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Double domination number of the shadow (2,3)-distance graphs

AYSUN AYTAÇ*[1] AND AYŞEN MUTLU[1]

[1] *Faculty of Science, Department of Mathematics, Ege University, İzmir, 35100, Turkey.*

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Abstract. Let $G = (V, E)$ be a graph with the vertex set $V(G)$ and S be a subset of $V(G)$. If every vertex of V is dominated by S at least twice, then the set S is called a double domination set of the graph. The number of elements of the double domination set with the smallest cardinality is called double domination number and denoted by $\gamma_{\times 2}(G)$ notation. In this paper, we discussed the double domination parameter on some types of shadow distance graphs such as cycle, path, star, complete bipartite and wheel graphs.

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1. Introduction and Background

Many real-life problems can be modeled mathematically by using differential equations, integral equations, algebraic relations, etc. However, the graphical representation of such problems, showing how the various components are related, appeals to anyone working on it. Although the beginning of these graphic representations dates back many years, its emergence as a concrete mathematical structure was shaped by the finding of a new branch of mathematics, graph theory. As one of the most important characterizations, graph domination, has been associated with various application areas such as analyzing chemical structures, electrical and communication networks, and database management. Thus, graph domination has attracted interest from many mathematicians due to its application potential to apply many problems such as design and analysis of communication networks as well as defense supervision [4, 14, 19].

Now, we provide some basic information and definitions that will form the basis of this study. In general, we follow [8, 15]. Let $G = (V(G), E(G))$ be a graph. The open neighborhood of a vertex $v \in V(G)$ is $N(v) = N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$, and its closed neighborhood $N[v] = N(v) \cup \{v\}$. The degree of v, denoted by $\deg(v)$, is the size of its open neighborhood. One degree vertex is called as a pendant vertex or a leaf, and its neighbor is called a support vertex. An edge incident to a leaf (or a pendant vertex) is called a pendant edge.

Let D be a subgraph of the vertex set of a graph G. If D is a dominating set in a graph G then every vertex in $V(G)\backslash D$ is adjacent to at least one vertex in D, and the number of elements of the minimum cardinality

*Corresponding author. Email address: **aysun.aytac@ege.edu.tr** (Aysun AYTAÇ), **ayshhen.mutlu@gmail.com** (Ayşen MUTLU)

domination set is called the domination number of G, denoted by $\gamma(G)$ [15]. Many variants of domination parameter are studied in the literature [1–3, 6, 7, 11, 13, 16, 17].

In this paper, we focused on the double domination parameter. Double dominating set (abbreviated DDS) is introduced in [12]. A set $S \subseteq V$ is a double dominating set for G if each vertex in V is dominated by at least two vertices in S. The smallest cardinality of a double dominating set is called the double domination number $\gamma_{\times 2}(G)$. If S is a DDS of G of size as double domination number, then it is called as $\gamma_{\times 2}(G)$-set [12, 13]. Frankly, double domination is defined only for graphs without isolated vertices.

Let D be the set of all distances between distinct pairs of vertices in G and let $D_s \subseteq D$ is called the distance set. The distance graph of G denoted by $D\left(G, D_s\right)$ is the graph having the same vertex set with G and if $d(u, v) \in D_s$ then two vertices u and v are adjacent in $D\left(G, D_s\right)$. The shadow distance graph of G, denoted by $D_{\text{sd}}\ \left(G, D_s\right)$ is formed from G to satisfy the following properties [12, 18, 20] :

P1 : G has two copies say G itself and G'

P2 : if $u \in V(G)$ is first copy then the corresponding vertex as $u' \in V\left(G'\right)$ is second copy

P3 : the vertex set of shadow distance graph, $D_{sd}\left(G, D_s\right)$, is $V(G) \cup V\left(G'\right)$

P4 : the edge set of shadow distance graph , $D_{sd}\left(G, D_s\right)$, is $E(G) \cup E\left(G'\right) \cup E_{ds}$ where E_{ds} is the set of all edges between two distinct vertices $u \in V(G)$ and $v' \in V\left(G'\right)$ that satisfy the condition $d(u, v) \in D_s$ in G.

2. Main Results

We recall the following results related to the double domination number of a graph.

Theorem 2.1. *[10] Let G be a graph with no isolated vertices. Then* $2 \leq \gamma_{\times 2}(G) \leq n$.

Theorem 2.2. *[10] If G is any graph without isolated vertices, then* $\gamma(G) \leq \gamma_{\times 2}(G)-1$.

Theorem 2.3. *[5, 10, 12]*

a) If $G \cong P_n$ *is a path graph for* $n \geq 2$*, then* $\gamma_{\times 2}\left(P_n\right)=\left\lceil\frac{2n+2}{3}\right\rceil$

b) If $G \cong C_n$ *is a cycle graph for* $n \geq 3$*, then* $\gamma_{\times 2}\left(C_n\right)=\left\lceil\frac{2n}{3}\right\rceil$

c) If $G \cong K_{1,m}$ *is a star graph for* $m>1$*,* $\gamma_{\times 2}\left(K_{1,m}\right)=m+1$.

Observation 2.4. *[9] Each* $DD-set$ *generated for any graph must contain all leaves and support vertices of the graph.*

We begin our results with the some distance shadow graphs.

Theorem 2.5. *If* $G \cong P_n$ *for* $n \geq 8$*, then*

$$\gamma_{\times 2}\left(D_{sd}(G,\{2\})\right)=\begin{cases}\left\lceil\dfrac{4(n+1)}{5}\right\rceil & , n \equiv 3,\ 4\ \textit{(mod 5)}\\ \left\lceil\dfrac{4(n+1)}{5}\right\rceil+1 & , n \equiv 0\ , 2\ \textit{(mod 5)}\\ \left\lceil\dfrac{4(n+1)}{5}\right\rceil+2 & , n \equiv 1\ \textit{(mod 5)}\end{cases}$$

Proof. Consider two copies of G, one G itself and the other denoted by G'. Let $V_1 = \{1, 2, \ldots, n\}$ be the vertices of G and let $V_2 = \{n+1, n+2, \ldots, 2n\}$ be the vertices of G'. We first establish upper bounds on $\gamma_{\times 2}(D_{sd}(G, \{2\}))$. Let

$D_1 = \bigcup_{i=0}^{\lfloor \frac{n}{5} \rfloor - 1} \{(5i+2), (5i+3)\}$, $D_2 = \bigcup_{i=0}^{\lfloor \frac{n}{5} \rfloor - 1} \{(n+5i+2), (n+5i+3)\}$ and $D = D_1 \cup D_2$.

