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Perrin's polynomial bracelets

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Abstract. The present research performs a study of the combinatorial approach for the Tetrarrin sequence and its polynomial form. Primarily, we initially have the definition of the Tetrarrin sequence, which is an extension of the Perrin sequence. Next, we present the theorem referring to its combinatorial interpretation via bracelets. The combinatorial approach referring to the polynomial Tetrarrin sequence is established based on the combinatorial model of Tridovan, assigning weights to the pieces to then form the polynomial Tetrarrin bracelets. Finally, these models present a way of visualizing the terms of the sequence, allowing a differentiated approach for the study of recurrent numerical sequences.

AMS Subject Classifications: 65Q30, 11B37.

Keywords: tetrarrin bracelets, tetrarrin sequence, tetrarrin polynomial sequence.

Contents

1. Introduction and Background

The present work defines a combinatorial interpretation related to the Tetrarrin sequence and another for its polynomial form. Thus, the reader will be able to see discussions referring to Tetrarrin and polynomial Tetrarrin sequences, thus enhancing an extension of these Perrin numbers.

Thus, it should be noted that the primordial, known as the Perrin sequence, is a third-order, recurrent numeric sequence given by the recurrence: $R_n = R_{n-2} + R_{n-3}, R_0 = 3, R_1 = 0, R_2 = 2, n \geq 3$ [**?**]. These numbers have a close relationship with the Padovan sequence $\{P_n\}$, with recurrence: $P_n = P_{n-2} + P_{n-3}$, differing in their initial values, which are given by: $P_0 = P_1 = P_2 = 1$ [**?**].

The Tridovan sequence ($\{T_n\}$), defined by [**?**], is a fourth order sequence, derived from the Padovan sequence with recurrence $T_n = T_{n-2} + T_{n-3} + T_{n-4}$ and initial values given by $T_0 = 1, T_1 = 0, T_2 = T_3 = 1$. Therefore, in this work an extension of the Perrin numbers is carried out, naming the Tetrarrin sequence, Te_n (fourth order), which will be addressed in the next section.

That said, we highlight the combinatorial study, with the definition of board given by [**?**], which depicts that a board is formed by squares called houses, cells or positions. These positions are enumerated and these enumerations describe the position. A given board with n squares will just be called n-board.

From this, it is important to present the work of [**?**], in which the combinatorial model of Tridovan is defined, based on a construction rule with the pieces: black square of size 1×1, blue domino of size 1×2, gray triminoes of size 1×3 and green tetraminoes of size 1×4, all with weight 1. It is also configured that the black square is

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intended to complement the empty tiles, subject to the rule of being inserted only at the beginning and, only once on each tile. The particular rules mentioned are defined for the theorem concerning Tridovan tiling [**?**].

In Figure **??**, on the left side, some examples are provided in order to fill in the n-board corresponding to the Tridovan sequence. On the right side are the terms corresponding to the Tridovan numbers. With this, it is possible to perceive the term t_n as being the amount of tile shapes on the n-board, following the aforementioned rules, determines the relationship: $t_n = T_n + T_{n-1}, n \geqslant 0$.

Figure 1: Tridovan tiling. Source: [**?**].

figuras/ladrilhotridovan.pdf

In view of this, Tetrarrin bracelets and their polynomial form will be defined in a primordial way in this research, introducing the Tetrarrin combinatorial and Tetrarrin polynomial model, before a combinatorial interpretation for the recurrent numerical sequence derived from the Perrin sequence.

2. The Tetrarrin sequence and its polynomial form

Based on the study by [**?**], who carried out an extension of the Padovan sequence, expanding the order of this sequence and defining new sequences arising from the Padovan numbers, we have the study for the Perrin numbers. With this, an extension of the Perrin sequence is performed, defining the Tetrarrin sequence.

The Tetrarrin sequence is a linear and recurrent sequence of the fourth order, studied primarily in this research.

