\,,\mathbb{R}\right)\to\mathbb{R}$ continuous and $\psi\in C\left([-r,0]\,,\mathbb{R}\right)$.

Notation 2.12. *For* $0<\alpha\leq1$ *the space* $C^\alpha\left([-r,\mathfrak{T}]\,,\mathbb{R}\right)$ *is defined as follows*

$$C^\alpha\left([-r,\mathfrak{T}]\,,\mathbb{R}\right)=\left\{\mathfrak{u}\in C\left([-r,T]\,,\mathbb{R}\right):\ D_\alpha\mathfrak{u}\in C\left(J,\mathbb{R}\right)\right\}.$$

The following result is an immediate consequence of Lemma 2.5 and Lemma 2.9.

Lemma 2.13. *Let* $0<\alpha\leq1$*. If* $\mathfrak{u}\in C^\alpha\left([-r,T]\,,\mathbb{R}\right)$*, then* $\mathfrak{u}$ *is a solution of the following integral equation*

$$\begin{cases}\mathfrak{u}(\tau)=\psi(0)+\int\limits_0^\tau s^{\alpha-1}\widetilde{g}(s,\mathfrak{u}(s)\,,\max\limits_{t\in[s-r,s]}\mathfrak{u}(t))ds,\ \textit{for all }\tau\in J,\\ \mathfrak{u}(\tau)=\psi(\tau)\,,\ \textit{for all }\tau\in[-r,0]\,,\end{cases}$$

if, and only if, $\mathfrak{u}$ *is a solution of the Cauchy problem (2.2).*

Now, we have the following result.

Theorem 2.14. *Assume that the following hypothesis are satisfied*

(H) There exists a positive constants L_1 and L_2 such that

$$\left|\widetilde{g}\left(t,\mathfrak{u}_1,\mathfrak{v}_1\right)-\widetilde{g}\left(t,\mathfrak{u}_2,\mathfrak{v}_2\right)\right|\leq L_1\left|\mathfrak{u}_1-\mathfrak{u}_2\right|+L_2\left|\mathfrak{v}_1-\mathfrak{v}_2\right|,$$

for all $t\in J$, $\mathfrak{u}_i\in\mathbb{R}$ and $\mathfrak{v}_i\in\mathbb{R}$ for $i=1,2$.

Then the problem (2.2) admits a unique solution $\mathfrak{u}\in C^{\alpha}\left(\left[-r,\mathfrak{T}\right],\mathbb{R}\right)$.

Proof. Let $\mathfrak{u}\in C^{\alpha}\left(\left[-r,\mathfrak{T}\right],\mathbb{R}\right)$ and consider the following equation

$$\begin{cases}\mathfrak{u}\left(\tau\right)=\psi\left(0\right)+\int\limits_0^{\tau}s^{\alpha-1}\widetilde{g}(s,\mathfrak{u}\left(s\right),\max\limits_{t\in[s-r,s]}\mathfrak{u}\left(t\right))ds,\text{ for all }\tau\in J,\\ \mathfrak{u}(\tau)=\psi\left(\tau\right),\text{ for all }\tau\in\left[-r,0\right].\end{cases}$$

Now we define the operator

$$\begin{aligned}A:\ &C^{\alpha}\left(\left[-r,\mathfrak{T}\right],\mathbb{R}\right)\to C^{\alpha}\left(\left[-r,\mathfrak{T}\right],\mathbb{R}\right)\\ &u\quad\mapsto\left(A\mathfrak{u}\right)\left(\tau\right)=\begin{cases}\psi\left(0\right)+\int\limits_0^{\tau}s^{\alpha-1}\widetilde{g}(s,\mathfrak{u}\left(s\right),\max\limits_{t\in[s-r,s]}\mathfrak{u}\left(t\right))ds,\text{ for all }\tau\in J,\\ \mathfrak{u}(\tau)=\psi\left(\tau\right),\text{ for all }\tau\in\left[-r,0\right],\end{cases}\end{aligned}$$

and we define the following norm

$$\left\|v\right\|=\max_{\tau\in[-r,\mathfrak{T}]}e^{-\frac{\lambda}{\alpha}|\tau|^{\alpha}}\left|v\left(\tau\right)\right|,$$

where $v\in C^{\alpha}\left(\left[-r,\mathfrak{T}\right],\mathbb{R}\right)$ and $\lambda>0$.

Since the norms $\|.\|_*$ and $\|.\|_0$ are equivalent, then $\left(C^{\alpha}\left(\left[-r,\mathfrak{T}\right],\mathbb{R}\right),\|.\|_*\right)$ is a Banach space.

Now let $\mathfrak{u}_1,\mathfrak{u}_2\in C^{\alpha}\left(\left[-r,\mathfrak{T}\right],\mathbb{R}\right)$, then for all $\tau\in J$, one has

$$\begin{aligned}&e^{-\frac{\lambda}{\alpha}\tau^{\alpha}}\left|\left(A\mathfrak{u}_1\right)\left(\tau\right)-\left(A\mathfrak{u}_2\right)\left(\tau\right)\right|\\ &=e^{-\frac{\lambda}{\alpha}\tau^{\alpha}}\left|\int\limits_0^{\tau}s^{\alpha-1}\left(\widetilde{g}(s,\mathfrak{u}_1\left(s\right),\max_{t\in[s-r,s]}\mathfrak{u}_1\left(t\right))-\widetilde{g}(s,\mathfrak{u}_2\left(s\right),\max_{t\in[s-r,s]}\mathfrak{u}_2\left(t\right))\right)ds\right|\\ &\leq e^{-\frac{\lambda}{\alpha}\tau^{\alpha}}\int\limits_0^{\tau}s^{\alpha-1}\left|\widetilde{g}(s,\mathfrak{u}_1\left(s\right),\max_{t\in[s-r,s]}\mathfrak{u}_1\left(t\right))-\widetilde{g}(s,\mathfrak{u}_2\left(s\right),\max_{t\in[s-r,s]}\mathfrak{u}_2\left(t\right))\right|ds\\ &\leq e^{-\frac{\lambda}{\alpha}\tau^{\alpha}}\int\limits_0^{\tau}s^{\alpha-1}\left(L_1\left|\mathfrak{u}_1\left(s\right)-\mathfrak{u}_2\left(s\right)\right|+L_2\left|\max_{t\in[s-r,s]}\mathfrak{u}_1\left(t\right)-\max_{t\in[s-r,s]}\mathfrak{u}_2\left(t\right)\right|\right)ds.\end{aligned}$$

From Lemma 2.10, we obtain

$$
\begin{aligned}
& e^{-\frac{\lambda}{\alpha}\tau^{\alpha}} \left|\left(A\mathfrak{u}_1\right)(\tau)-\left(A\mathfrak{u}_2\right)(\tau)\right| \\
& e^{-\frac{\lambda}{\alpha}\tau^{\alpha}} \int\limits_0^{\tau} s^{\alpha-1}\left(L_1\left|\mathfrak{u}_1(s)-\mathfrak{u}_2(s)\right|+L_2 \max_{t\in[s-r,s]}\left|\mathfrak{u}_1(t)-\mathfrak{u}_2(t)\right|\right) ds \\
& \leq e^{-\frac{\lambda}{\alpha}\tau^{\alpha}}\left(L_1+L_2\right)\left\|\mathfrak{u}_1-\mathfrak{u}_2\right\| \int\limits_0^{\tau} e^{\frac{\lambda}{\alpha}s^{\alpha}} s^{\alpha-1} ds \\
& = e^{-\frac{\lambda}{\alpha}\tau^{\alpha}}\left(L_1+L_2\right)\left\|\mathfrak{u}_1-\mathfrak{u}_2\right\|\left(\frac{e^{\frac{\lambda}{\alpha}\tau^{\alpha}}-1}{\lambda}\right) \\
& = \left(L_1+L_2\right)\left\|\mathfrak{u}_1-\mathfrak{u}_2\right\|\left(\frac{1-e^{-\frac{\lambda}{\alpha}\tau^{\alpha}}}{\lambda}\right) \\
& < \frac{\left(L_1+L_2\right)}{\lambda}\left\|u_1-u_2\right\|.
\end{aligned}
$$

