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Formal derivation and existence of global weak solutions of an energetically consistent viscous sedimentation model

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Abstract. The purpose of this paper is to derive a viscous sedimentation model from the Navier-Stokes system for incompressible flows with a free moving boundary. The derivation is based on the different properties of the fluids; thus, we perform a multiscale analysis in space and time, and a different asymptotic analysis to derive a system coupling two different models: the sediment transport equation for the lower layer and the shallow water model for the upper one. We finally prove the existence of global weak solutions in time for model containing some additional terms.
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Contents

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1. Introduction

Sediment transport models are used to model watercourse beds. They are bilayer models of two immiscible layers that have a model of the shallow water in the first layer and Reynolds lubrication equation at the second layer. In the literature, many works has been done on sediment transport, proposing models to stimulate sediment transport by water. We can quote [6, 7, 22, 24].
Recently in [7], Fernandez and al. are derived a non-viscous sediment model. In their work, they are limited to a first-order approximation for obtaining the model of shallow water which does not allow to obtain a viscous model. To carry out our work we relied on the papers [6, 7, 19, 21].

From theoretical point of view, many studies have been done, particularly for the existence of global weak solutions of shallow-water equations model. As an example, we refer to [5], where such results were given for an isothemal model of compressible fluids with capillarity.

In [21], only the stability of weak solutions has been proved, since the construction of approximate weak solutions which preserve the 'mathematical BD entropy' seems to be an open problem. In the present work this problem does not exist, as we do not need the multiplier $|u|^k u$ to get the 'BD entropy'.

In the analysis we propose in this work, our contribution is twofold. On the one hand, we propose a constructive approach inspired by [7, 16] to arrive at a viscous sedimentation model. Our purpose is to study the evolution of this system which consists of two layers of Newtonian viscous fluids with different properties. On the other hand, our study is concerned with the existence of global weak solutions of a model similar to the one we obtained. This is done in a bounded domain of $\mathbb{R}^2$ with periodic boundary conditions.

In our model we add some additional regularizing terms, namely
$-\kappa\nabla\cdot(1+\dfrac{h_2}{rh_1})\nabla(h_1+\dfrac{1}{r}h_2)$, the cold pressure $\delta h_1\nabla h_1^{-\alpha}$ and the interface tension $\bar{\kappa}h_1\nabla\Delta^{2s+1}h_1$ with $\alpha,\kappa,\bar{\kappa}$ positive constants and $\alpha\neq 0$. Those terms are useful to bound h_1 away form zero (see [3, 11, 24]).

Our paper is organized as follows. In the section 2, we did the formal derivation of the model. First of all we write the equations in non-dimensional variables. Next, we perform the hydrostatic approximation and use am asymptotic analysis to deduce the shallow water system for the upper layer. Also by an asymptotic analysis, we deduce the transport equation for the lower layer. In addition in the section 3, we present our final model. To finish, in Section 3 we study the existence of global weak solutions for a model similar to that obtained in Section 2. We start by giving the definition of global weak solutions, next we establish a classical energy equality and the 'mathematical BD entropy', which entail some regularities on the unknowns. We also give an existence theorem of global weak solutions.

2. Formal derivation

2.1. Physical domain and governing equations

This section is devoted to the formal derivation of the model. Thus, we consider a superposition of two immiscible layers of different materials. The upper layer contains water and the lower layer is formed of sediment. Each layer is governed by the incompressible three dimensional Navier Stokes equations. We consider a cartesian coordinate system where x represents the horizontal 2D direction and z the vertical one. Taking into account the different physical properties for each layer, we derive shallow water model for the upper layer and the Reynolds lubrification equation for the lower layer. Let us define the physical domain for the fluid and sediment by $\Omega_1(t)$ and $\Omega_2(t)$ respectively; t being the time variable. Here, we suppose that the sediment domain is composed by a one layer. We assume that the bottom is defined by the function $b(x)$ and we denote by $\eta(t,x)$ the interface. The free surface is given by $\xi(t,x)$. The global domain $\Omega(t)$ is defined as

$$\Omega(t) = \Omega_1(t) \cup \Omega_2(t) \cup \Gamma_b(t) \cup \Gamma_{1,2}(t) \cup \Gamma_s(t),$$

$\Omega_1(t) = \{(x,z) \in \mathbb{R}^3 : x \in \omega, \eta(x,t) < z < \xi(x,t)\}$,

$\Omega_2(t) = \{(x,z) \in \mathbb{R}^3 : x \in \omega, b(x) < z < \eta(x,t)\}$,

$\Gamma_{1,2}(t) = \{(x,z) \in \mathbb{R}^3 : x \in \omega, z = \eta(x,t)\}$,

$\Gamma_s(t) = \{(x,z) \in \mathbb{R}^3 : x \in \omega, z = \xi(x,t)\}$,

and
$\Gamma_b = \{(x,z) \in \mathbb{R}^3 : x \in \omega, z = b(x)\}$.