If $n \equiv 0 \pmod 5$, let $S = D \cup \{(n-1), (2n-1)\}$. If $n \equiv i \pmod 5$ where $i \in \{1, 2, 3, 4\}$, let $S = D \cup \{(n-1), (n-2), (2n-1), (2n-2)\}$. In all cases, the set S is a $DD-set$ of $D_{sd}(G, \{2\})$. Further if $n \equiv 0, 2 \pmod 5$, then $|S| = \left\lceil \frac{4(n+1)}{5} \right\rceil + 1$, while if $n \equiv 1 \pmod 5$, then $|S| = \left\lceil \frac{4(n+1)}{5} \right\rceil + 2$. Finally, if $n \equiv 3, 4 \pmod 5$, then $|S| = \left\lceil \frac{4(n+1)}{5} \right\rceil$. Hence, $\gamma_{\times 2}(D_{sd}(G, \{2\})) \leq \left\lceil \frac{4(n+1)}{5} \right\rceil$ if $n \equiv 3, 4 \pmod 5$, $\gamma_{\times 2}(D_{sd}(G, \{2\})) \leq \left\lceil \frac{4(n+1)}{5} \right\rceil + 1$ if $n \equiv 0, 2 \pmod 5$ and $\gamma_{\times 2}(D_{sd}(G, \{2\})) \leq \left\lceil \frac{4(n+1)}{5} \right\rceil + 2$ if $n \equiv 1 \pmod 5$.

Now let's prove the lower bounds on $\gamma_{\times 2}(D_{sd}(G, \{2\}))$. Let's assume that the set $X = \{u_1, u_2, \ldots, u_i, \ldots, u_m, u_{m+1}, \ldots, u_j, \ldots, u_x\}$ is a $\gamma_{\times 2} - set$. Here; u_i and u_j are any two positive integers such that $u_1 < u_2 < \cdots < u_i < \cdots < u_m < u_{m+1} < \cdots < u_j < \cdots < u_x$, where $1 \leq u_i \leq n$ $i \in \{1, 2, \ldots, m\}$ and $n+1 \leq u_j \leq 2n$ $j \in \{n+1, \ldots, x\}$. We have $f_t = u_{t+2} - u_t$ for $t \in \{1, 2, \ldots, x-2\}$ and $t \neq m-1$. To show the inverse of the inequality, we need to show that $f_t \leq 5$.

Suppose $f_t \geq 6$ for at least one value of x. Without loss of generality, assume that $f_t = 6$. In accordance with this claim; the following sets are obtained.

$$D_1' = \{2, 3, 8, 9\} \cup \left\{ \bigcup_{i=0}^{\lceil \frac{n-12}{5} \rceil - 1} \{(5i+13), (5i+14)\} \right\} \text{ and}$$

$$D_2' = \{(n+2), (n+3), (n+4), (n+8), (n+9), (n+10)\} \cup \left\{ \bigcup_{i=0}^{\lceil \frac{n-12}{5} \rceil - 1} \{(n+5i+13), (n+5i+14)\} \right\}$$

In this case, $X = D_1' \cup D_2'$ and $|X| = 10 + 4\left\lceil \frac{n-12}{5} \right\rceil$. If $n \equiv 3 \pmod 5$, then $|X| = 10 + 4\left(\frac{n-8}{5}\right) = \frac{4n+18}{5}$. However, this value contradicts the upper value we found earlier as $|S| = \frac{4n+8}{5}$ for $n \equiv 3 \pmod 5$. A similar situation can easily be seen that the values obtained for $n \equiv 0, 1, 2, 4 \pmod 5$ according to the X set contradict the upper limits we obtained earlier. For all values of n according to $mod\, 5$, it is easily seen that $u_1 + u_2 + u_{m+1} + u_{m+2} = 2n + 10$ since $u_1 = 2$, $u_2 = 3$, $u_{m+1} = n+2$ and $u_{m+2} = n+3$.

If $n \equiv 0 \pmod 5$, then $\sum_{t_1=1}^{m-3} f_{t_1} + \sum_{t_2=m+1}^{x-3} f_{t_2} + f_{m-2} + f_{x-2} \leq 5(x-6) + 4$. Thus, we get $\sum_{t_1=1}^{m-3} f_{t_1} + \sum_{t_2=m+1}^{x-3} f_{t_2} = (u_{m-1} + u_{m-2} + u_{x-1} + u_{x-2}) - (2n+10) + f_{m-2} + f_{x-2}$. For $n \equiv 0 \pmod 5$, $u_{m-1} = n-2$, $u_{m-2} = n-3$, $u_{x-1} = 2n-2$ and $u_{x-2} = 2n-3$, $f_{m-2} = f_{x-2} = 2$. So, we have $6n - 10 - 2n - 10 + 4 \leq 5x - 30 + 4$ and $x \geq \left\lceil \frac{4n+10}{5} \right\rceil$. In this case, $|X| = x \geq \left\lceil \frac{4n+10}{5} \right\rceil = \left\lceil \frac{4(n+1)}{5} \right\rceil + 1$. This implies that $\gamma_{\times 2}(D_{sd}(G, \{2\})) \geq \left\lceil \frac{4(n+1)}{5} \right\rceil + 1$.

If $n \equiv 1 \pmod 5$, then $\sum_{t_1=1}^{m-4} f_{t_1} + \sum_{t_2=m+1}^{x-4} f_{t_2} + f_{m-3} + f_{m-2} + f_{x-3} + f_{x-2} \leq 5(x-8) + 8$. Thus, we get $\sum_{t_1=1}^{m-4} f_{t_1} + \sum_{t_2=m+1}^{x-4} f_{x_2} = (u_{m-2} + u_{m-3} + u_{x-2} + u_{x-3}) - (2n+10) + f_{m-3} + f_{m-2} + f_{x-3} + f_{x-2}$. For $n \equiv 1 \pmod 5$, $u_{m-2} = n-3$, $u_{m-3} = n-4$, $u_{x-2} = 2n-3$, $u_{x-3} = 2n-4$ and $f_{m-3} = f_{m-2} = f_{x-3} = f_{x-2} = 2$. So, we have $6n - 14 - 2n - 10 \leq 5x - 40$ and $x \geq \left\lceil \frac{4n+16}{5} \right\rceil$. In this case, $|X| = x \geq \left\lceil \frac{4n+16}{5} \right\rceil = \left\lceil \frac{4(n+1)}{5} \right\rceil + 2$. This implies that $\gamma_{\times 2}(D_{sd}(G, \{2\})) \geq \left\lceil \frac{4(n+1)}{5} \right\rceil + 2$.

If $n \equiv 2 \pmod 5$, then $\sum_{t_1=1}^{m-4} f_{t_1} + \sum_{t_2=m+1}^{x-4} f_{t_2} + f_{m-3} + f_{m-2} + f_{x-3} + f_{x-2} \leq 5(x-8) + 16$. Thus, we get $\sum_{t_1=1}^{m-4} f_{t_1} + \sum_{t_2=m+1}^{x-4} f_{t_2} = (u_{m-2} + u_{m-3} + u_{x-2} + u_{x-3}) - (2n+10) + f_{m-3} + f_{m-2} + f_{x-3} + f_{x-2}$. For $n \equiv 2 \pmod 5$, $u_{m-2} = n-4$, $u_{m-3} = n-5$, $u_{x-2} = 2n-4$, $u_{x-3} = 2n-5$ and $f_{m-3} = f_{m-2} = f_{x-3} = f_{x-2} = 4$. So, we have $6n - 18 - 2n - 10 \leq 5x - 40$ and $x \geq \left\lceil \frac{4n+12}{5} \right\rceil = \left\lceil \frac{4(n+1)}{5} \right\rceil + 1$. In this case, $|X| = x \geq \left\lceil \frac{4n+12}{5} \right\rceil = \left\lceil \frac{4(n+1)}{5} \right\rceil + 1$. This implies that $\gamma_{\times 2}(D_{sd}(G,\{2\})) \geq \left\lceil \frac{4(n+1)}{5} \right\rceil + 1$.