If we choose $\lambda \geq (L_1+L_2)$, we obtain A is a contraction on $\left(C^{\alpha}\left([-r,\mathfrak{T}],\mathbb{R}\right),\|.\|\right)$ and therefore by Banach's fixed point theorem, the operator A admits a unique fixed point and consequently from Lemma 2.13, it follows that the problem (2.2) admits a unique solution $u \in C^{\alpha}\left([-r,\mathfrak{T}],\mathbb{R}\right)$. ■

Lemma 2.15. *Let $u \in C^{\alpha}\left([-r,\mathfrak{T}],\mathbb{R}\right)$ satisfying*

$$
\begin{cases}
\left(\mathfrak{D}_{\alpha}\mathfrak{u}\right)(\tau) \leq -M_1\mathfrak{u}(\tau) - N_1 \min\limits_{s\in[\tau-r,\tau]}\mathfrak{u}(s), \ \tau \in J, \\
\mathfrak{u}(0) \leq \mathfrak{u}(\tau) \leq 0, \textit{for all } \tau \in [-r,0],
\end{cases}
\tag{2.3}
$$

where $0<\alpha\leq 1$ and M_1 and N_1 are positive real numbers.

If

$$
\left(M_1+N_1\right)\frac{\mathfrak{T}^{\alpha}}{\alpha} \leq 1,
$$

then $\mathfrak{u} \leq 0$ in $[-r,\mathfrak{T}]$.

Proof. Assume that there exists $t_0 \in (0,\mathfrak{T}]$ such that

$$
u(t_0) > 0. \tag{2.4}
$$

We put by definition

$$
\mathfrak{u}(\eta) = \min_{t\in[-r,t_0]} u(t) \leq 0,
$$

where $\eta \in [0,t_0)$.

From Lemma 2.6, there exists $\sigma \in (\eta,t_0)$ such that

$$
\mathfrak{u}(t_0)-\mathfrak{u}(\eta) = \left(\frac{t_0^{\alpha}-\eta^{\alpha}}{\alpha}\right)\left(\mathfrak{D}_{\alpha}u\right)(\sigma).
$$

Then by using (2.3) and (2.4), we obtain

$$
-\mathfrak{u}(\eta) < -\left(M_1\mathfrak{u}(\sigma)+N_1\min_{s\in[\sigma-r,\sigma]}\mathfrak{u}(\sigma)\right)\left(\frac{t_0^{\alpha}-\eta^{\alpha}}{\alpha}\right).
$$

Which implies that

$$
\begin{aligned}
-\mathfrak{u}(\eta) &< -\left(M_1+N_1\right)\mathfrak{u}(\eta)\left(\frac{t_0^\alpha-\eta^\alpha}{\alpha}\right) \\
&< -\left(M_1+N_1\right)\mathfrak{u}(\eta)\frac{\mathfrak{T}^\alpha}{\alpha}.
\end{aligned}
$$

That is

$$
\left(M_1+N_1\right)\frac{\mathfrak{T}^\alpha}{\alpha} > 1 \text{ if } u(\eta) < 0.
$$

Which is a contradiction with the assumption

$$
\left(M_1+N_1\right)\frac{\mathfrak{T}^\alpha}{\alpha} \leq 1.
$$

If $\mathfrak{u}(\eta)=0$, we obtain also a contradiction.

Then, we have

$$
\mathfrak{u}(t) \leq 0, \text{ for all } t \in [-r,\mathfrak{T}].
$$

■

Remark 2.16. *The idea of the proof of the preceding Lemma 2.15 is similar to that of [26, Lemma 2.1 part i)].*

Lemma 2.17. *Assume that* $\mathfrak{u} \in C^\alpha([-r,\mathfrak{T}],\mathbb{R})$ *satisfying*

$$
\begin{cases}
\left(\mathfrak{D}_\alpha \mathfrak{u}\right)(t) \leq -\widetilde{M_1}\mathfrak{u}(t) - \widetilde{N_1}\max\limits_{s\in[t-r,t]}\mathfrak{u}(s),\ t \in J, \\
\mathfrak{u}(t) \leq 0, \textit{ for all } t \in [-r,0],
\end{cases}
$$

where $0<\alpha\leq 1$, $\widetilde{M_1}\leq 0$ *and* $\widetilde{N_1}\leq 0$.

If

$$
-\left(\widetilde{M_1}+\widetilde{N_1}\right)\frac{\mathfrak{T}^\alpha}{\alpha} < 1,
$$

then $\mathfrak{u}(t)\leq 0$, *for all* $t\in[-r,\mathfrak{T}]$.

Proof. Assume that there exists $t_1 \in (0,T]$ such that

$$
\mathfrak{u}(t_1) > 0.
$$

We put by definition

$$
\mathfrak{u}\left(\widetilde{t}\right) = \max_{t\in[-r,t_1]}\mathfrak{u}(t) > 0,
$$

where $\widetilde{t} \in (0,t_1]$.

We have

$$
\left(\mathfrak{D}_\alpha \mathfrak{u}\right)(t) \leq -\widetilde{M_1}\mathfrak{u}(t) - \widetilde{N_1}\max_{s\in[t-r,t]}\mathfrak{u}(s),\ t\in J.
$$

Which implies

$$
\left(\mathfrak{D}_\alpha \mathfrak{u}\right)(t) \leq -\left(\widetilde{M_1}+\widetilde{N_1}\right)\mathfrak{u}\left(\widetilde{t}\right)
$$

Applying the operator I_α to the both sides of the previous inequality, we obtain

$$
\mathfrak{u}\left(\widetilde{t}\right) - \mathfrak{u}(0) \leq -\left(\widetilde{M_1}+\widetilde{N_1}\right)u\left(\widetilde{t}\right)\int_0^{\widetilde{t}} s^{\alpha-1}ds
$$

That is
$$\mathfrak{u}\left(\widetilde{t}\right)-\mathfrak{u}\left(0\right)\leq-\frac{\left(\widetilde{M_1}+\widetilde{N_1}\right)\mathfrak{u}\left(\widetilde{t}\right)}{\alpha}\widetilde{t}^{\alpha}.$$
Which implies
$$\mathfrak{u}\left(\widetilde{t}\right)\leq-\frac{\left(\widetilde{M_1}+\widetilde{N_1}\right)\mathfrak{u}\left(\widetilde{t}\right)}{\alpha}\mathfrak{T}^{\alpha}.$$
Since $\mathfrak{u}\left(\widetilde{t}\right)>0$, we obtain
$$1\leq-\frac{\left(\widetilde{M_1}+\widetilde{N_1}\right)}{\alpha}\mathfrak{T}^{\alpha}.$$
Which is a contradiction with the assumption
$$-\frac{\left(\widetilde{M_1}+\widetilde{N_1}\right)}{\alpha}\mathfrak{T}^{\alpha}<1,$$
and then, we get
$$\mathfrak{u}\left(t\right)\leq 0,\text{ for all } t\in\left[-r,\mathfrak{T}\right].$$