The domain $\Omega(t) \subset \mathbb{R}^3$ is periodic. For each layer ($i = 1,2$), we start from the 3D Navier-Stokes equations for incompressible fluid and sediment components see [6, 7, 15]

$$\mathrm{div}(U_i) = 0, \tag{2.1a}$$
$$\rho_i \partial_t(U_i) + (\rho_i U_i \nabla) U_i - \mathrm{div}(\sigma_i) = -\rho_i g \vec{e}_z, \tag{2.1b}$$

where we denote by $U_i = {}^t(\mathbf{u}_i, w_i)$ the velocity field with $\mathbf{u}_i = (u_i, v_i)$, σ_i the stress tensor associated to each layer, ρ_i the density and g the gravitational vector with $\vec{e}_z = {}^t\ (0,0,1)$.
If we rewrite the equation for each component of the velocity, the previous system is equivalent to the following one:

$$\mathrm{div}_x \mathbf{u}_i + \partial_z w_i = 0, \tag{2.2a}$$
$$\rho_i \partial_t \mathbf{u}_i + \rho_i \mathbf{u}_i \nabla \mathbf{u}_i + \rho_i w_i \partial_z(\mathbf{u}_i) + \nabla p_i = 2\nu_i \mathrm{div}(D(\mathbf{u}_i)) + \nu_i \partial_z^2 \mathbf{u}_i + \nu_i \nabla_x(\partial_z w_i), \tag{2.2b}$$
$$\rho_i \partial_t w_i + \rho_i \mathbf{u}_i \nabla w_i + \rho_i w_i \partial_z w_i = \nu_i \Delta w_i + 2\nu_i \partial_z^2 w_i + \nu_i \partial_z(\mathrm{div}\mathbf{u}_i) - \partial_z p_i - \rho_i g. \tag{2.2c}$$

for $i = 1,2$,
where ρ_i is the density, p_i the pressure and g the gravity constant.Moreover μ_i and $\nu_i = \mu_i/\rho_i$, denote the dynamic and kinematic viscosity coefficients respectively. We also introduce the ratio of the densites r, respectively the stress tensor given by

$$r = \frac{\rho_1}{\rho_2}, \qquad \sigma_i(\mathbf{u}_i) = 2\nu_i D(\mathbf{u}_i) - p_i Id, \quad \text{where} \quad D(\mathbf{u}_i) = \frac{\nabla \mathbf{u}_i +^t \nabla \mathbf{u}_i}{2},$$

and Id is the identity matrix.
From now on, subscript 1 will correspond to the layer located on the top and subscript 2 to those located belox. We denote by $h_1(t,x) = \xi(t,x) - b(x)$ the tichness of the layer 1 and by $h_2(t,x) = \eta(t,x) - b(x)$ the tichness of the sediment layer. See Figure 1.
The system (2.2a)-(2.2c) is completed by the following boundaries conditions:
• At the free surface $z = \xi(x,t) = b(x) + h_2(x,t) + h_1(x,t)$:

- The surface tension condition. Let N_s the unitary outward normal vector to the free surface and k the mean curvature of the surface with $k = -\mathrm{div}(N_s)$. The surface tension is given by the equality

$$\sigma_1 N_s = -\delta k N_s, \tag{2.3}$$

where $N_s = \dfrac{1}{\sqrt{1 + |\nabla_x \xi|^2}} \begin{pmatrix} -\nabla_x \xi \\ 1 \end{pmatrix}$