Definition 2.1. *The Tetrarrin sequence, represented by $Te_{(n)}$ with $n \geqslant 0$ and $n \in \mathbb{N}$, has the following recurrence formula:*

$$Te_{(n)} = Te_{(n-2)} + Te_{(n-3)} + Te_{(n-4)},$$

with the following initial values: $Te_{(0)} = 3$, $Te_{(1)} = 0$, $Te_{(2)} = 2$ and $Te_{(3)} = 3$.

Thus, the first terms of this sequence are: $3, 0, 2, 3, 5, 5, 10, 13, \ldots$

Based on [**?**], in which they presented a relationship between the Padovan sequence and Perrin, we then sought to obtain a linear combination of the terms of the Tetrarrin sequence ($Te_{(n)}$) and Tridovan ($T_{(n)}$). Taking as a premise that this linear combination is possible, the following system of equations was modeled: $Ax = y$, presenting the following definitions:

$$A = \begin{bmatrix} T_{(0)} & T_{(1)} & T_{(2)} & T_{(3)} \\ T_{(1)} & T_{(2)} & T_{(3)} & T_{(4)} \\ T_{(2)} & T_{(3)} & T_{(4)} & T_{(5)} \end{bmatrix}, y = \begin{bmatrix} Te_{(4)} \\ Te_{(5)} \\ Te_{(6)} \\ Te_{(7)} \end{bmatrix}$$

and x is a vector of coefficients satisfying the system. Thus, it was possible to obtain the relation:

$$Te_{(n)} = 3T_{(n-4)} + 3T_{(n-3)} + 4T_{(n-2)}. \tag{2.1}$$

Since other identities can be obtained, arising from arithmetic operations on the mathematical relation presented in Equation **??**, we have: $Te_{(n)} = 2T_{(n-2)} + 3T_{(n-3)} + 3T_{(n-4)}$. From there, Tetrarrin's bracelet, $te_{(n)}$, will be defined in the next section.

So, based on the extension of the polynomial Padovan sequence, which is called the polynomial Tridovan, based on the definitions established by [**? ? ?**], thus defining the polynomial sequence of Tetrarrin.

Definition 2.2. *The Tetrarrin polynomial sequence, $Te_{(n)}(x)$, satisfies the following recurrence formula, for $n \in \mathbb{N}$ and $n \geqslant 4$.*

$$Te_{(n)}(x) = x^2 Te_{(n-2)}(x) + x Te_{(n-3)}(x) + Te_{(n-4)}(x),$$

with the initial terms: $Te_{(0)}(x) = 3, Te_{(1)}(x) = 0, Te_{(2)}(x) = 2x^2, Te_{(3)}(x) = 3x$.

Thus, we have the Table **??** with the first terms of the Tetrarrin polynomial sequence.

Table 1: First ten polynomial terms of Tetrarrin. Source: Prepared by the authors.

n	$Te_{(n)}(x)$
0	3
1	0
2	$2x^2$
3	$3x$
4	$2x^4 + 3$
5	$5x^3$
6	$2x^5 + 8x^2$
7	$7x^5 + 6x$
8	$2x^7 + 15x^4 + 3$
9	$7x^7 + 2x^6 + 13x^3 + 6x^2$

With this, the relationship between the polynomial sequences of Tridovan and Tetrarrin is investigated, through the resolution of linear systems, obtaining:

$$Te_{(n+2)}(x) = 2T_{(n-2)}(x) + 3T_{(n-3)}(x) + 3T_{(n-4)}(x) \tag{2.2}$$

In view of this, the study of the polynomial combinatorial model of Tetrarrin can be established, based on the Theorem referring to the polynomial combinatorial model of Tridovan studied by [**?**].

3. Tetrarrin combinatorial and Tetrarrin polynomial model

Based on the discussions carried out in the introduction to this research and in its highlighted sources, we will approach the Tetrarrin combinatorial model theorem and its polynomial form.