If $n \equiv 3 \pmod 5$, then $\sum_{t_1=1}^{m-2} f_{t_1} + \sum_{t_2=m+1}^{x-2} f_{t_2} \leq 5(x-4)$. Thus, we get $\sum_{t_1=1}^{m-2} f_{t_1} + \sum_{t_2=m+1}^{x-2} f_{t_2} = (u_m + u_{m-1} + u_x + u_{x-1}) - (2n+10)$. For $n \equiv 3 \pmod 5$, $u_m = n$, $u_{m-1} = n-1$, and $u_x = 2n$, $u_{x-1} = 2n-1$. So, we have $6n - 2 - 2n - 10 \leq 5x - 20$ and $x \geq \left\lceil \frac{4n+8}{5} \right\rceil$. In this case, $|X| = x \geq \left\lceil \frac{4n+8}{5} \right\rceil = \left\lceil \frac{4(n+1)}{5} \right\rceil$. This implies that $\gamma_{\times 2}(D_{sd}(G,\{2\})) \geq \left\lceil \frac{4(n+1)}{5} \right\rceil$.

If $n \equiv 4 \pmod 5$, then $\sum_{t_1=1}^{m-2} f_{x_1} + \sum_{t_2=m+1}^{x-2} f_{t_2} \leq 5(x-4)$. Thus, we get $\sum_{t_1=1}^{m-2} f_{t_1} + \sum_{t_2=m+1}^{x-2} f_{t_2} = (u_m + u_{m-1} + u_x + u_{x-1}) - (2n+10)$. For $n \equiv 4 \pmod 5$, $u_m = n-1$, $u_{m-1} = n-2$, $u_x = 2n-1$ and $u_{x-1} = 2n-2$ So, we have $6n - 6 - 2n - 10 \leq 5x - 20$ and $x \geq \left\lceil \frac{4n+4}{5} \right\rceil$. In this case, $|X| = x \geq \left\lceil \frac{4n+4}{5} \right\rceil = \left\lceil \frac{4(n+1)}{5} \right\rceil$. This implies that $\gamma_{\times 2}(D_{sd}(G,\{2\})) \geq \left\lceil \frac{4(n+1)}{5} \right\rceil$. Thus, the desired equality is obtained as a result of the lower and upper bounds on $\gamma_{\times 2}(D_{sd}(G,\{2\}))$.

This completes the proof.

■

Theorem 2.6. *If $G \cong C_n$ for $n \geq 11$, then*

$$\gamma_{\times 2}(D_{sd}(G,\{2\})) = \begin{cases} \left\lceil \dfrac{4n}{5} \right\rceil & , n \equiv 0,\ 4 \ (\textit{mod } 5) \\ \left\lceil \dfrac{4n}{5} \right\rceil + 1 & , n \equiv 1, 3 \ (\textit{mod } 5) \\ \left\lceil \dfrac{4n}{5} \right\rceil + 2 & , n \equiv 2 \ (\textit{mod } 5) \end{cases}$$

Proof. Let the vertices of the $D_{sd}(G,\{2\})$ graph be divided into two sets of $V(D_{sd}(G,\{2\})) = V_1 \cup V_2$ where $V_1 = \{1, 2, \ldots, n\}$ and $V_2 = \{n+1, n+2, \ldots, 2n\}$.We first establish upper bounds on $\gamma_{\times 2}(D_{sd}(G,\{2\}))$. Let

$$D_1 = \{1, n\} \cup \left\{ \bigcup_{i=0}^{\left\lceil \frac{n-6}{5} \right\rceil - 1} \{(5i+5), (5i+6)\} \right\},$$

$$D_2 = \{(n+1), (2n)\} \cup \left\{ \bigcup_{i=0}^{\left\lceil \frac{n-6}{5} \right\rceil - 1} \{(n+5i+5), (n+5i+6)\} \right\} \text{ and } D = D_1 \cup D_2.$$

If $n \equiv 1 \pmod 5$, let $S = D \cup \{(n-1), (2n-1)\}$, in other cases $S = D$. In all cases, the set S is a $DD-set$ of $D_{sd}(G,\{2\})$. Further if $n \equiv 0, 4 \pmod 5$, then $|S| = \left\lceil \frac{4n}{5} \right\rceil$, while if $n \equiv 1, 3 \pmod 5$, then $|S| = \left\lceil \frac{4n}{5} \right\rceil + 1$.

Finally, if $n \equiv 2 \pmod 5$, then $|S| = \left\lceil \frac{4n}{5} \right\rceil + 2$. Hence, $\gamma_{\times 2}\left(D_{sd}\left(G, \{2\}\right)\right) \leq \left\lceil \frac{4n}{5} \right\rceil$ if $n \equiv 0, 4 \pmod 5$, $\gamma_{\times 2}\left(D_{sd}\left(G, \{2\}\right)\right) \leq \left\lceil \frac{4n}{5} \right\rceil + 1$ if $n \equiv 1, 3 \pmod 5$ and $\gamma_{\times 2}\left(D_{sd}\left(G, \{2\}\right)\right) \leq \left\lceil \frac{4n}{5} \right\rceil + 2$ if $n \equiv 2 \pmod 5$.
Now let's prove the lower bounds on $\gamma_{\times 2}\left(D_{sd}\left(G, \{2\}\right)\right)$. Let's assume that the set $X = \{u_1, u_2, \ldots, u_i, \ldots, u_m, u_{m+1}, \ldots, u_j, \ldots, u_x\}$ is a $\gamma_{\times 2} - set$. Here; u_i and u_j are any two positive integers such that $u_1 < u_2 < \cdots < u_i < \cdots < u_m < u_{m+1} < \cdots < u_j < \cdots < u_x$, where $1 \leq u_i \leq n$ $i \in \{1, 2, \ldots, m\}$ and $n + 1 \leq u_j \leq 2n$ $j \in \{n+1, \ldots, x\}$. We have $f_t = u_{t+2} - u_t$ for $t \in \{1, 2, \ldots, x-2\}$and $t \neq m-1$. To show the inverse of the inequality, we need to show that $f_t \leq 5$.
Suppose $f_t \geq 6$ for at least one value of t. Without loss of generality, assume that$f_t = 6$. In accordance with this claim; the following sets are obtained.

$$D_1{}' = \{1\ ,\ n\} \cup \left\{ \bigcup_{i=0}^{\left\lceil \frac{n-7}{5} \right\rceil - 1} \{(5i+6), (5i+7)\} \right\} \text{ and}$$

$$D_2{}' = \{(n+1), (n+2), (n+5), (2n)\} \cup \left\{ \bigcup_{i=0}^{\left\lceil \frac{n-7}{5} \right\rceil - 1} \{(n+5i+6), (n+5i+7)\} \right\}.$$