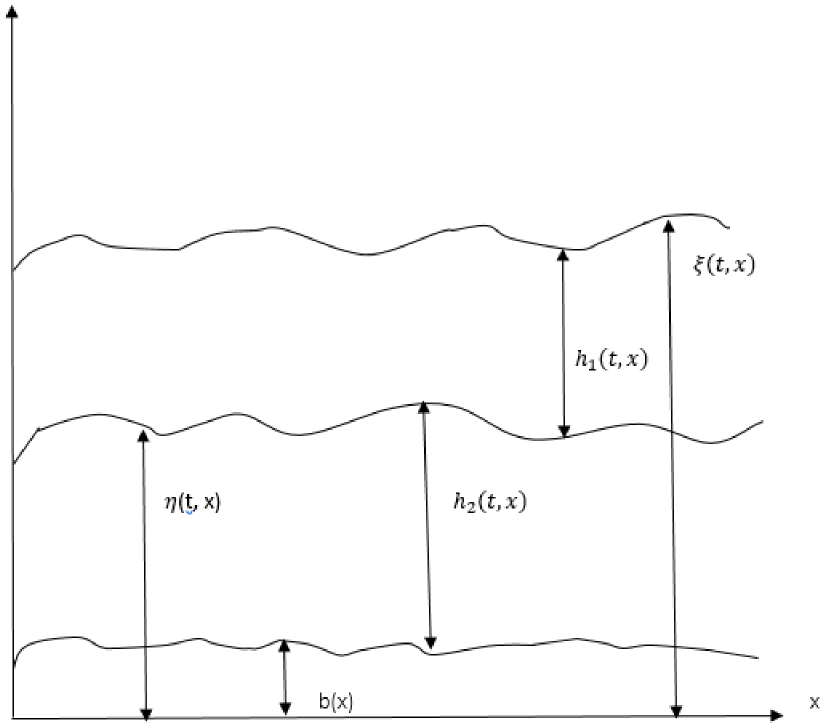

Figure 1: Sediment and water heights

and δ being a constant.

- The kinematic condition:

$$\partial_t \xi = U_1.N_s. \tag{2.4}$$

• At the fluid/sediment interface, $\eta(t,x) = b(x) + h_2(x,t)$:

- The kinematic conditions corresponding to both velocities:

$$\partial_t \eta = U_1.N_\eta = U_2.N_\eta \tag{2.5}$$

where $N_\eta = \dfrac{1}{\sqrt{1+|\nabla_x \eta|^2}}\begin{pmatrix} -\nabla_x \eta \\ 1 \end{pmatrix}$.

- The continuity of the normal component of the tensors:

$$(\sigma_1 N_\eta)_n - (\sigma_2 N_\eta)_n = (\delta_\eta k_\eta N_\eta), \tag{2.6}$$

where δ_η is the interfacial tension coefficient, $k_\eta = -\text{div} N_\eta$ is the mean curvature of the interface.

- The friction law (Navier-slip boundary condition) at the fluid-sediment interface asserting that:

$$(\sigma_i N_\eta)_\tau = \text{fric}(U_1 - U_2)_\tau. \tag{2.7}$$

We note that the friction coefficient is denoted by c and the subscript τ is the tangential component of the vector.
In the sequel we denote by $\text{fric}(U_1 - U_2) = C\rho_1(U_1 - U_2)$ the friction term between the two layers.

• At the bottom, $z = b(x)$:

- The no penetration condition:

$$U_2.N_b = 0, \tag{2.8}$$

where the unitary normal vector to the bottom is
$N_b = \frac{1}{\sqrt{1+|\nabla_x b|^2}} \begin{pmatrix} -\nabla_x b \\ 1 \end{pmatrix}.$

Remark 2.1. 1. In [7], a coulomb condition is considered between the static and the moving sediment particules. Here, we consider this condition at the interface $z = \eta(t,x)$.

2. To obtain the model, firstly we shall write these equations under a dimensionless. Secondly we shall develop the vertical integration in each layer to obtain the shallow water system. In addition, we shall perform the asymptotic analysis studding both, first and second order approximative for the the shallow water system. Finally, we will find for the sediment layer, the transport equation.

2.2. Dimensionless equations

In order to compare the terms that occur in the equations, we introduce dimensionless variables. For this, we note by H, and L the characteristic height and length respectively. In the considered flows, we assume that the characteristic height is very small compared to the characteristic length and we note by $\varepsilon = \frac{H}{L}$ the aspect ratio between the characteristic height and length. The characteristic velocities are U for the layer 1 and U_2 for the sediment layer. Consequently, the characteristic times are respectively $T = \frac{L}{U_1}$ and $T_2 = \frac{L}{V}$ for each layer. In particular we assume that

$$U_2 = \varepsilon^2 U, \quad \text{so consequently,} \quad T_2 = \frac{L}{U_2} = \frac{1}{\varepsilon^2} T.$$

This hypothesis also affects the definitions of the Froude and Reynolds numbers. For the sake of clarity we indicate separately these variables. We consider the ”asterisk” notation for the dimensionless variables.
General dimensionless variables:

$$x = L\bar{x}, \qquad z = H\bar{z}, \qquad \text{fric} = \rho_1 U^2 \bar{\text{fric}}$$

Non-dimensionalization for layer 1:

$$\mathbf{u}_1 = U\bar{\mathbf{u}}_1, \qquad w_1 = \varepsilon U \bar{w}_1, \qquad t_1 = \frac{L}{U}\bar{t}_1, \qquad p_1 = \rho_1 U^2 \bar{p}_1$$

$$F_{r_1} = \frac{U}{\sqrt{gH}}, \qquad Re_1 = \frac{UL}{\nu_1}, \qquad h_1 = H\bar{h}_1$$

Non-dimensionalization for layer 2:

$$\mathbf{u}_2 = \varepsilon^2 U\bar{\mathbf{u}}_2, \qquad w_2 = \varepsilon^3 U \bar{w}_2, \qquad t = \frac{1}{\varepsilon^2} T\bar{t}_2, \qquad p_2 = \frac{\rho_2 \nu_2 U}{\varepsilon H}\bar{p}_2$$

$$F_{r_2} = \frac{\varepsilon^2 U}{\sqrt{gH}}, \qquad Re_2 = \frac{\varepsilon^2 UL}{\nu_2}, \qquad h_2 = H_2\bar{h}_2 \qquad \text{with} \quad H_2 = \varepsilon H.$$

We also define the ratio of the densities,

$$r = \frac{\rho_1}{\rho_2} \qquad \text{with} \qquad r < 1.$$

Remark 2.2. We set $C = U\bar{C}$.
Assuming that H is the characteristic height for the bottom, $b = H\bar{b}$.