Initially, we can rescue the sequence of Fibonacci and Lucas, where they present the same recurrence relation and different initial values [**?**]. In this way, [**?**] study the Fibonacci combinatorial model and, in a complementary way, the Lucas combinatorial model. Lucas's combinatorial interpretation takes place in the form of circles, called bracelets, where l_n is the number of ways to tile a circular board composed of n cells marked with 1×1 squares and 1×2 dominoes. The Figure **??** visually portrays the amount of tiles on Lucas' bracelet of size 4, that is, l_4, obtaining a total of 7 ways to tile the bracelet.

For $n \geqslant 0$, we have l_n the number of ways to tile a circular board of size n, with squares and dominoes. Then l_n, being the nth Lucas number, we have [**?**]:

$$l_n = L_n.$$

Figure 2: Lucas bracelets size 4. Source: [**?**].

figuras/lucas.png

From the studies carried out on the combinatorial model of Tridovan [**?**], bracelet of the sequence of Lucas [**?**], sequence of Tridovan [**?**], sequence of Tetrarrin and of the algebraic relations between the sequences of Tridovan and Tetrarrin, there is the investigation of the combinatorial model of Tetrarrin.

Set te_n the amount of coverage of a circular board with n positions labeled clockwise, using the pieces: black curved squares, blue curved dominoes, gray curved triminoes and green curved tetraminos. It is also defined that the black curved square, if it appears, covers only one position among the first positions $1, 2, 3$.

It is called a n-bracelet, a covering of a circular n-board. Note that a bracelet is said to be *out of phase* if there is a domino in position $(n, 1)$. Otherwise it is said to be *in phase*. The present definition is also valid for the cases of gray curved tetraminoes in positions $(n-1, n, 1)$ or $(n, 1, 2)$ and green curved tetraminoes in positions $(n-2, n-1, n, 1)$, $(n-1, n, 1, 2)$ or $(n, 1, 2, 3)$.

Theorem 3.1. *For $n \geq 2$, let te_n the number of orientated bracelets of a $1n$ board with curved black square, curved blue domino, curved gray triminoes and curved green tetraminos, all weighing 1 and such that the curved black square appears only once and in the three first positions depending of the position of last tile. Then $te_n = Te_n + Te_{n-1}$, where Te_n is the $n-$th term of the Tetrarrin sequence.*

Proof. For $n = 3$ we have $te_3 = 5$ that counted the $3-$bracelets, 2 in-phase and 3 out-of-phase. Similarly, we obtain $te_4 = 8, te_5 = 10$.

Consider the last tile in a n bracelet counted by te_n. Observe that this last tile is not a curved black square, since $n \geq 2$. Then, the $n-$bracelet ends with one curved blue dominoes, with one curved gray triminoes or one curved green tetraminos.

In the case of last tile is a curved blue dominoes, we have the bracelet in-phase with the dominoes in position $(n-3, n-2)$, or the bracelet out-of-phase, with the dominoes in position $(n-2, 1)$. Thus, there are $n-2$ positions left that must be covered in $te_(n-2)$ ways.

When the $n-$bracelet ends with one curved gray triminoes, we have the bracelets in-phase with this tile in position $(n-4, n-3, n-2)$, (in-phase), or out-of-phase, with this tile in one of these positions $(n-2, 1, 2)$, $(n-3, n-2, 1)$. This implies that there are $n-3$ positions left that must be covered in $te_(n-3)$ ways.

When the $n-$bracelet ends with one curved green tetraminos, we have the bracelets in-phase with this tile in position $(n-5, n-4, n-3, n-2)$, (in-phase), or out-of-phase, with this tile in one of these positions

$(n-2,1,2,3)$, $(n-4,n-3,n-2,1)$. This implies that there are $n-4$ positions left that must be covered in $te_(n-4)$ ways.