■

3. Main Results

Definition 3.1. *We say that $\underline{\mathfrak{u}}\in C^{\alpha}\left(\left[-r,\mathfrak{T}\right],\mathbb{R}\right)$ is a lower solution of (1.1) if*
$$\begin{cases}(\mathfrak{D}_{\alpha}\underline{\mathfrak{u}})(\tau)\leq\mathfrak{f}(\tau,\underline{\mathfrak{u}}\left(\tau\right),\max\limits_{s\in[\tau-r,\tau]}\underline{\mathfrak{u}}\left(s\right)),\ \tau\in J,\\ \underline{\mathfrak{u}}\left(\tau\right)\leq\varphi\left(\tau\right),\ \tau\in\left[-r,0\right].\end{cases}$$

Definition 3.2. *We say that $\overline{\mathfrak{u}}\in C^{\alpha}\left(\left[-r,\mathfrak{T}\right],\mathbb{R}\right)$ is an upper solution of (1.1) if*
$$\begin{cases}\left(\mathfrak{D}_{\alpha}\overline{\mathfrak{u}}\right)\left(\tau\right)\geq\mathfrak{f}\left(\tau,\overline{\mathfrak{u}}\left(\tau\right),\max\limits_{s\in[\tau-r,\tau]}\overline{\mathfrak{u}}\left(s\right)\right),\ \tau\in J,\\ \overline{\mathfrak{u}}\left(\tau\right)\geq\varphi\left(\tau\right),\ \tau\in\left[-r,0\right].\end{cases}$$

Definition 3.3. *If $u\in C^{\alpha}\left(\left[-r,\mathfrak{T}\right],\mathbb{R}\right)$ and fulfills (1.1), then we say that $\mathfrak{u}$ is a solution of (1.1).*

We have the following result.

Theorem 3.4. *Assume that there two constants $M\geq 0$, $N\geq 0$ satisfying*

(H1) $\mathfrak{f}(\tau,x_1,y_1)-\mathfrak{f}\left(\tau,x_2,y_2\right)\geq-M\left(x_1-x_2\right)-N\left(y_1-y_2\right)$, *for all* $\tau\in J$, $\underline{\mathfrak{u}}\left(t\right)\leq x_2\leq x_1\leq\overline{\mathfrak{u}}\left(t\right)$ *and* $\max\limits_{s\in[t-r,t]}\underline{\mathfrak{u}}\left(s\right)\leq y_2\leq y_1\leq\max\limits_{s\in[t-r,t]}\overline{\mathfrak{u}}\left(s\right)$, *where* $\underline{\mathfrak{u}}$ *and* $\overline{\mathfrak{u}}$ *are lower and upper solutions respectively for problem (1.1) such that* $\underline{\mathfrak{u}}\leq\overline{\mathfrak{u}}$ *in* $[-r,\mathfrak{T}]$.

(H2) $\overline{\mathfrak{u}}\left(\tau\right)-\overline{\mathfrak{u}}\left(0\right)\leq\varphi\left(t\right)-\varphi\left(0\right)\leq\underline{\mathfrak{u}}\left(\tau\right)-\underline{\mathfrak{u}}\left(0\right)$, *for all* $\tau\in\left[-r,0\right]$.

(H3) $\left(M+N\right)\dfrac{\mathfrak{T}^{\alpha}}{\alpha}\leq 1$.

Then the problem (1.1) has a minimal solution $\mathfrak{u}_-$ and a maximal solution $\mathfrak{u}^+$ such that for every solution $\mathfrak{u}$ of (1.1) with $\underline{\mathfrak{u}}\leq\mathfrak{u}\leq\overline{\mathfrak{u}}$ in $[-r,\mathfrak{T}]$, we have
$$\underline{\mathfrak{u}}\leq\mathfrak{u}_-\leq\mathfrak{u}\leq\mathfrak{u}^+\leq\overline{\mathfrak{u}}\text{ in }\left[-r,\mathfrak{T}\right].$$

Proof. We take $\underline{\mathfrak{u}}_0 = \underline{\mathfrak{u}}$, and we define the sequences $(\underline{\mathfrak{u}}_n)_{n\geq 1}$ by

$$\begin{cases} (\mathfrak{D}_\alpha \underline{\mathfrak{u}}_{n+1})(\tau) + M\underline{\mathfrak{u}}_{n+1}(\tau) + N \min_{s\in[\tau-r,\tau]} \underline{\mathfrak{u}}_{n+1}(s) = \mathfrak{f}_n(\tau), \ \tau \in J, \\ \underline{\mathfrak{u}}_{n+1}(\tau) = \varphi(\tau), \ \tau \in [-r, 0], \end{cases} \tag{3.1}$$

where

$$\mathfrak{f}_n(\tau) = \mathfrak{f}(\tau, \underline{\mathfrak{u}}_n(\tau), \max_{s\in[\tau-r,\tau]} \underline{\mathfrak{u}}_n(s)) + M\underline{\mathfrak{u}}_n(\tau) + N \min_{s\in[\tau-r,\tau]} \underline{\mathfrak{u}}_n(s).$$

Analogously, we take $\overline{\mathfrak{u}}_0 = \overline{\mathfrak{u}}$ and we define the sequences $(\overline{\mathfrak{u}}_n)_{n\geq 1}$ by

$$\begin{cases} (\mathfrak{D}_\alpha \overline{\mathfrak{u}}_{n+1})(\tau) + M\overline{\mathfrak{u}}_{n+1}(\tau) + N \min_{s\in[\tau-r,\tau]} \overline{\mathfrak{u}}_{n+1}(s) = \widetilde{f}_n(\tau), \ \tau \in J, \\ \overline{\mathfrak{u}}_{n+1}(\tau) = \varphi(\tau), \ \tau \in [-r, 0], \end{cases} \tag{3.2}$$

where

$$\widetilde{f}_n(\tau) = \mathfrak{f}(\tau, \overline{\mathfrak{u}}_n(\tau), \max_{s\in[\tau-r,\tau]} \overline{\mathfrak{u}}_n(s)) + M\overline{\mathfrak{u}}_n(\tau) + N \min_{s\in[\tau-r,\tau]} \overline{\mathfrak{u}}_n(s).$$

Step 1: For all $n \in \mathbb{N}$, we have

$$\underline{\mathfrak{u}}_n \leq \underline{\mathfrak{u}}_{n+1} \leq \overline{\mathfrak{u}}_{n+1} \leq \overline{\mathfrak{u}}_n \text{ in } [-r, \mathfrak{T}].$$

Let

$$\mathfrak{v}_0(\tau) := \underline{\mathfrak{u}}_0(\tau) - \underline{\mathfrak{u}}_1(\tau), \tau \in [-r, \mathfrak{T}].$$

By (3.1) and using the definition of lower solution and the hypothesis (H2), we have

$$\begin{cases} (\mathfrak{D}_\alpha \mathfrak{v}_0)(\tau) + M\mathfrak{v}_0(\tau) + N\left(\max_{s\in[\tau-r,\tau]} \underline{\mathfrak{u}}_0(s) - \max_{s\in[\tau-r,\tau]} \underline{\mathfrak{u}}_1(s)\right) \leq 0, \tau \in J, \\ \mathfrak{v}_0(0) \leq \mathfrak{v}_0(\tau) \leq 0, \text{ for all } \tau \in [-r, 0]. \end{cases}$$

Then from Lemma 2.11, we obtain

$$\begin{cases} (\mathfrak{D}_\alpha \mathfrak{v}_0)(\tau) + M\mathfrak{v}_0(\tau) + N \min_{s\in[\tau-r,\tau]} \mathfrak{v}_0(s) \leq 0, \tau \in J, \\ v_0(0) \leq v_0(\tau) \leq 0, \text{ for all } \tau \in [-r, 0]. \end{cases}$$