In this case, $X = D_1{}' \cup D_2{}'$ and $|X| = 6 + 4\left\lceil \frac{n-7}{5} \right\rceil$. If $n \equiv 0 \pmod 5$, then $|X| = 6 + 4\left(\frac{n-5}{5}\right) = \frac{4n+10}{5}$. However, this value contradicts the upper value we found earlier as $|S| = \frac{4n}{5}$ for $n \equiv 0 \pmod{5}$. A similar situation can easily be seen that the values obtained for $n \equiv i \pmod 5$, $i \in \{1, 2, 3, 4\}$ according to the X set contradict the upper limits we obtained earlier. This contradicts our claim. Thus, it must be$f_x \leq 5$. In this case, we have $\sum\limits_{t_1=1}^{m-2} f_{t_1} + \sum\limits_{t_2=m+1}^{x-2} f_{t_2} \leq 5(x-4)$. Furthermore, for all values of n according to $mod\ 5$, it is easily seen that $u_1 + u_2 + u_{m+1} + u_{m+2} = 2n + 13$ since $u_1 = 1$, $u_2 = 6$, $u_{m+1} = n + 1$ and $u_{m+2} = n + 5$.

If $n \equiv 0 \pmod 5$, then $\sum\limits_{t_1=1}^{m-3} f_{t_1} + \sum\limits_{t_2=m+1}^{x-3} f_{t_2} + f_{m-2} + f_{x-2} \leq 5(x-6) + 8$. Thus, we get $\sum\limits_{t_1=1}^{m-3} f_{t_1} + \sum\limits_{t_2=m+1}^{x-3} f_{t_2} + f_{m-2} + f_{x-2} = (u_{m-1} + u_{m-2} + u_{x-1} + u_{x-2}) - (2n+13) + f_{m-2} + f_{x-2}$. For $n \equiv 0 \pmod 5$, $u_{m-1} = n-4$, $u_{m-2} = n-5$, $u_{x-1} = 2n-4$, and $u_{x-2} = 2n-5$, $f_{m-2} = f_{x-2} = 4$. So, we have $6n - 18 - 2n - 13 \leq 5(x-6)$ and $x \geq \left\lceil \frac{4n-1}{5} \right\rceil$. In this case, $|X| = x \geq \left\lceil \frac{4n-1}{5} \right\rceil = \left\lceil \frac{4n}{5} \right\rceil$. This implies that $\gamma_{\times 2}\left(D_{sd}\left(G, \{2\}\right)\right) \geq \left\lceil \frac{4n}{5} \right\rceil$.

If $n \equiv 1 \pmod 5$, then $\sum\limits_{t_1=1}^{m-2} f_{t_1} + \sum\limits_{t_2=m+1}^{x-2} f_{t_2} \leq 5(x-4)$. Thus, we get $\sum\limits_{t_1=1}^{m-2} f_{t_1} + \sum\limits_{t_2=m+1}^{x-2} f_{t_2} = (u_m + u_{m-1} + u_x + u_{x-1}) - (2n+13)$. For $n \equiv 1 \pmod 5$, $u_m = n$, $u_{m-1} = n-1$, $u_x = 2n$ and $u_{x-1} = 2n-1$. So, we have $6n - 2 - 2n - 13 \leq 5x - 20$ and $x \geq \left\lceil \frac{4n+5}{5} \right\rceil = \left\lceil \frac{4n}{5} \right\rceil + 1$. In this case, $|X| = x \geq \left\lceil \frac{4n}{5} \right\rceil + 1$. This implies that $\gamma_{\times 2}\left(D_{sd}\left(G, \{2\}\right)\right) \geq \left\lceil \frac{4n}{5} \right\rceil + 1$.

If $n \equiv 2 \pmod 5$, then $\sum\limits_{t_1=1}^{m-3} f_{t_1} + \sum\limits_{t_2=m+1}^{x-3} f_{t_2} + f_{m-2} + f_{x-2} \leq 5(x-6) + 4$. Thus, we get $\sum\limits_{t_1=1}^{m-3} f_{t_1} + \sum\limits_{t_2=m+1}^{x-3} f_{t_2} + f_{m-2} + f_{x-2} = (u_{m-1} + u_{m-2} + u_{x-1} + u_{x-2}) + f_{m-2} + f_{x-2} - (2n+13)$. For $n \equiv 2 \pmod 5$, $u_{m-1} = n-1$, $u_{m-2} = n-2$, $u_{x-1} = 2n-1$, $u_{x-2} = 2n-2$. So, we have $6n - 6 - 2n - 13 \leq 5(x-6)$ and $x \geq \left\lceil \frac{4n+12}{5} \right\rceil = \left\lceil \frac{4n}{5} \right\rceil + 2$. In this case, $|X| = x \geq \left\lceil \frac{4n}{5} \right\rceil + 2$. This implies that $\gamma_{\times 2}\left(D_{sd}\left(G, \{2\}\right)\right) \geq \left\lceil \frac{4n}{5} \right\rceil + 2$.

If $n \equiv 3 \pmod 5$, then $\sum\limits_{t_1=1}^{m-3} f_{t_1} + \sum\limits_{t_2=m+1}^{x-3} f_{t_2} + f_{m-2} + f_{x-2} \leq 5(x-6) + 6$. Thus, we get $\sum\limits_{t_1=1}^{m-3} f_{t_1} + \sum\limits_{t_2=m+1}^{x-3} f_{t_2} + f_{m-2} + f_{x-2} = (u_{m-1} + u_{m-2} + u_{x-1} + u_{x-2}) - (2n+13) + f_{m-2} + f_{x-2}$. For $n \equiv 3 \pmod 5$, $u_{m-1} = n-2$, $u_{m-2} = n-3$, $u_{x-1} = 2n-2$, $u_{x-2} = 2n-3$ and $f_{m-2} = f_{x-2} = 3$. So, we have $6n - 10 - 2n - 13 \leq 5(x-6)$ and $x \geq \left\lceil \frac{4n+7}{5} \right\rceil = \left\lceil \frac{4n}{5} \right\rceil + 1$. In this case, $|X| = x \geq \left\lceil \frac{4n}{5} \right\rceil + 1$. This implies that $\gamma_{\times 2}\left(D_{sd}\left(G, \{2\}\right)\right) \geq \left\lceil \frac{4n}{5} \right\rceil + 1$.

If $n \equiv 4 \pmod 5$, then $\sum_{t_1=1}^{m-3} f_{t_1} + \sum_{t_2=m+1}^{x-3} f_{t_2} + f_{m-2} + f_{x-2} \leq 5(x-6) + 8$. Thus, we get $\sum_{t_1=1}^{m-3} f_{t_1} + \sum_{t_2=m+1}^{x-3} f_{t_2} + f_{m-2} + f_{x-2} = (u_{m-1} + u_{m-2} + u_{x-1} + u_{x-2}) - (2n+13) + f_{m-2} + f_{x-2}$. For $n \equiv 4 \pmod 5$, $u_{m-1} = n-3$, $u_{m-2} = n-4$, $u_{x-1} = 2n-3$, $u_{x-2} = 2n-4$ and $f_{m-2} = f_{x-2} = 4$. So, we have $6n - 14 - 2n - 13 \leq 5(x-6)$ and $x \geq \left\lceil \frac{4n+3}{5} \right\rceil = \left\lceil \frac{4n}{5} \right\rceil$. In this case, $|X| = x \geq \left\lceil \frac{4n}{5} \right\rceil$. This implies that $\gamma_{\times 2}(D_{sd}(G,\{2\})) \geq \left\lceil \frac{4n}{5} \right\rceil$.
Thus, the desired equality is obtained as a result of the lower and upper bounds on $\gamma_{\times 2}(D_{sd}(G,\{2\}))$.