Thus, $te_n = te_{n-2} + te_{n-3} + te_{n-4}$, for $n \geq 2$, and $te_3 = 5, te_4 = 8$ and $te_5 = 10$. Therefore, $te_n = Te_n$. ■

In order to exemplify the model, we have Figure **??** for the case of $n = 2$. Where you have 2 cases for a 2-tetrarrin bracelet with rotating blue curved domino.

Figure 3: Tetrarrin bracelets for case $n = 2$. Source: Prepared by the authors.

figuras/te2elen.pdf

It is important to note that the bracelet finished with a curved blue domino, is rotated only once, generating two bracelets. In case the bracelet ends with a gray curved trimino, it is rotated twice, thus generating three bracelets. For bracelets ending with a curved green tetramino, the piece is rotated twice, generating three bracelets.

Figure **??** presents the cases for values of $n = 6, 7, 8$.

Figure 4: Tetrarrin bracelets for cases $n = 3, 4, 5$. Source: Prepared by the authors.

Henceforth, we have the definition of the Tetrarrin polynomial combinatorial model, carried out based on the study previously discussed in this article.

To do this, define $te_n(x)$ the amount of coverage of a circular board with $(n-2)$ positions labeled clockwise, using black curved squares of weight 1, blue curved dominoes of weight x^2 , gray curved triminoes of weight x

figuras/tetrarrinvarios.pdf

and green curved tetraminoes of weight 1, so that the black curved square, if it appears, covers only one position among the first positions $1, 2, 3$.

In this way, the previously mentioned denomination referring to the n-bracelet follows, bearing in mind that it has a covering of a circular n-tray. Similarly, one can say that a bracelet is said to be *out of phase* if there is a domino in position $(n, 1)$. Otherwise it is said to be *in phase*. It is noteworthy that this definition can be extended to the cases of gray curved tetraminos in positions $(n-1, n, 1)$ or $(n, 1, 2)$ and green curved tetraminos in positions $(n-2, n-1, n, 1)$, $(n-1, n, 1, 2)$ or $(n, 1, 2, 3)$. The black curved squares, when they appear, must be rotated, occupying the first three positions. The other pieces, blue curved domino, gray curved trimino and green curved tetramino, rotate only twice, when arranged without the presence of the black curved square piece.

Theorem 3.2. *For $n \geq 2$, let $te_n(x)$ the number of orientated bracelets of a $1n$ board with curved black square weighing 1, curved blue domino weighing x^2, curved gray triminoes weighing x and curved green tetraminos weighing 1, such that the curved black square appears only once and in the three first positions depending of the position of last tile. Then $te_n(x) = Te_n(x) + Te_{n-1}(x)$, where $Te_n(x)$ is the $n-$th term of the Tetrarrin polynomial sequence.*

Proof. The proof follows analogous to the validation of the Theorem **??** ■

An example of the model is Figure **??** for the case of $n = 5$. Where we have 5 cases for a 3-tetrarrin polynomial bracelet.

Figure 5: Braceletes polinomiais de Tetrarrin para o caso $n = 5$. Source: Prepared by the authors.

figuras/tetra2polielen.pdf

Figure **??** presents the cases for values of $n = 6, 7, 8$.

Figure 6: Tetrarrin polynomial bracelets for cases $n = 6, 7, 8$. Source: Prepared by the authors.

figuras/tetrarrinvariospolielen.pdf

4. Conclusions

The present research allowed a study of the combinatorial interpretation of Tetrarrin, introducing the combinatorial model of Tetrarrin numbers and its polynomial form in an unprecedented way. In this way, it is possible to explore the combinatorial approach of the recurrent numerical sequence, contributing to the evolution of Perrin numbers and their visualization.

From the Perrin sequence, it was possible to insert elements in the main recurrence, defining the Tetrarrin sequence and its polynomial form. After that, the combinatorial models of Tetrarrin and polynomial sequences were introduced.

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