From Lemma 2.15, one has

$$\mathfrak{v}_0 \leq 0 \text{ in } [-r, \mathfrak{T}].$$

Which means that

$$\underline{\mathfrak{u}}_0 \leq \underline{\mathfrak{u}}_1 \text{ in } [-r, \mathfrak{T}]. \tag{3.3}$$

Similarly, we can prove that

$$\overline{\mathfrak{u}}_1 \leq \overline{\mathfrak{u}}_0 \text{ in } [-r, \mathfrak{T}]. \tag{3.4}$$

Now, we put by definition

$$w_1(t) = \underline{u}_1(t) - \overline{u}_1(t), t \in [-r, \mathfrak{T}].$$

Using (3.1) and (3.2), we have

$$\begin{aligned} &(\mathfrak{D}_\alpha w_1)(\tau) + Mw_1(t) + N \min_{s\in[\tau-r,\tau]} w_1(s) \\ &= \mathfrak{f}_0(\tau) - \widetilde{f}_0(\tau) - N \max_{s\in[\tau-r,\tau]} \underline{u}_1(\tau) + N \max_{s\in[\tau-r,\tau]} \overline{u}_1(\tau) + N \min_{s\in[\tau-r,\tau]} w_1(s). \end{aligned}$$

From Lemma 2.11, we obtain

$$(\mathfrak{D}_\alpha w_1)(\tau) + Mw_1(\tau) + N \min_{s\in[\tau-r,\tau]} w_1(s) \leq \mathfrak{f}_0(\tau) - \widetilde{f}_0(\tau), \tau \in J.$$

Since $\underline{\mathfrak{u}}_0 = \underline{\mathfrak{u}} \leq \overline{\mathfrak{u}} = \overline{\mathfrak{u}}_0$ in $[-r, 0]$ and using the hypothesis (H1) , we obtain

$$\left(\mathfrak{D}_\alpha w_1\right)(\tau) + M w_1(\tau) + N \min_{s \in [\tau - r, \tau]} w_1(s) \leq 0, \ \tau \in J. \tag{3.5}$$

On the other hand, we have

$$w_1(\tau) = 0, \text{ for all } \tau \in [-r, 0].$$

That is

$$w_1(0) = w_1(\tau) = 0, \text{ for all } \tau \in [-r, 0]. \tag{3.6}$$

By the previous equality and (3.5), we have

$$\begin{cases} \left(\mathfrak{D}_\alpha w_1\right)(\tau) + M w_1(\tau) + N \min\limits_{s \in [\tau - r, \tau]} w_1(s) \leq 0, \ \tau \in J, \\ w_1(0) = w_1(\tau) = 0, \text{ for all } \tau \in [-r, 0]. \end{cases}$$

Then by hypothesis (H3) Lemma 2.15 implies

$$w_1 \leq 0 \text{ in } [-r, \mathfrak{T}].$$

Which means that

$$\underline{\mathfrak{u}}_1 \leq \overline{\mathfrak{u}}_1 \text{ in } [-r, \mathfrak{T}]. \tag{3.7}$$

Then by (3.3), (3.4) and (3.7), we have

$$\underline{u}_0 \leq \underline{u}_1 \leq \overline{u}_1 \leq \overline{u}_0 \text{ in } [-r, \mathfrak{T}].$$

Now we assume for fixed $n \geq 1$, we have

$$\underline{u}_n \leq \underline{u}_{n+1} \leq \overline{u}_{n+1} \leq \overline{u}_n \text{ in } [-r, \mathfrak{T}],$$

and we show that

$$\underline{u}_{n+1} \leq \underline{u}_{n+2} \leq \overline{u}_{n+2} \leq \overline{u}_{n+1} \text{ in } [-r, \mathfrak{T}].$$

We put by definition

$$v_{n+1}(\tau) := \underline{\mathfrak{u}}_{n+1}(\tau) - \underline{\mathfrak{u}}_{n+2}(\tau), \tau \in [-r, \mathfrak{T}].$$

By (3.1), we have

$$\begin{cases} \left(\mathfrak{D}_\alpha v_{n+1}\right)(\tau) + M v_{n+1}(\tau) + N \min\limits_{s \in [\tau - r, \tau]} v_{n+1}(s) = \mathfrak{g}_n(\tau), \ \tau \in J, \\ v_{n+1}(0) = v_{n+1}(\tau) = 0, \tau \in [-r, 0], \end{cases}$$

where

$$\mathfrak{g}_n(\tau) = \mathfrak{f}_n(\tau) - \mathfrak{f}_{n+1}(\tau), \text{ for all } \tau \in J.$$

Since by the hypothesis of recurrence, we have $\underline{\mathfrak{u}}_n \leq \underline{\mathfrak{u}}_{n+1}$ in J and from Lemma 2.11 and using the hypothesis (H1), we obtain

$$\begin{cases} \left(\mathfrak{D}_\alpha v_{n+1}\right)(\tau) + M v_{n+1}(\tau) + N \min\limits_{s \in [\tau - r, \tau]} v_{n+1}(s) \leq 0, \ \tau \in J, \\ v_{n+1}(0) = v_{n+1}(\tau) = 0, \tau \in [-r, 0], \end{cases}$$

and then from Lemma 2.15, we get

$$v_{n+1} \leq 0 \text{ in } [-r, \mathfrak{T}].$$

That is

$$\underline{\mathfrak{u}}_{n+1} \leq \underline{\mathfrak{u}}_{n+2} \text{ in } [-r, \mathfrak{T}]. \tag{3.8}$$

Similarly, we can prove that
$$\overline{\mathfrak{u}}_{n+2} \leq \overline{\mathfrak{u}}_{n+1} \text{ in } [-r, \mathfrak{T}], \tag{3.9}$$
and
$$\underline{\mathfrak{u}}_{n+2} \leq \overline{\mathfrak{u}}_{n+2} \text{ in } [-r, \mathfrak{T}]. \tag{3.10}$$
Then by (3.8), (3.9) and (3.10), we obtain
$$\underline{\mathfrak{u}}_{n+1} \leq \underline{\mathfrak{u}}_{n+2} \leq \overline{\mathfrak{u}}_{n+2} \leq \overline{\mathfrak{u}}_{n+1} \text{ in } [-r, \mathfrak{T}].$$
Hence for all $n \in \mathbb{N}$, we have
$$\underline{\mathfrak{u}}_{n} \leq \underline{\mathfrak{u}}_{n+1} \leq \overline{\mathfrak{u}}_{n+1} \leq \overline{\mathfrak{u}}_{n} \text{ in } [-r, \mathfrak{T}].$$

Step 2: The consequence $(\underline{\mathfrak{u}}_n)_{n\in\mathbb{N}}$ converges to a minimal solution of (1.1).

By **Step 1** and using Dini theorem, it follows that the sequence of functions $(\mathfrak{u}_n)_{n\in\mathbb{N}}$ converges uniformly to $\mathfrak{u}_-$.

Let $n \in \mathbb{N}^*$ and $t \in J$, then from Lemma 2.13 we get
$$\underline{\mathfrak{u}}_{n+1}(\tau) - \underline{\mathfrak{u}}_{n+1}(0) = \int_0^\tau s^{\alpha-1} \mathfrak{F}_n(s)\, ds,$$
where
$$\mathfrak{F}_n(s) = \mathfrak{f}_n(s) - M\underline{\mathfrak{u}}_{n+1}(s) - N \max_{t\in[s-r,s]} \underline{\mathfrak{u}}_{n+1}(t).$$
Now, as n tends to $+\infty$, we obtain
$$\mathfrak{F}_n(s) \to \mathfrak{f}(s, \mathfrak{u}_-(s), \max_{t\in[s-r,s]} \mathfrak{u}_-(s)).$$
Which implies
$$-u_-(\tau) - \mathfrak{u}_-(0) = \int_0^\tau s^{\alpha-1} \mathfrak{f}(s, \mathfrak{u}_-(s), \max_{t\in[s-r,s]} \mathfrak{u}_-(s)) ds,$$
and from Lemma 2.13, we deduce
$$(\mathfrak{D}_\alpha \mathfrak{u}_-)(t) = \mathfrak{f}(\tau, \mathfrak{u}_-(\tau), \max_{t\in[\tau-r,\tau]} \mathfrak{u}_-(t)),\ \tau \in J.$$
On the other hand, we have
$$\mathfrak{u}_- = \varphi \text{ in } [-r, 0],$$
and consequently it follows that $\mathfrak{u}_-$ is a solution of (1.1).