This completes the proof.

■

Theorem 2.7. *For $m \geq 1$ and $n \geq 2$, let $G \cong K_{m,n}$ be a bipartite complete graph with $(m+n)$-vertices. Then, the double dominance number of the graph $(D_{sd}(G,\{2\})$ is $\gamma_{\times 2}(D_{sd}(G,\{2\})) = 4$.*

Proof. Let the vertices of the $D_{sd}(G,\{2\})$ graph be divided into four sets of $V(D_{sd}(G,\{2\})) = V_1 \cup V_2 \cup V_1' \cup V_2'$, where $V_1 = \{v_1, v_2, \ldots, v_m\}$, $V_2 = \{v_1, v_2, \ldots, v_n\}$, $V_1' = \{v_1', v_2', \ldots, v_m'\}$ and $V_2' = \{v_1', v_2', \ldots, v_n'\}$. We first establish upper bounds on $\gamma_{\times 2}(D_{sd}(G,\{2\}))$. If $S = \{v_1, v_1, v_1', v_1'\}$, then the set S is the $DD-set$ of the graph $D_{sd}(G,\{2\})$. Thus, $\gamma_{\times 2}(D_{sd}(G,\{2\})) \leq 4$.
For the lower bound, let the set T be the $\gamma_{\times 2}(D_{sd}(G,\{2\})) - set$. Assume that $|T| = 3$. This requires that every vertex in T has at least one neighbor still in T. Taking into account that $V_1 \cong V_1'$and $V_2 \cong V_2'$, the following cases are obtained.

Case 1. Let $u_i \in V_1$, $v_j \in V_2$, $v_t' \in V_2'$. Assume that $T = \{u_i, v_j, v_t'\}$ $i \in \{1, \ldots, m\}$, $j, t \in \{1, \ldots, n\}$ and $j \neq t$. However, in this case, there will be vertices in the graph $D_{sd}(G,\{2\})$ that are not double dominated.

Case 2. Let $u_i, u_j \in V_1$, $u_t' \in V_1'$. Assume that $T = \{u_i, u_j, u_t'\}$ $i, j, t \in \{1, \ldots, m\}$ and $i \neq j \neq t$. However, in this case, there will be vertices in the graph $D_{sd}(G,\{2\})$ that are not double dominated.

Case 3. Let $v_i, v_j \in V_2$, $v_t' \in V_2'$. Assume that $T = \{v_i, v_j, v_t'\}$ $i, j, t \in \{1, \ldots, n\}$ and $i \neq j \neq t$. However, in this case, there will be vertices in the graph $D_{sd}(G,\{2\})$ that are not double dominated.

Case 4. Let $u_i \in V_1$, $\mathrm{u}_j' \in V_1'$, $v_t' \in V_2'$. Assume that $T = \left\{u_i, u_j', v_t'\right\}$ $i, j \in \{1, \ldots, m\}$, $t \in \{1, \ldots, n\}$ and $i \neq j$. However, in this case, there will be vertices in the graph $D_{sd}(G,\{2\})$ that are not double dominated.

Case 5. Let $v_j \in V_2$, $v_t' \in V_2'$, $\mathrm{u}_i \in V_1'$. Assume that $T = \{v_j, v_t', u_i\}$ $j, t \in \{1, \ldots, n\}$, $i \in \{1, \ldots, m\}$ and $j \neq t$. However, in this case, there will be vertices in the graph $D_{sd}(G,\{2\})$ that are not double dominated.

Case 6. Let $u_i \in V_1$, $v_j \in V_2$, $\mathrm{u}_t' \in V_1'$. Assume that $T = \{u_i, v_j, u_t'\}$ $i, t \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\}$ and $i \neq t$. However, in this case, there will be vertices in the graph $D_{sd}(G,\{2\})$ that are not double dominated.

In all cases, some vertices of the graph cannot be double dominated. Thus, we get $\gamma_{\times 2}(D_{sd}(G,\{2\})) = |T| \geq 4$. Thus, the desired equality is obtained as a result of the lower and upper bounds on $\gamma_{\times 2}(D_{sd}(G,\{2\}))$.

This completes the proof.

■

Corollary 2.8. *Let $G \cong S_{1,n}$ be a star graph with $(n+1)$-vertices. Then, the double dominance number of the graph $(D_{sd}(G,\{2\})$ is $\gamma_{\times 2}(D_{sd}(G,\{2\})) = 4$.*

Proof. If $m = 1$ and $n \geq 2$, then $K_{m,n} \cong K_{1,n}$. Thus, the proof of the result is easily seen from Theorem 2.7. ■

Theorem 2.9. *Let $G \cong W_{1,n}$ be a wheel graph with $(n+1)$-vertices. Then, the double dominance number of the graph $(D_{sd}(G,\{2\})$ is $\gamma_{\times 2}(D_{sd}(G,\{2\})) = 4$.*

Proof. Let the vertices of the $D_{sd}(G,\{2\})$ graph be divided into two sets of $V(D_{sd}(G,\{2\})) = V(G) \cup V(G')$, where $V(G) = \{c_1, u_1, \ldots, u_n\}$ and $V(G') = \{c_1', u_1', \ldots, u_n'\}$. Let c_1 be the central vertex of the graph G. We first establish upper bounds on $\gamma_{\times 2}(D_{sd}(G,\{2\}))$. If $S = \{c_1, u_1, c_1', u_1'\}$, then the set S is the $DD - set$ of the graph $D_{sd}(G,\{2\})$. Thus, $\gamma_{\times 2}(D_{sd}(G,\{2\})) \leq 4$.

To complete the proof, we need to prove the lower bound. Let the set T be the $\gamma_{\times 2}(D_{sd}(G,\{2\})) - set$. Assume that $|T| = 3$. For double dominating of vertices in T, at least one neighbor of each vertices must be in T. Thus, we have the following states.

Case 1. Let every vertex in T be at $V(G)$. Since $\deg(c_1) = n$, one of the vertices must be c_1 (or every vertex in T be at $V(G')$). However, in this case, there will be vertices in the graph $D_{sd}(G,\{2\})$ that are not double dominated.

Case 2. Let two vertices in T be at $V(G)$ and the other at $V(G')$. Since $\deg(c_1) = n$, one of the vertices must be c_1 (or two vertices in T be at $V(G')$and the other at $V(G)$). However, in this case, there will be vertices in the graph $D_{sd}(G,\{2\})$ that are not double dominated.

In all cases, some vertices of the graph cannot be double dominated. Thus, we get $\gamma_{\times 2}(D_{sd}(G,\{2\})) = |T| \geq 4$. The desired bounds are obtained as a result of the upper bounds on $\gamma_{\times 2}(D_{sd}(G,\{2\}))$ that were established earlier.