Thus, the equations and the boundary conditions written in dimensionless form read as follows (we omit the "asterisk" to simplify the notation):

• Layer 1:

$$\mathrm{div}_x \mathbf{u}_1 + \partial_z w_1 = 0, \tag{2.9a}$$

$$\partial_{t_1}\mathbf{u}_1 + \mathbf{u}_1\nabla_x\mathbf{u}_1 + w_1\partial_z\mathbf{u}_1 + \nabla_x p_1 = \frac{1}{Re_1}(2\mathrm{div}_x(D_x(\mathbf{u}_1)) + \frac{1}{\varepsilon^2}\partial_z^2\mathbf{u}_1 + \nabla_x(\partial_z w_1)), \tag{2.9b}$$

$$\varepsilon^2(\partial_{t_1}w_1 + \mathbf{u}_1\nabla_x w_1 + w_1\partial_z w_1) = \frac{1}{Re_1}(\varepsilon^2\Delta_x w_1 + 2\partial_z^2 w_1 + \partial_z(\mathrm{div}_x\mathbf{u}_1)) - \partial_z p_1 - \frac{1}{Fr_1^2}. \tag{2.9c}$$

• Layer 2:

$$\mathrm{div}_x\mathbf{u}_2 + \partial_z w_2 = 0, \tag{2.10a}$$

$$\varepsilon^8 Re_2(\partial_{t_2}\mathbf{u}_2 + \mathbf{u}_2\nabla_x\mathbf{u}_2 + w_2\partial_z\mathbf{u}_2) + \nabla_x p_2 = 2\varepsilon^4\mathrm{div}_x(D_x(\mathbf{u}_2)) + \partial_{z^2}^2\mathbf{u}_2 + \varepsilon^4\nabla(\partial_z w_2) \tag{2.10b}$$

$$\begin{aligned}&\varepsilon^8 Re_2(\partial_{t_2}w_2 + \mathbf{u}_2\nabla_x w_2 + w_2\partial_z w_2) = \varepsilon^4(\varepsilon^4\Delta w_2 + \partial_z(\mathrm{div}_x\mathbf{u}_2) + 2\partial_z^2 w_2)\\&-\varepsilon^4\frac{Re_2}{Fr_2^2} - \partial_z p_2.\end{aligned} \tag{2.10c}$$

• Conditions at the free surface

$$\partial_{t_1}\xi + \mathbf{u}_1.\nabla_x\xi = w_1, \tag{2.11a}$$

$$\left(\frac{-2}{Re_1}D_x(\mathbf{u}_1) + \rho_1 p_1 - \rho_1\frac{\varepsilon}{Re_1}C^{-1}\Delta\xi\right)\nabla_x\xi + \frac{1}{Re_1}\nabla_x w_1 + \frac{1}{\varepsilon^2}\frac{1}{Re_1}\partial_z\mathbf{u}_1 = 0, \tag{2.11b}$$

$$-\frac{1}{Re_1}(\varepsilon^2\nabla_x w_1 + \partial_z\mathbf{u}_1)\nabla_x\xi + \frac{2}{Re_1}\partial_z w_1 + \rho_1\varepsilon\frac{1}{Re_1}C^{-1}\Delta\xi - \rho_1 p_1 = 0. \tag{2.11c}$$

• Conditions at the interface

$$\partial_{t_1}\eta + \mathbf{u}_1.\nabla_x\eta = w_1, \tag{2.12a}$$

$$\partial_{t_1}\eta + \varepsilon^2\mathbf{u}_2.\nabla_x\eta = \varepsilon^3 w_2, \tag{2.12b}$$

$$\partial_{t_2}\eta + \mathbf{u}_2.\nabla_x\eta = w_2, \tag{2.12c}$$

$$\frac{1}{Re_1}\left(\nabla w_1 + \frac{1}{\varepsilon^2}\partial_z\mathbf{u}_1\right) = -r\frac{1}{\varepsilon}\mathrm{fric}(\mathbf{u}_1 - \varepsilon^2\mathbf{u}_2)\sqrt{1+\varepsilon^2|\nabla\eta|^2}, \tag{2.12d}$$