Now, we prove that if $\mathfrak{u}$ is another solution of (1.1) such that $\underline{\mathfrak{u}} \leq \mathfrak{u} \leq \overline{\mathfrak{u}}$, then $\mathfrak{u}_- \leq \mathfrak{u}$.

Since $\mathfrak{u}$ is an upper solution of (1.1), then by **Step 1**, we have
$$\forall n \in \mathbb{N},\ \underline{\mathfrak{u}}_n \leq \mathfrak{u}.$$
Which implies that
$$\mathfrak{u}_- = \lim_{n\to+\infty} \underline{\mathfrak{u}}_n \leq \mathfrak{u}.$$
This means that $\mathfrak{u}_-$ is a minimal solution of (1.1).

The second step's proof is finished.

In a similar way, we can prove that the sequence $(\overline{\mathfrak{u}}_n)_{n\in\mathbb{N}}$ converges to a maximal solution $\mathfrak{u}^+$ of (1.1).

The proof of Theorem 3.4 is complete. ■

For the uniqueness of solutions for the problem (1.1), it is necessary to impose additional conditions on $\mathfrak{f}$.

(H4) There exists a negative real number M_1 such that the function $x \longmapsto \mathfrak{f}(\tau, x, y) + M_1 y$ is decreasing for all $\tau \in J$ and $y \in \mathbb{R}$.

(H5) There exists a negative real number N_1 such that the function $y \longmapsto \mathfrak{f}(\tau, x, y) + N_1 y$ is decreasing for all $\tau \in J$ and $x \in \mathbb{R}$.

(H6) $-(M_1 + N_1)\dfrac{\mathfrak{T}^\alpha}{\alpha} < 1.$

We have the following result.

Theorem 3.5. *Assume that hypothesis (Hi) for i = 1,...,6 are satisfied, then the problem (1.1) admits a unique solution* $\mathfrak{u}$ *such that* $\underline{\mathfrak{u}} \leq \mathfrak{u} \leq \overline{\mathfrak{u}}$ *in* $[-r, \mathfrak{T}]$.

Proof. By Theorem 3.4, the problem (*1.1*) admits a minimal and a maximal solutions $\mathfrak{u}_-$ and $\mathfrak{u}^+$ such that

$$\underline{\mathfrak{u}} \leq \mathfrak{u}_- \leq \mathfrak{u}^+ \leq \overline{\mathfrak{u}} \text{ in } [-r, \mathfrak{T}].$$

We put by definition

$$\mathfrak{z}(\tau) = \mathfrak{u}^+(\tau) - \mathfrak{u}_-(\tau), \tau \in [-r, \mathfrak{T}].$$

We have

$$\mathfrak{z} \geq 0 \text{ in } [-r, \mathfrak{T}]. \tag{3.11}$$

Now, we are going to prove that

$$\mathfrak{z} \leq 0 \text{ in } [-r, \mathfrak{T}].$$

As we have

$$\begin{cases} (\mathfrak{D}_\alpha \mathfrak{z})(\tau) = \mathfrak{f}(\tau, \mathfrak{u}^+(\tau), \max\limits_{t \in [\tau - r, \tau]} \mathfrak{u}^+(t)) - \mathfrak{f}(\tau, \mathfrak{u}_-(\tau), \max\limits_{t \in [\tau - r, \tau]} u_-(t)),\ \tau \in J, \\ \mathfrak{z}(0) = \mathfrak{z}(\tau) = 0,\ \tau \in [-r, 0]. \end{cases}$$

By using the hypothesis (H4), we obtain

$$\begin{cases} (\mathfrak{D}_\alpha \mathfrak{z})(\tau) + M_1 \mathfrak{z}(\tau) \leq \\ \mathfrak{f}(\tau, \mathfrak{u}_-(\tau), \max\limits_{t \in [\tau - r, \tau]} \mathfrak{u}^+(t)) - \mathfrak{f}(\tau, \mathfrak{u}_-(\tau), \max\limits_{t \in [\tau - r, \tau]} u_-(t)),\ \tau \in J, \\ \mathfrak{z}(0) = \mathfrak{z}(\tau) = 0,\ \tau \in [-r, 0]. \end{cases}$$

Now from Lemma 2.10, we have

$$\max_{t \in [\tau - r, \tau]} z(t) = \max_{t \in [\tau - r, \tau]} \left| \mathfrak{u}^+(t) - \mathfrak{u}_-(t) \right| \geq \max_{t \in [\tau - r, \tau]} \mathfrak{u}^+(t) - \max_{t \in [\tau - r, \tau]} u_-(t),$$

and then according to hypothesis (H4), we obtain

$$\begin{cases} (\mathfrak{D}_\alpha \mathfrak{z})(\tau) + M_1 z(\tau) + N_1 \max\limits_{t \in [\tau - r, \tau]} \mathfrak{z}(t) \leq 0,\ \tau \in J, \\ \mathfrak{z}(0) = \mathfrak{z}(\tau) = 0,\ \tau \in [-r, 0]. \end{cases}$$

From Lemma 2.17, we get

$$\mathfrak{z}(t) \leq 0 \text{ in } [-r, \mathfrak{T}],$$

and therefore, there is a unique solution to problem (1.1).

■

4. Applications

4.1. Example 1

Consider the problem

$$\begin{cases} \mathfrak{D}_{\frac{1}{2}}\mathfrak{u}(\tau)=\sqrt{\tau}\mathfrak{u}(\tau)-\max\limits_{s\in[\tau-1,\tau]}\mathfrak{u}(s)+\cos\tau,\ \tau\in\left[0,\dfrac{1}{4}\right], \\ \mathfrak{u}(\tau)=\tau,\ \tau\in[-1,0]. \end{cases} \tag{4.1}$$

Let $\underline{\mathfrak{u}}(\tau)=\tau$ and $\overline{\mathfrak{u}}(\tau)=\tau+1$ in $\left[-1,\dfrac{1}{4}\right]$.

For the problem (4.1), $\underline{\mathfrak{u}}$ is a lower solution if

$$\begin{cases} \mathfrak{D}_{\frac{1}{2}}\underline{\mathfrak{u}}(\tau)\leq\sqrt{\tau}\underline{\mathfrak{u}}(\tau)-\max\limits_{s\in[\tau-1,\tau]}\underline{\mathfrak{u}}(s)+\cos\tau,\ \tau\in\left[0,\dfrac{1}{4}\right], \\ \underline{\mathfrak{u}}(\tau)\leq\tau,,\ \tau\in[-1,0]. \end{cases}$$

That is

$$\begin{cases} \sqrt{\tau}\leq\tau^{\frac{3}{2}}-\tau+\cos\tau,\ \tau\in\left[0,\dfrac{1}{4}\right], \\ \tau\leq\tau,\ \tau\in[-1,0]. \end{cases}$$

Since $\varphi_1(\tau)=\sqrt{\tau}-\tau^{\frac{3}{2}}+\tau-\cos\tau\leq 0$ in $\left[0,\dfrac{1}{4}\right]$,

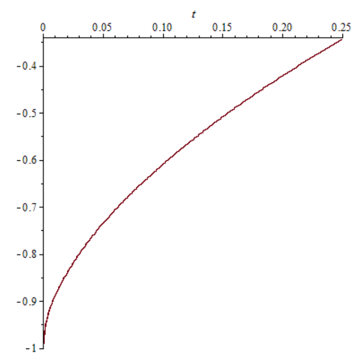

Figure 1: Graph of the function φ_1

we conclude that $\underline{\mathfrak{u}}$ is a lower solution for the problem (4.1).