This completes the proof. ■

Theorem 2.10. *If $G \cong P_n$ for $n \geq 10$, then*

$$\gamma_{\times 2}(D_{sd}(G,\{3\})) = \begin{cases} \left\lceil \dfrac{4n+8}{5} \right\rceil + 1 & , n \equiv 1 \ (mod\ 5) \\ \left\lceil \dfrac{4n+8}{5} \right\rceil & , otherwise \end{cases}$$

Proof. We first establish upper bounds on $\gamma_{\times 2}(D_{sd}(G,\{3\}))$. We have $\deg(u_1) = \deg(u_n) = \deg(u_{n+1}) = \deg(u_{2n}) = 2$, $\deg(u_i) = 2$, $i \in \{2,\ 3,\ n-1,\ n-2,\ n+2,\ n+3,\ 2n-1,\ 2n-2\}$ and $\deg(u_j) = 4$, $j \in \{4\ ,\ \ldots\ ,\ n-3\ ,\ n+4\ ,\ 2n-3\}$. Let the set D be a $DD - set$ of the graph $D_{sd}(G,\{3\})$. Therefore, in order to double dominate the vertex u_1, it must have neighbors as well. Similarly, this is valid for the vertex u_{n+1}. So, $\{u_2, u_4, u_{n+2}, u_{n+4}\} \in D$. In order for the vertices in D to be double dominated, u_5 and its duplicate, u_{n+5}, must be added to S. In this case the vertices u_6 , u_7 and similarly the vertices u_{n+6} , u_{n+7} that are copies of these peaks are double dominated by the set D. For double dominating of the vertices u_6 and u_7, the vertices u_{n+9} , u_{n+10} are added to D since D is a $DD - set$. Add the vertices u_9 , u_{10} for u_{n+6} and u_{n+7}. Continuing in this way, upper limits on $\gamma_{\times 2}(D_{sd}(G,\{3\}))$ are obtained. Let

$$D = \bigcup_{i=0}^{\lfloor \frac{n}{5} \rfloor - 1} \{u_{5i+4}\ ,\ u_{5i+5}\ ,\ u_{n+5i+4}\ ,\ u_{n+5i+5}\} \cup \{u_2\ ,\ u_{n+2}\}.$$

If $n \equiv 0 \ (\text{mod}\ 5\ )$, let $S = D$. If $n \equiv 1,2,3 \ (\text{mod}\ 5)$, let $S = D \cup \{u_n, u_{2n}\}$. Otherwise, $n \equiv 4 \ (\text{mod}\ 5\ )$, $S = D \cup \{u_n, u_{n-1}, u_{2n}, u_{2n-1}\}$. In all cases, the set S is a $DD - set$ of $D_{sd}(G,\{3\})$. Further if $n \equiv 1 \ (\text{mod}\ 5\ )$, then $|S| = \left\lceil \frac{4n+8}{5} \right\rceil + 1$, while if $n \neq 1 \ (\text{mod}\ 5)$, then $|S| = \left\lceil \frac{4n+8}{5} \right\rceil$. Hence, $\gamma_{\times 2}(D_{sd}(G,\{3\})) \leq \left\lceil \frac{4n+8}{5} \right\rceil + 1$ if $n \equiv 1 \ (\text{mod}\ 5\ )$ and otherwise$\gamma_{\times 2}(D_{sd}(G,\{3\})) \leq \left\lceil \frac{4n+8}{5} \right\rceil + 1$.

Now let's prove the lower bounds on $\gamma_{\times 2}(D_{sd}(G,\{3\}))$. Let's assume that the set $X = \{u_1, u_2, \ldots, u_i, \ldots, u_m, u_{m+1}, \ldots, u_j, \ldots, u_x\}$ is a $\gamma_{\times 2} - set$. Here; u_i and u_j are any two positive integers such that $u_1 < u_2 < \ldots < u_i < \ldots < u_m < u_{m+1} < \ldots < u_j < \ldots < u_x$, where $1 \leq u_i \leq n$ $i \in \{1,2,\ldots,m\}$ and $n+1 \leq u_j \leq 2n$ $j \in \{n+1,\ldots,x\}$. We have $f_t = u_{t+2} - u_t$ for $t \in \{1,2,\ldots,x-2\}$ and $t \neq m-1,\ m$. To show the inverse of the inequality, we need to show that $f_t \leq 5$. Suppose $f_t \geq 6$ for at least one value of t. Without loss of generality, assume that $f_t = 6$. In accordance with this claim; the following sets are obtained.

$$D' = \{2,4,5,(n+2),(n+4),(n+6),(n+9)\} \cup \left\{ \bigcup_{i=0}^{\lceil \frac{n-9}{5} \rceil - 1} \{(5i+10),(5i+11),(n+5i+10),(n+5i+11)\} \right\}$$