$$\frac{1}{Re_1}\left(\varepsilon^3\nabla w_2 + \varepsilon\partial_z\mathbf{u}_2\right) = -\frac{1}{\varepsilon}\mathrm{fric}(\mathbf{u}_1 - \varepsilon^2\mathbf{u}_2)\sqrt{1+\varepsilon^2|\nabla\eta|^2}, \tag{2.12e}$$

$$\begin{aligned}&\frac{1}{Re_1}\left(-2D(\mathbf{u}_1)\cdot\nabla\eta + (\nabla w_1 + \frac{1}{\varepsilon^2}\partial_z\mathbf{u}_1)(1-\varepsilon^2|\nabla\eta|^2) + 2\partial_z w_1\nabla\eta\right)\\&\qquad = r\frac{1}{\varepsilon}\mathrm{fric}\big((\mathbf{u}_1 - \varepsilon^2\mathbf{u}_2) + \varepsilon^2(w_1 - \varepsilon^2 w_2)\nabla\eta\big)\sqrt{1+\varepsilon^2|\nabla\eta|^2},\end{aligned} \tag{2.12f}$$

$$\begin{aligned}&\frac{1}{Re_2}\left(-2\varepsilon^3 D(\mathbf{u}_2)\cdot\nabla\xi_\varepsilon + (\varepsilon^3\nabla w_2 + \partial_z\mathbf{u}_2)(1-\varepsilon^2|\nabla\eta|^2) + 2\varepsilon^2\partial_z w_2\nabla\eta\right)\\&\qquad = \frac{1}{\varepsilon}\mathrm{fric}\big((\mathbf{u}_1 - \varepsilon^2\mathbf{u}_2) + \varepsilon^2(w_1 - \varepsilon^2 w_2)\nabla\eta\big)\sqrt{1+\varepsilon^2|\nabla\eta|^2},\end{aligned} \tag{2.12g}$$

$$\begin{aligned}&\rho_1\varepsilon^2[\frac{2}{Re_1}D(\mathbf{u}_1) - p_1]|\nabla\eta|^2 - 2\rho_1(\partial_z\mathbf{u}_1 + \varepsilon^2\nabla w_1)\nabla\eta + \rho_1(\frac{2}{Re_1}\partial_z w_1 - p_1)\\&\qquad = \varepsilon^2\frac{\rho_2}{Re_2}(\varepsilon^4 D(\mathbf{u}_2) - p_2)|\nabla\eta|^2 - 2\rho_2\frac{1}{Re_2}\varepsilon^3(\partial_z\mathbf{u}_2 + \varepsilon^3\nabla w_2)\nabla\eta\\&\qquad\quad + \frac{1}{Re_2}\rho_2(2\varepsilon^3\partial_z w_2 - p_2) - \varepsilon\rho_1\frac{C_\eta^{-1}}{Re_1}\mathrm{div}(\eta)(1+\varepsilon^2|\nabla\eta|^2).\end{aligned} \tag{2.12h}$$

• Condition at the bottom

$$-\mathbf{u}_2 \nabla_x b + w_2 = 0. \tag{2.13}$$

2.3. Layer Ω_1: Shallow water

To get the Saint-Venant-Exner system, we first take the hydrostatic approximation and then develop the asymptotic analysis of equations.

2.3.1. Hydrostatic approximation

Since the length of the flow is supposed to be very large compared to the depth of the water, we assume that ε to be small. Let us take the formal expression of system (2.2a)-(2.8) at $O(\varepsilon^2)$ (see [1, 9, 10, 12] for the usual derivations of hydrostatic approximations), and keep the terms of order zero and one. We obtain successively,

• Layer 1:

$$\mathrm{div}_x \mathbf{u}_1 + \partial_z w_1 = 0, \tag{2.14a}$$

$$\partial_t \mathbf{u}_1 + \mathbf{u}_1 \nabla \mathbf{u}_1 + \partial_z (w_1 \mathbf{u}_1) + \nabla p_1 = \frac{1}{Re_1}(2\mathrm{div}(D(\mathbf{u}_1)) + \frac{1}{\varepsilon^2}\partial_z^2 \mathbf{u}_1 + \nabla(\partial_z w_1), \tag{2.14b}$$

$$\partial_z p_1 = -\frac{1}{Fr_1^2} + \frac{1}{Re_1}(2\partial_z^2 w_1 + \partial_z(\mathrm{div}\mathbf{u}_1)). \tag{2.14c}$$