Similarly if we have

$$\begin{cases} \mathfrak{D}_{\frac{1}{2}}\overline{\mathfrak{u}}(\tau)\geq\sqrt{\tau}\overline{\mathfrak{u}}(\tau)-\max\limits_{s\in[\tau-1,\tau]}\overline{\mathfrak{u}}(s)+\cos\tau,\ \tau\in\left[0,\dfrac{1}{4}\right], \\ \overline{\mathfrak{u}}(\tau)\geq\tau, \tau\in[-1,0]. \end{cases}$$

we obtain $\overline{\mathfrak{u}}$ is an upper solution for the problem (4.1).

That is

$$\begin{cases} \tau^{\frac{3}{2}}-\tau-1+\cos\tau\leq 0,\ \tau\in\left[0,\dfrac{1}{4}\right], \\ \tau+1\geq\tau, \tau\in[-1,0]. \end{cases}$$

Since $\varphi_2(\tau)=\tau^{\frac{3}{2}}-\tau-1+\cos\tau\leq 0$, for all $\tau\in\left[0,\dfrac{1}{4}\right]$,

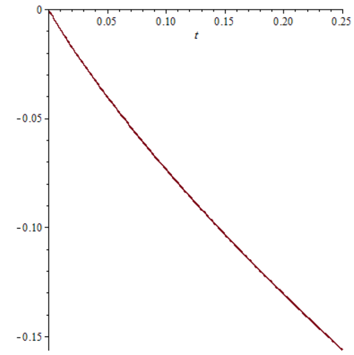

Figure 2: Graph of the function φ_2

we obtain the desired upper solution for the problem (4.1).

Now, if we select $N=1$ and $M=0$, then

$$(M+N)\frac{T^{\alpha}}{\alpha}=\frac{\left(\frac{1}{4}\right)^{\frac{1}{2}}}{\frac{1}{2}}=1\leq 1,$$

and if we choose $M_1=-\frac{1}{2}$ and $N_1=0$, we have

$$-(M_1+N_1)\frac{T^{\alpha}}{\alpha}=\frac{\left(\frac{1}{4}\right)^{\frac{1}{2}}}{2\times\frac{1}{2}}=\frac{1}{2}<1.$$

Nevertheless, it is evident that the function $\tau\mapsto\sqrt{\tau}u(\tau)-\max\limits_{s\in[\tau-1,\tau]}u(s)+\cos\tau$ satisfies the remaining assumptions of Theorem 3.5. As a result, the problem (4.1) admits a unique solution $\mathfrak{u}$ such that $\underline{\mathfrak{u}}\leq u\leq\overline{\mathfrak{u}}$.

4.2. Example 2

Consider the problem

$$\begin{cases}\mathfrak{D}_{\frac{2}{3}}\mathfrak{u}(\tau)=\dfrac{\tau^{\frac{2}{3}}}{4}\mathfrak{u}(\tau)-\dfrac{\max\limits_{s\in[\tau-1,\tau]}\mathfrak{u}(s)}{8}+\dfrac{2}{3}(\tau^{\frac{2}{3}}+1)+\dfrac{1}{8},\ \tau\in\left[0,\frac{1}{2}\right],\\ \mathfrak{u}(\tau)=\tau^{\frac{2}{3}},,\ \tau\in[-1,0].\end{cases}\tag{4.2}$$

Let $\underline{\mathfrak{u}}(\tau)=\tau^{\frac{2}{3}}$ and $\overline{\mathfrak{u}}(\tau)=2\tau^{\frac{2}{3}}+1$, in $\left[-1,\frac{1}{2}\right]$.

For the problem (4.2), $\underline{\mathfrak{u}}$ is a lower solution if we have

$$\begin{cases}\mathfrak{D}_{\frac{2}{3}}\underline{\mathfrak{u}}(\tau)\leq\dfrac{\tau^{\frac{2}{3}}}{4}\underline{\mathfrak{u}}(\tau)-\dfrac{\max\limits_{s\in[\tau-1,\tau]}\mathfrak{u}(s)}{8}+\dfrac{2}{3}(\tau^{\frac{2}{3}}+1)+\dfrac{1}{8},\ \tau\in\left[0,\dfrac{1}{2}\right],\\ \underline{\mathfrak{u}}(\tau)\leq\tau^{\frac{2}{3}},\tau\in[-1,0].\end{cases}$$

That is

$$\begin{cases}\dfrac{2}{3}\leq\dfrac{\tau^{\frac{4}{3}}}{4}-\dfrac{(\tau-1)^{\frac{2}{3}}}{8}+\dfrac{2}{3}(\tau^{\frac{2}{3}}+1)+\dfrac{1}{8},\ \tau\in\left[0,\dfrac{1}{2}\right],\\ \tau^{\frac{2}{3}}\leq\tau^{\frac{2}{3}},\tau\in[-1,0].\end{cases}$$

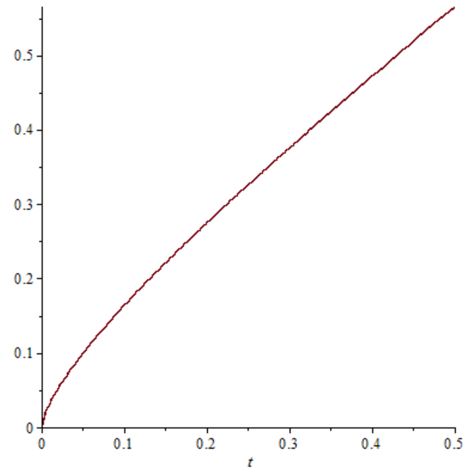

Figure 3: Graph of the function φ_3

Since $\varphi_3(\tau) = \frac{\tau^{\frac{4}{3}}}{4} - \frac{(\tau-1)^{\frac{2}{3}}}{8} + \frac{2}{3}\tau^{\frac{2}{3}} + \frac{1}{8} \geq 0$, for all $\tau \in \left[0, \frac{1}{2}\right]$,

we get $\underline{\mathrm{u}}$ is a lower solution for the problem (4.2).

Similarly if

$$\begin{cases} \mathfrak{D}_{\frac{2}{3}}\overline{\mathrm{u}}\,(\tau) \geq \frac{\tau^{\frac{2}{3}}}{4}\overline{\mathrm{u}}\,(\tau) - \frac{\max\limits_{s\in[\tau-1,\tau]} \overline{u}\,(s)}{8} + \frac{2}{3}(\tau^{\frac{2}{3}}+1) + \frac{1}{8},\ \tau \in \left[0, \frac{1}{2}\right], \\ \overline{\mathrm{u}}(\tau) \geq \tau^{\frac{2}{3}}, \tau \in [-1, 0]\,. \end{cases}$$

we obtain $\overline{\mathrm{u}}$ is an upper solution for the problem (4.2).