In this case, $X = D'$ and $|X| = 8 + 4\left\lceil \frac{n-9}{5} \right\rceil$. If $n \equiv 0 \pmod 5$, then $|X| = 8 + 4\left(\frac{n-5}{5}\right) = \frac{4n+20}{5}$. However, this value contradicts the upper value we found earlier as $|S| = \frac{4n+10}{5}$ for $n \equiv 0 \pmod 5$. A similar situation can easily be seen that the values obtained for $n \equiv i \pmod 5$, $i \in \{1,2,3,4\}$ according to the X set contradict the upper limits we obtained earlier. This contradicts our claim. It must be $f_t \leq 5$. So, we have $\sum_{t_1=1}^{m-2} f_{t_1} + \sum_{t_2=m+1}^{x-2} f_{t_2} \leq 5(x-4)$. For all values of n according to $mod\ 5$, it is easily seen that $u_1 + u_2 + u_m + u_{m+1} = 2n + 12$ since $u_1 = 2$, $u_2 = 4$, $u_m = n+2$, $u_{m+1} = n+4$.
If $n \equiv 0 \pmod 5$, then $\sum_{t_1=1}^{m-2} f_{t_1} + \sum_{t_2=m+1}^{x-2} f_{t_2} = (u_m + u_{m-1} + u_x + u_{x-1}) - (2n+12)$. For $n \equiv 0 \pmod 5$, $u_m = n$, $u_{m-1} = n-1$, $u_x = 2n$ and $u_{x-1} = 2n-1$. So, we have $6n - 2 - 2n - 12 \leq 5(x-4)$ and $x \geq \left\lceil \frac{4n+6}{5} \right\rceil$. In this case, $|X| = x \geq \left\lceil \frac{4n+6}{5} \right\rceil = \left\lceil \frac{4n+8}{5} \right\rceil$. This implies that $\gamma_{\times 2}(D_{sd}(G,\{3\})) \geq \left\lceil \frac{4n+8}{5} \right\rceil$.
If $n \equiv 1,2,3,4 \pmod 5$, then $\sum_{t_1=1}^{m-3} f_{t_1} + \sum_{t_2=m+1}^{x-3} f_{t_2} + f_{m-2} + f_{x-2} \leq 5(x-6) + f_{m-2} + f_{x-2}$. Moreover, $\sum_{t_1=1}^{m-3} f_{t_1} + \sum_{t_2=m+1}^{x-3} f_{t_2} + f_{m-2} + f_{x-2} = (u_{m-2} + u_{m-1} + u_{x-2} + u_{x-1}) - (2n+12) + f_{m-2} + f_{x-2}$.
If $n \equiv 1 \pmod 5$, then we have $4n - 18 \leq 5(x-6)$ and $x \geq \left\lceil \frac{4n+12}{5} \right\rceil$since $u_{m-2} = n-2$, $u_{m-1} = n-1$, $u_{x-2} = 2n-2$, $u_{x-1} = 2n-1$ and $f_{m-2} = f_{x-2} = 2$. In this case, $|X| = x \geq \left\lceil \frac{4n+12}{5} \right\rceil = \left\lceil \frac{4n+8}{5} \right\rceil + 1$. This implies that $\gamma_{\times 2}(D_{sd}(G,\{3\})) \geq \left\lceil \frac{4n+8}{5} \right\rceil + 1$.
If $n \equiv 2 \pmod 5$, then we have $6n - 10 - 2n - 12 \leq 5(x-6)$ and $x \geq \left\lceil \frac{4n+8}{5} \right\rceil$since $u_{m-2} = n-3$, $u_{m-1} = n-2$, $u_{x-2} = 2n-3$, $u_{x-1} = 2n-2$ and $f_{m-2} = f_{x-2} = 3$. In this case, $|X| = x \geq \left\lceil \frac{4n+8}{5} \right\rceil$. This implies that $\gamma_{\times 2}(D_{sd}(G,\{3\})) \geq \left\lceil \frac{4n+8}{5} \right\rceil$.
If $n \equiv 3 \pmod 5$, then we have $6n - 14 - 2n - 12 \leq 5(x-6)$ and $x \geq \left\lceil \frac{4n+4}{5} \right\rceil$since $u_{m-2} = n-4$, $u_{m-1} = n-3$, $u_{x-2} = 2n-4$, $u_{x-1} = 2n-3$ and $f_{m-2} = f_{x-2} = 4$. In this case, $|X| = x \geq \left\lceil \frac{4n+5}{5} \right\rceil = \left\lceil \frac{4n+8}{5} \right\rceil$. This implies that $\gamma_{\times 2}(D_{sd}(G,\{3\})) \geq \left\lceil \frac{4n+8}{5} \right\rceil$.

If $n \equiv 4 \pmod 5$, then

$$\begin{aligned} &\sum_{t_1=1}^{m-4} f_{t_1} + \sum_{t_2=m+1}^{x-4} f_{t_2} + f_{m-3} + f_{m-2} + f_{x-3} + f_{x-2} \\ &\leq 5(x-8) + f_{m-3} + f_{m-2} + f_{x-3} + f_{x-2} \end{aligned}$$

Moreover, $\sum_{t_1=1}^{m-4} f_{t_1} + \sum_{t_2=m+1}^{x-4} f_{t_2} = (u_{m-2} + u_{m-3} + u_{x-2} + u_{x-3}) - (2n+12)$. For $n \equiv 4 \pmod 5$, we have $6n - 18 - 2n - 12 \leq 5x - 40$ and $x \geq \left\lceil \frac{4n+10}{5} \right\rceil$ since $u_{m-2} = n-4$, $u_{m-3} = n-5$, $u_{x-2} = 2n-4$,

$u_{x-3}=2n-5$ and $f_{m-3}=f_{m-2}=f_{x-3}=f_{x-2}=4$. In this case, $|X|=x\geq\left\lceil\frac{4n+10}{5}\right\rceil=\left\lceil\frac{4n+8}{5}\right\rceil$. This implies that $\gamma_{\times2}\left(D_{sd}\left(G,\{3\}\right)\right)\geq\left\lceil\frac{4n+8}{5}\right\rceil$.
Thus, the desired equality is obtained as a result of the lower and upper bounds on $\gamma_{\times2}\left(D_{sd}\left(G,\{3\}\right)\right)$.
This completes the proof. ∎

Theorem 2.11. *If $G\cong C_n$ for $n\geq 10$, then*

$$\gamma_{\times2}\left(D_{sd}\left(G,\{3\}\right)\right)=\begin{cases}\left\lceil\dfrac{4n+10}{5}\right\rceil-1 & ,\ n\equiv1\ \textit{(mod 5)}\\ \left\lceil\dfrac{4n+10}{5}\right\rceil+1 & ,\ n\equiv3\ \textit{(mod 5)}\\ \left\lceil\dfrac{4n+10}{5}\right\rceil & ,\ \textit{otherwise}\end{cases}$$

Proof. Let the vertices of the $D_{sd}\left(G,\{3\}\right)$ graph be divided into two sets of $V\left(D_{sd}\left(G,\{3\}\right)\right)=V_1\cup V_2$ where$V_1=\{1,2,\ldots,n\}$ and $V_2=\{n+1,n+2,\ldots,2n\}$. We first establish upper bounds on $\gamma_{\times2}\left(D_{sd}\left(G,\{3\}\right)\right)$. Let

$$D_1=\{n,(n-1),(n-2)\}\cup\left\{\bigcup_{i=0}^{\left\lceil\frac{n-7}{5}\right\rceil-1}\{(5i+4),(5i+5)\}\right\},$$

$$D_2=\{2n\ ,\ 2n-1\ ,\ 2n-2\}\cup\left\{\bigcup_{i=0}^{\left\lceil\frac{n-7}{5}\right\rceil-1}\{(n+5i+4),(n+5i+5)\}\right\}\text{ and }D=D_1\cup D_2.$$

If $n\equiv0,1,3,4\,(\text{mod }5)$, let $S=D$. If $n\equiv2\,(\text{mod }5)$, let $S=D\cup\{n-3\ ,\ 2n-3\}$. Otherwise, $n\equiv4\,(\text{mod }5)$, $S=D\cup\{u_n,u_{n-1},u_{2n},u_{2n-1}\}$. In all cases, the set S is a $DD-set$ of $D_{sd}\left(G,\{3\}\right)$. Further if $n\equiv1\,(\text{mod }5)$, then $|S|=\left\lceil\frac{4n+10}{5}\right\rceil-1$, while if $n\equiv3\,(\text{mod }5)$, then $|S|=\left\lceil\frac{4n+10}{5}\right\rceil+1$ and otherwise $|S|=\left\lceil\frac{4n+10}{5}\right\rceil$. Hence, $\gamma_{\times2}\left(D_{sd}\left(G,\{3\}\right)\right)\leq\left\lceil\frac{4n+10}{5}\right\rceil-1$ if $n\equiv1$ (mod 5), $\gamma_{\times2}\left(D_{sd}\left(G,\{3\}\right)\right)\leq\left\lceil\frac{4n+10}{5}\right\rceil+1$ if $n\equiv3$ (mod 5) and otherwise $\gamma_{\times2}\left(D_{sd}\left(G,\{3\}\right)\right)\leq\left\lceil\frac{4n+10}{5}\right\rceil$.
Now let's prove the lower bounds on $\gamma_{\times2}\left(D_{sd}\left(G,\{3\}\right)\right)$. Let's assume that the set $X=\{u_1,u_2,\ldots,u_i,\ldots,u_m,u_{m+1},\ldots,u_j,\ldots,u_x\}$ is a $\gamma_{\times2}-set$. Here; u_i and u_j are any two positive integers such that $u_1<u_2<\ldots<u_i<\ldots<u_m<u_{m+1}<\ldots<u_j<\ldots<u_x$, where $1\leq u_i\leq n$ $i\in\{1,2,\ldots,m\}$ and $n+1\leq u_j\leq 2n$ $j\in\{n+1,\ldots,x\}$. We have $f_t=u_{t+2}-u_t$ for $t\in\{1,2,\ldots,x-2\}$ and $t\neq m\ ,\ m-1\ ,\ m-2$. To show the inverse of the inequality, we need to show that $f_t\leq 5$.
Suppose $f_t\geq 6$ for at least one value of t. Without loss of generality, assume that$f_t=6$. In accordance with this claim; the following sets are obtained. Let