• Layer 2:

$$\mathrm{div}_x \mathbf{u}_2 + \partial_z w_2 = 0, \tag{2.15a}$$

$$\nabla_x p_2 = \partial_z^2 \mathbf{u}_2, \tag{2.15b}$$

$$\partial_z p_2 = O(\varepsilon). \tag{2.15c}$$

• Conditions at the free surface

$$\partial_{t_1} \xi + \mathbf{u}_1 . \nabla_x \xi = w_1, \tag{2.16a}$$

$$\left(\frac{-2}{Re_1} D_x(\mathbf{u}_1) + \rho_1 p_1 - \rho_1 \frac{\varepsilon}{Re_1} C^{-1} \nabla \xi\right) \nabla_x \xi + \frac{1}{Re_1} \nabla_x w_1 + \frac{1}{\varepsilon^2}\frac{1}{Re_1} \partial_z \mathbf{u}_1 = 0, \tag{2.16b}$$

$$-\frac{1}{Re_1} \partial_z \mathbf{u}_1 \nabla_x \xi + \frac{2}{Re_1} \partial_z w_1 + \frac{\rho_1 \varepsilon C^{-1} \Delta \xi}{Re_1} - \rho_1 p_1 = 0. \tag{2.16c}$$

• Conditions at the interface

$$\partial_{t_1}\eta + \mathbf{u}_1.\nabla_x\eta = w_1, \tag{2.17a}$$
$$\partial_{t_1}\eta = O(\varepsilon), \tag{2.17b}$$
$$\partial_{t_2}\eta + \mathbf{u}_2.\nabla_x\eta = w_2, \tag{2.17c}$$
$$\frac{1}{Re_1}\left(\nabla w_1 + \frac{1}{\varepsilon^2}\partial_z\mathbf{u}_1\right) = -r\frac{1}{\varepsilon}\text{fric}(\mathbf{u}_1 - \varepsilon^2\mathbf{u}_2)\sqrt{1+\varepsilon^2|\nabla\eta|^2}, \tag{2.17d}$$
$$\frac{1}{Re_1}\left(\varepsilon^4\nabla w_2 + \varepsilon\partial_z\mathbf{u}_2\right) = -\frac{1}{\varepsilon}\text{fric}(\mathbf{u}_1 - \varepsilon^2\mathbf{u}_2)\sqrt{1+\varepsilon^2|\nabla\eta|^2}, \tag{2.17e}$$
$$\begin{aligned}\frac{1}{Re_1}\Big(&-2D(\mathbf{u}_1)\cdot\nabla\eta + (\nabla w_1 + \frac{1}{\varepsilon^2}\partial_z\mathbf{u}_1)(1-\varepsilon^2|\nabla\eta|^2) + 2\partial_z w_1\nabla\eta\Big)\\ &= r\frac{1}{\varepsilon}\text{fric}\big((\mathbf{u}_1 - \varepsilon^2\mathbf{u}_2) + \varepsilon^2(w_1 - \varepsilon^3 w_2)\nabla\eta\big)\sqrt{1+\varepsilon^2|\nabla\eta|^2},\end{aligned} \tag{2.17f}$$
$$\begin{aligned}\frac{1}{Re_2}\Big(&-2\varepsilon^3 D(\mathbf{u}_2)\cdot\nabla\eta + (\varepsilon^3\nabla w_2 + \partial_z\mathbf{u}_2)(1-\varepsilon^2|\nabla\eta|^2) + 2\varepsilon^2\partial_z w_2\nabla\eta\Big)\\ &= \frac{1}{\varepsilon}\text{fric}\big((\mathbf{u}_1 - \varepsilon^2\mathbf{u}_2) + \varepsilon^2(w_1 - \varepsilon^2 w_2)\nabla\eta\big)\sqrt{1+\varepsilon^2|\nabla\eta|^2},\end{aligned} \tag{2.17g}$$
$$\begin{aligned}\varepsilon^2[&\frac{2}{Re_1}D(\mathbf{u}_1) - p_1]|\nabla\eta|^2 - 2r(\partial_z\mathbf{u}_1 + \varepsilon^2\nabla w_1)\nabla\eta + (\frac{2}{Re_1}\partial_{z_1}w_1 - p_1)\\ &= \varepsilon^2\frac{r}{Re_2}(\varepsilon^4 D(\mathbf{u}_2) - p_2)|\nabla\eta|^2 - 2r\frac{1}{Re_2}\varepsilon^3(\partial_{z_2}\mathbf{u}_2 + \varepsilon^3\nabla w_2)\nabla\eta\\ &\quad + \frac{r}{Re_2}(2\varepsilon^3\partial_z w_2 - p_2) - \varepsilon\frac{C_\eta^{-1}}{Re_1}\text{div}(\eta)(1+\varepsilon^2|\nabla\eta|^2).\end{aligned} \tag{2.17h}$$