That is

$$\begin{cases} \frac{4}{3} \geq \frac{\tau^{\frac{2}{3}}}{4}\left(2\tau^{\frac{2}{3}}+1\right) - \left(\frac{2\,(\tau-1)^{\frac{2}{3}}+1}{8}\right) + \frac{2}{3}(t^{\frac{2}{3}}+1) + \frac{1}{8},\ \tau \in \left[0, \frac{1}{2}\right], \\ 2\tau^{\frac{2}{3}}+1 \geq \tau^{\frac{2}{3}},\ \ \tau \in [-1, 0]\,. \end{cases}$$

That is

$$\begin{cases} \frac{\tau^{\frac{4}{3}}}{2} + \frac{11}{8}\tau^{\frac{2}{3}} - \frac{(\tau-1)^{\frac{2}{3}}}{4} - \frac{2}{3} \leq 0,\ \tau \in \left[0, \frac{1}{2}\right], \\ \tau^{\frac{2}{3}} + 1 \geq 0,\ ,\ \tau \in [-1, 0]\,. \end{cases}$$

Since $\varphi_4(\tau) = \frac{\tau^{\frac{4}{3}}}{2} + \frac{11}{8}\tau^{\frac{2}{3}} - \frac{(\tau-1)^{\frac{2}{3}}}{4} - \frac{2}{3} \leq 0$, for all $\tau \in \left[0, \frac{1}{2}\right]$,

we obtain the desired upper solution for the problem (4.2).

Now, if we select $M = 0$ and $N = \frac{1}{8}$, then

$$(M+N)\,\frac{T^{\alpha}}{\alpha} = \frac{1}{8}\frac{\left(\frac{1}{2}\right)^{\frac{2}{3}}}{\frac{2}{3}} = 0.11812 \leq 1,$$

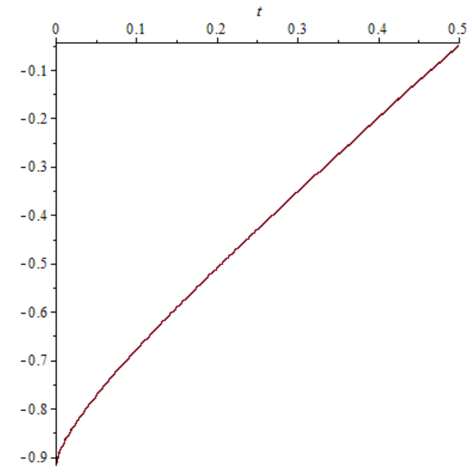

Figure 4: Graph of the function φ_4

and if we choose $M_1 = -\frac{1}{2}$ and $N_1 = 0$, we have

$$-(M_1 + N_1)\frac{T^\alpha}{\alpha} = \frac{\left(\frac{1}{2}\right)^{\frac{2}{3}}}{2 \times \frac{2}{3}} = 0.47247 < 1.$$

Nevertheless, it is evident that the function $\tau \mapsto \frac{\tau^{\frac{2}{3}}}{4} u(\tau) - \frac{\max\limits_{s\in[\tau-1,\tau]} u(s)}{8} + \frac{2}{3}(\tau^{\frac{2}{3}} + 1) + \frac{1}{8}$ satisfies the remaining assumptions of Theorem 3.5. As a result, the problem (4.2) admits a unique solution u such that $\underline{\mathfrak{u}} \leq \mathfrak{u} \leq \overline{\mathfrak{u}}$.

4.3. Example 3

Consider the problem

$$\begin{cases} \mathfrak{D}_{\frac{1}{2}} \mathfrak{u}(\tau) = -\frac{\mathfrak{u}(\tau)}{2} - \frac{\max\limits_{s\in\left[\tau-\frac{\pi}{2},\tau\right]} \mathfrak{u}(s)}{2} + \sqrt{\tau}\cos(\tau) + \sin(\tau), \ \tau \in \left[0, \frac{\pi}{16}\right], \\ \mathfrak{u}(\tau) = 1 + \tau, \tau \in \left[-\frac{\pi}{2}, 0\right]. \end{cases} \tag{4.3}$$

Let $\underline{\mathfrak{u}}(\tau) = \sin(\tau)$ and $\overline{\mathfrak{u}}(\tau) = 1$, for all $\tau \in \left[0, \frac{\pi}{16}\right]$.

First $\underline{u}$ is a lower solution if

$$\begin{cases} \mathfrak{D}_{\frac{1}{2}} \underline{\mathfrak{u}}(\tau) \leq -\frac{\underline{\mathfrak{u}}(\tau)}{2} - \frac{\max\limits_{t\in\left[\tau-\frac{\pi}{2},\tau\right]} \underline{\mathfrak{u}}(t)}{2} + \sqrt{\tau}\cos(\tau) + \sin(\tau), \ \tau \in \left[0, \frac{\pi}{16}\right], \\ \underline{\mathfrak{u}}(\tau) \leq 1 + \tau, \tau \in \left[-\frac{\pi}{2}, 0\right]. \end{cases}$$

That is

$$\begin{cases} \sqrt{\tau}\cos(\tau) \leq \left(1 - \frac{2}{2}\right)\frac{\sin(\tau)}{2} + \sqrt{\tau}\cos(\tau), \ \tau \in \left[0, \frac{\pi}{16}\right], \\ \sin(\tau) \leq 1 + \tau, \tau \in \left[-\frac{\pi}{2}, 0\right]. \end{cases}$$

Since $\sin(\tau)-1-\tau\leq 0$, for all $\tau\in\left[-\frac{\pi}{2},0\right]$, we conclude that $\underline{\mathfrak{u}}$ is a lower solution for the problem (4.3).

Similarly if we have,

$$\begin{cases}\mathfrak{D}_{\frac{1}{2}}\overline{\mathfrak{u}}(\tau)\geq-\dfrac{\overline{u}(\tau)}{2}-\dfrac{\max\limits_{s\in\left[\tau-\frac{\pi}{2},\tau\right]}\mathfrak{u}(s)}{2}+\sqrt{\tau}\cos(\tau)+\sin(\tau)\,,\ \tau\in\left[0,\frac{\pi}{16}\right],\\ \overline{\mathfrak{u}}(\tau)\geq 1+\tau,\tau\in\left[-\frac{\pi}{2},0\right].\end{cases}$$

we obtain $\overline{\mathfrak{u}}$ is an upper solution for the problem (4.3).

That is

$$\begin{cases}0\geq-1+\sqrt{\tau}\cos(\tau)+\sin(\tau)\,,\ \tau\in\left[0,\frac{\pi}{16}\right],\\ 1\geq 1+\tau,\tau\in\left[-\frac{\pi}{2},0\right].\end{cases}$$

Since $\varphi_5(\tau)=-1+\sqrt{\tau}\cos(\tau)+\sin(\tau)\leq 0$, for all $\tau\in\left[0,\frac{\pi}{16}\right]$,

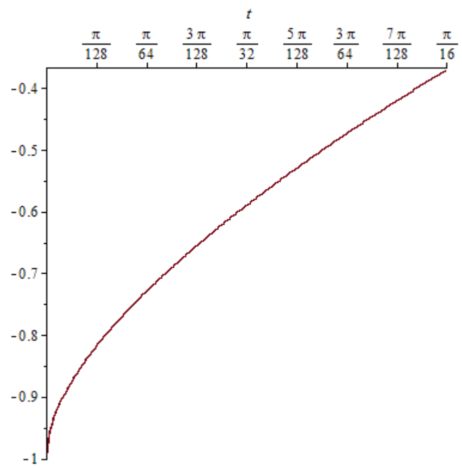

Figure 5: Graph of the function φ_5

we obtain the desired upper solution for the problem (4.3).