$$D'=\{n,\ (n-1)\,,\ (n-2)\,,\ 2n,\ 2n-1,\ 2n-2\}\cup\left\{\bigcup_{i=0}^{\left\lceil\frac{n-7}{6}\right\rceil-1}\{(6i+4)\,,(6i+5)\,,(n+6i+4)\,,(n+6i+5)\}\right\}$$

However, the vertices $(6i+6)$, $(n+6i+6)$ cannot be double dominated with this set. In this case, some vertices must be added to the set D'. This contradicts the upper bound we found earlier. Hence, it must be $f_t\leq 5$. So, we get

$$\sum_{t_1=1}^{m-5}f_{t_1}+\sum_{t_2=m+1}^{x-5}f_{t_2}+f_{m-4}+f_{m-3}+f_{m-2}+f_{x-4}+f_{x-3}+f_{x-2}$$
$$\leq 5\,(x-10)+f_{m-4}+f_{m-3}+f_{m-2}+f_{x-4}+f_{x-3}+f_{x-2}.$$

Also, the right-hand side of the inequality is equal to $(u_{m-4}+u_{m-3}+u_{x-4}+u_{x-3})+f_{m-4}+f_{m-3}+f_{m-2}+f_{x-4}+f_{x-3}+f_{x-2}$. For all values of n according to $mod\ 5$, it is easily seen that $u_1+u_2+u_{m+1}+u_{m+2}=2n+18$ since $u_1=4$, $u_2=5$, $u_{m+1}=n+4$, $u_{m+2}=n+5$.
For $n\equiv 0$ (mod 5), we have $6n-22-2n-18\leq 5x-50$ and $x\geq \lceil\frac{4n+10}{5}\rceil$ since $u_{m-4}=n-6$, $u_{m-3}=n-5$, $u_{x-4}=2n-6$, $u_{x-3}=2n-5$ and $f_{m-4}=f_{x-4}=f_{m-3}=f_{x-3}=4$, $f_{m-2}=f_{x-2}=2$. In this case, $|X|=x\geq\lceil\frac{4n+10}{5}\rceil$. This implies that $\gamma_{\times 2}(D_{sd}(G,\{3\}))\geq\lceil\frac{4n+10}{5}\rceil$.
For $n\equiv 1$ (mod 5), we have $6n-26-2n-18\leq 5(x-10)$ and $x\geq\lceil\frac{4n+6}{10}\rceil$since $u_{m-4}=n-7$, $u_{m-3}=n-6$, $u_{x-4}=2n-7$, $u_{x-3}=2n-6$ and $f_{m-4}=f_{x-4}=f_{m-3}=f_{x-3}=3$, $f_{m-2}=f_{x-2}=2$. In this case, $|X|=x\geq\lceil\frac{4n+6}{10}\rceil=\lceil\frac{4n+10}{6}\rceil-1$. This implies that $\gamma_{\times 2}(D_{sd}(G,\{3\}))\geq\lceil\frac{4n+10}{5}\rceil+1$.
If $n\equiv 2$ (mod 5) , then $\sum\limits_{t_1=1}^{m-4}f_{t_1}+\sum\limits_{t_2=m+1}^{x-4}f_{t_2}+f_{m-3}+f_{m-2}+f_{x-3}+f_{x-2}\leq 5(x-8)+f_{m-3}+f_{m-2}+f_{x-3}+f_{x-2}$. For $n\equiv 2$ (mod 5), $f_{m-3}=f_{m-2}=f_{x-3}=f_{x-2}=2$. Then, we have $\sum\limits_{t_1=1}^{m-4}f_{t_1}+\sum\limits_{t_2=m+1}^{x-4}f_{t_2}+28=(u_{m-3}+u_{m-2}+u_{x-3}+u_{x-2})-(2n+18)+8$. Furthermore, we get $6n-10-2n-18\leq 5x-40$ and $x\geq\lceil\frac{4n+12}{5}\rceil$ since $u_{m-3}=n-3$, $u_{m-2}=n-2, u_{x-3}=2n-3$ $u_{x-2}=2n-2$. In this case, $|X|=x\geq\lceil\frac{4n+12}{5}\rceil=\lceil\frac{4n+10}{5}\rceil$. This implies that $\gamma_{\times 2}(D_{sd}(G,\{3\}))\geq\lceil\frac{4n+10}{5}\rceil$.
If $n\equiv 3$ (mod 5), then the formula in $n\equiv 0,1$ (mod 5) is valid. For $n\equiv 3$ (mod 5), we have $6n-14-2n-18\leq 5(x-10)$ and $x\geq\lceil\frac{4n+18}{5}\rceil$since $u_{m-4}=n-4$, $u_{m-3}=n-3$, $u_{x-4}=2n-4$, $u_{x-3}=2n-3$ and $f_{m-4}=f_{m-3}=f_{m-2}=f_{x-4}=f_{x-3}=f_{x-2}=2$. In this case, $|X|=x\geq\lceil\frac{4n+18}{5}\rceil=\lceil\frac{4n+10}{5}\rceil+1$. This implies that $\gamma_{\times 2}(D_{sd}(G,\{3\}))\geq\lceil\frac{4n+10}{5}\rceil+1$.
If $n\equiv 4$ (mod 5), then the formula in $n\equiv 0,1,3$ (mod 5) is valid. For $n\equiv 4$ (mod 5), we have $6n-18-2n-18\leq 5(x-10)$ and $x\geq\lceil\frac{4n+14}{5}\rceil=\lceil\frac{4n+10}{5}\rceil$since $u_{m-4}=n-5$, $u_{m-3}=n-4$, $u_{x-4}=2n-5$, $u_{x-3}=2n-4$ and $f_{m-4}=f_{m-3}=f_{x-4}=f_{x-3}=3$ and $f_{m-2}=f_{x-2}=2$. In this case, $|X|=x\geq\lceil\frac{4n+10}{5}\rceil$. This implies that $\gamma_{\times 2}(D_{sd}(G,\{3\}))\geq\lceil\frac{4n+10}{5}\rceil$.
Thus, the desired equality is obtained as a result of the lower and upper bounds on $\gamma_{\times 2}(D_{sd}(G,\{3\}))$.
This completes the proof. ■

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