• Conditions at the bottom

$$-\mathbf{u}_2\nabla_x b + w_2 = 0. \tag{2.18}$$

2.3.2. Asymptotic analysis and shallow water system

To obtain the shallow water equation, we assume that the height is small with respect to the length of the domain, that is $\varepsilon \ll 1$.
We first integrate each equations of (2.14a)-(2.14c) from the layer 1 from η to ξ. For equation (2.14a), by using (2.11a) (2.17a) and (2.17b), we get

$$\partial_{t_1}h_1 + \text{div}\int_\eta^\xi \mathbf{u}_1 dz = 0. \tag{2.19}$$

The condition (2.17a) allows us by integrating the equation (2.14b) to get

$$\begin{aligned}&\partial_{t_1}\int_\eta^\xi \mathbf{u}_1 dz + \text{div}\int_\eta^\xi \mathbf{u}_1\otimes\mathbf{u}_1 dz + \nabla_x\int_\eta^\xi p_1 - \frac{2}{Re_1}\text{div}\int_\eta^\xi D(\mathbf{u}_1)dz\\ &= \frac{1}{\varepsilon^2 Re_1}\partial_z\mathbf{u}_{1|z=\xi} - \frac{1}{\varepsilon^2 Re_1}\partial_z\mathbf{u}_{1|z=\eta} + \frac{1}{Re_1}\nabla_x w_{1|z=\xi} - \frac{1}{Re_1}\nabla_x w_{1|z=\eta}\\ &+ (w_1\mathbf{u}_1)_{|z=\xi} - (w_1\mathbf{u}_1)_{|z=\eta} - \mathbf{u}_1\partial_{t_1}\xi_{|z=\xi} + \mathbf{u}_1\partial_{t_1}\eta_{|z=\eta} - (\mathbf{u}_1.\nabla\xi)\mathbf{u}_{1|z=\xi}\\ &+ (\mathbf{u}_1.\nabla\eta)\mathbf{u}_{1|z=\eta} + p_1\nabla_x\xi_{|_{z=\xi}} - p_1\nabla_x\eta_{|_{|z=\eta}} - \frac{2}{Re_1}D(\mathbf{u}_1)\nabla_x\xi_{|z=\xi} + \frac{2}{Re_1}D(\mathbf{u}_1)\nabla_x\eta_{|z=\eta}\end{aligned} \tag{2.20a}$$

The expression of the pressure in (2.14c) is given by

$$\partial_z p_1 = -\frac{1}{Fr_1^2} + \frac{1}{Re_1}(2\partial_z^2 w_1 + \partial_z(\text{div}\mathbf{u}_1)).$$

By integrating this equation from z to ξ for $z \in [\eta, \xi]$, to obtain,

$$p_1 = p_{1|z=\xi} - \frac{1}{Fr^2}(z-\xi) + \frac{1}{Re_1}[2\partial_z w_1 + \text{div}(\mathbf{u}_1)] - \frac{1}{Re_1}[2\partial_z w_1 + \text{div}(\mathbf{u}_1)]_{|z=\xi}.$$

We use the divergence free condition, we get the following expression for P_1:

$$p_1 = p_{1|z=\xi} - \frac{1}{Fr^2}(z-\xi) - \frac{1}{Re_1}[\text{div}(\mathbf{u}_1) - \text{div}(\mathbf{u}_1)_{|z=\xi}]. \tag{2.21}$$

Due to conditions (2.16a), (2.17a), we can write

$$(\partial_{t_1}\xi + \mathbf{u}_1.\nabla_x \xi - w_1)\mathbf{u}_1|_{z=\xi} = 0 \quad \text{and} \quad (\partial_{t_1}\eta + \mathbf{u}_1.\nabla_x \eta - w_1)\mathbf{u}_1|_{z=\eta} = 0.$$

Thanks to conditions (2.16b), (2.16c), we have

$$\begin{aligned}\frac{1}{Re_1}\Big[-2D_x(\mathbf{u}_1)\nabla\xi + (\nabla_x w_1 + \frac{1}{\varepsilon^2}\partial_z \mathbf{u}_1)\Big] &= -\frac{1}{Re_1}\Big[\rho_1 p_1 - \rho_1 C^{-1}\Delta\xi\Big]\nabla\xi, \\ &= -\frac{1}{Re_1}\big[\partial_z u_1 - 2\partial_z w_1\big].\nabla\xi. \end{aligned} \tag{2.22a}$$

By using (2.17f), we have

$$\frac{1}{Re_1}\Big[-2D_x(\mathbf{u}_1)\nabla\eta + (\nabla_x w_1 + \frac{1}{\varepsilon^2}\partial_z \mathbf{u}_1)\Big] = \frac{1}{Re_1}\big[\partial_z u_1 \nabla\eta - 2\partial_z w_1\big].\nabla\eta - r\frac{1}{\varepsilon}\mathbf{u}_1\text{fric}. \tag{2.23}$$