Now, if we select $M=N+\frac{1}{2}$, then $\dfrac{\left(\frac{\pi}{16}\right)^{\frac{1}{2}}}{\frac{1}{2}}=0.88623\leq 1$ and the function $\tau\mapsto-\dfrac{\mathfrak{u}\left(\frac{\tau}{2}\right)}{2\pi}+\dfrac{\cos\left(\sqrt{\tau}\right)}{2}+\sin\left(\sqrt{\frac{\tau}{2}}\right)$ satisfies the remaining assumptions of Theorem 3.5. As a result, the problem (4.3) admits a unique solution u such that $\underline{\mathfrak{u}}\leq\mathfrak{u}\leq\overline{\mathfrak{u}}$.

References

[1] T. Abdeljawad, On conformable fractional calculus. *J. Comput. Appl. Math.*, **279**(2015), 57–66.

[2] E. A.-B. Abdel-Salam and M. I. Nouh, Conformable fractional polytropic gas spheres, *New Astronomy*, **76**(2020), Article 101322, 8 pages.

[3] D. Anderson, E. Camrud and D. Ulness, On the nature of the conformable derivative and its applications to physics, *J. Fract. Calc. Appl.*, **10**(2019), 92–135.

[4] V. G. ANGELOV AND D. D. BAINOV, On the functional differential equations with "maximums", *Appl. Anal.*, **16**(1983), 187–194.

[5] J.A.D. APPLEBY AND H. WU, Exponential growth and Gaussian–like fluctuations of solutions of stochastic differential equations with maximum functionals, *J. Phys.: Conf. Ser.*, **138**(012002) (2008), 1–25.

[6] V. AZHMYAKOV, A. AHMED AND E. I. VERRIEST, On the optimal control of systems evolving with state suprema, : Proceedings of the 2016 IEEE 55th Conference on Decision and Control, Las Vegas, USA, 2016, 3617–3623.

[7] D. D. BAINOV AND S. G. HRISTOVA, Monotone-iterative techniques of Lakshmikantham for a boundary value problem for systems of differential equations with maxima, *J. Math. Anal. Appl.*, **190**(1995), 391–401.

[8] D. D. BAINOV AND S. G. HRISTOVA, Differential equations with maxima, Chapman & Hall/CRC Pure and Applied Mathematics, 2011.

[9] B. BENDOUMA, A. CABADA, AND A HAMMOUDI, Existence results for conformable fractional problems with nonlinear functional boundary conditions, *Malaya Journal of Matematik*, **7**(2019), 700–708.

[10] B. BENDOUMA, A. CABADA, AND A HAMMOUDI, Existence results for systems of conformable fractional differential equations, *Arch. Math. (Brno).*, **55**(2019), 69–82.

[11] M. BOHNER AND V. F. HATIPOĞLU, Cobweb model with conformable fractional derivatives, *Math Meth Appl Sci.*, **41**(2018),1–8.

[12] T. A. BURTON, Volterra Integral and Differential Equations, Second Edition. Elsevier, Amsterdam, (2005).

[13] T. A. BURTON, Lyapunov Theory for Integral Equations with Singular Kernels and Fractional Differential Equations, Publisher Amazon.com, (2012).

[14] H. CHEN, S. MENG AND Y. CUI, Monotone iterative technique for conformable fractional differential equations with deviating arguments, *Discrete Dyn. Nat. Soc.*, **2018**(2018), Article ID 5827127, 9 pages.

[15] S. DASHKOVSKIY, O. KICHMARENKO AND K. SAPOZHNIKOVA, Approximation of solutions to the optimal control problems for systems with maximum, *J. Math. Sci. (N. Y.)*, **243**(2019), 192–203.

[16] M. DERHAB, Existence of extremal solutions for a class of conformable fractional differential equations with deviating arguments and with nonlocal initial Condition, *Comm. Appl. Nonlinear Anal.*, **29**(2022), 65 – 84.

[17] G. FERNÁNDEZ-ANAYA, S. QUEZADA-GARCÍA, M.A. POLO-LABARRIOS AND L.A. QUEZADA-TÉLLEZ, Novel solution to the fractional neutron point kinetic equation using conformable derivatives, *Annals of Nuclear Energy*, **160**(2021) 108407

[18] K. P. HADELER, On the theory of lateral inhibition, *Kybernetik*, **14**(1974), 161–165.

[19] A. HALANAY, Differential Equations Stability, Oscillations, Time Lags, Academic Press New-York and London, 1966.

[20] J.K. HALE AND S.M. VERDUYN LUNEL, Introduction to Functional Differential Equations, Applied mathematical sciences, Springer-Verlag New York Inc. 1993.

[21] I. KARAFYLLIS AND Z. P. JIANG, Stability and Stabilization of Nonlinear Systems, Communications and Control Engineering Series. Springer-Verlag London, Ltd., London, 2011.

[22] S. M. KHALED, E. R. EL-ZAHAR AND A. EBAID, Solution of Ambartsumian delay differential equation with conformable derivative, *Mathematics*, **7**(2019), 10 pages.

[23] R. KHALIL, M. AL HORANI, A. YOUSEF AND M. SABABHEH, A new definition of fractional derivative. *Journal of Computational and Applied Mathematics*, **264**(2014), 65–70.

[24] H. KISKINOVA, M. PETKOVAB AND A. ZAHARIEV, About the Cauchy problem for nonlinear system with conformable derivatives and variable delays, *AIP Conference Proceedings*, **2172, 050006**(2019), 9 pages.

[25] D. KUMAR, A.R. SEADAWY AND A.K JOARDAR, Modified Kudryashov method via new exact solutions for some conformable fractional differential equations arising in mathematical biology, *Chinese Journal of Physics*, **56**(2018), 75–85.

[26] V. LAKSHMIKANTHAM AND B.G. ZHANG, Monotone iterative technique for delay differential equations, *Appl. Anal.*, **22**(1986), 227–233.

[27] S. LIUA, H. WANG, X. LI AND H. LI, The extremal iteration solution to a coupled system of nonlinear conformable fractional differential equations, *J. Nonlinear Sci. Appl.*, **10**(2017), 5082–5089.

[28] E. LIZ AND S. TROFIMCHUK, On a dynamical model for happiness, *Math. Model. Nat. Phenom.*, **18**(2023), 15 pages.

[29] S. MENG AND Y. CUI, The extremal solution to conformable fractional differential equations involving integral boundary condition, *Mathematics* **7**(2019), 273-281.

[30] E. P. POPOV, Automatic Regulation and Control, Nauka, Moscow, 1966. (in Russian).

[31] C. THAIPRAYOON, S. K. NTOUYAS AND J. TARIBOON, Monotone iterative technique for nonlinear impulsive conformable fractional differential equations with delay, *Communications in Mathematics and Applications*, **12**(2021), 11–27.

[32] M. T. TEREKHIN AND V. V. KIRYUSHKIN, Nonzero solutions to a two-point boundary-value periodic problem for differential equations with maxima, *Russian Math. (Iz. VUZ)*, **54**(2010), 43–53.

[33] O. TROFYMCHUK, E. LIZ AND S. TROFIMCHUK, The peak-end rule and its dynamic realization through differential equations with maxima, *Nonlinearity*, **36**(2023), 507–536.

[34] W.-Z. WU, L. ZENG, C. LIU, W. XIE AND M. GOH, A time power-based grey model with conformable fractional derivative and its applications, Chaos, Solitons and Fractals, **155**(2022), 111657.

[35] S. YANG, L. WANG AND S. ZHANG, Conformable derivative: Application to non-Darcian flow in low-permeability porous media, *Appl. Math. Lett.*, **79**(2018), 105–110.

[36] W. ZHONG AND L. WANG, Basic theory of initial value problems of conformable fractional differential equations, *Adv. Difference Equ.*, **321**(2018), 14 pages.

[37] W. ZHONG AND L. WANG, Positive solutions of conformable fractional differential equations with integral boundary conditions, *Bound. Value Probl.*, **137**(2018), 12 pages.