So, for the first layer, we get the equation

$$\begin{aligned}&\partial_{t_1}\int_\eta^\xi \mathbf{u}_1 dz + \text{div}\int_\eta^\xi \mathbf{u}_1 \otimes \mathbf{u}_1 dz + \nabla_x \int_\eta^\xi p_1 - \frac{2}{Re_1}\text{div}\int_\eta^\xi D(\mathbf{u}_1)dz \\ &-p_1\nabla_x\eta_{|_{z=\eta}} + p_1\nabla_x\xi_{|_{z=\xi}} - \frac{1}{Re_1}(\partial_z \mathbf{u}_1\nabla\eta - 2\partial_z w_1)|_{z=\eta}.\nabla\eta \\ &= -\frac{1}{Re_1}\big(\partial_z u_1 \nabla\xi - 2\partial_z w_1\big)|_{z=\xi}.\nabla\xi - r\mathbf{u}_1\frac{1}{\varepsilon}\text{fric}\end{aligned} \tag{2.24}$$

2.3.3. Asymptotic analysis

We assume the problem to be in an asymptotic regime by supposing the following hypotheses on the data

$$\frac{1}{Re_i} = \varepsilon\mu_{01}, \quad \text{fric} = \varepsilon\text{fric}_0, \quad \nu_2 = \varepsilon^{-1}\bar{\nu}_2. \tag{2.25}$$

Thanks to the definition of the dimensionless variables for the layer 2, we have $Re_2 = \dfrac{\varepsilon^2 UL}{\nu_2}$,

$$Re_2 = \frac{\varepsilon^3}{\nu_{02}}, \quad \text{where} \quad \nu_{02} = \frac{\bar{\nu}_{02}}{UL} = O(1).$$

Since we look for a second-order approximation, we develop the unknowns at order 1 and define

$$\begin{aligned}&\mathbf{u}_1 = \mathbf{u}_1^0 + \varepsilon\mathbf{u}_1^1 + O(\varepsilon^2), \quad w_1 = w_1^0 + \varepsilon w_1^1 + O(\varepsilon^2), \quad p_1 = p_1^0 + \varepsilon p_1^1 + O(\varepsilon^2), \\ &\eta = \eta^0 + \varepsilon\eta^1 + O(\varepsilon^2), \quad \xi = \xi^0 + \varepsilon\xi^1 + O(\varepsilon^2), \quad \mathbf{u}_2 = \mathbf{u}_2^0 + \varepsilon\mathbf{u}_2^1 + O(\varepsilon^2), \\ &w_2 = w_2^0 + \varepsilon w_2^1 + O(\varepsilon^2), \quad p_2 = p_2^0 + \varepsilon p_2^1 + O(\varepsilon^2).\end{aligned} \tag{2.26}$$

For the development of h_2, we take into account that $\eta = h_2 + b$, so we can write

$$h_2 = h_2^0 + \varepsilon h_2^1 + O(\varepsilon^2), \tag{2.27}$$

where $h_2^0 = \eta^0 - b$ and $h_2^1 = \eta^1 - b$. In the some way, we can write

$$h_1 = h_1^0 + \varepsilon h_1^1 + O(\varepsilon^2), \tag{2.28}$$

with $h_1^0 = \xi^0 - \eta^0$ and $h_1^1 = \xi^1 - \eta^1$ (remember that $\xi = \eta + h_1$).

(a) First approximation

If we consider the terms of principal order (ε^0), we deduce from (2.9b), (2.11b) and (2.12f) the following expressions:

$$\partial_z^2 \mathbf{u}_1 = O(\varepsilon), \qquad \partial_z \mathbf{u}_{1|_{z=\xi}} = O(\varepsilon), \qquad \partial_z \mathbf{u}_{1|_{z=\eta}} = O(\varepsilon). \tag{2.29}$$

Then $\mathbf{u}_1$ does not depend on z at first order, so we can write $\mathbf{u}_1^0(t,x,z) = \mathbf{u}_1^0(t,x)$. This implies that we can rewrite the expressions above up to order one, to obtain the final equation for layer 1 at the first order. To begin with, by using the conditions (2.16a), (2.17a) and (2.17b), we write (2.19) as

$$\partial_{t_1} h_1^0 + \text{div}(h_1^0 \mathbf{u}_1^0) = 0. \tag{2.30}$$

To get the momentum equation, we simplify (2.21) by using the free surface condition (2.16a)-(2.16c) to have

$$p_1^0(z) = -\frac{1}{Fr_1^2}(z - \xi^0) - 2\varepsilon \nu_{01} \text{div}_x \mathbf{u}_1^0 + O(\varepsilon^2). \tag{2.31}$$

Therefore, computing the integral appearing in (2.24) yields

$$\nabla \int_\eta^{\xi^0} p_1^0 dz = h_1^0 \nabla(p_1^0(\xi^0)) + p_1^0(\xi^0)\nabla h_1 + \frac{1}{2}\frac{1}{Fr_1^2}\nabla(h_1^0)^2. \tag{2.32}$$

If we inject this expression into (2.24) and consider only the principal order terms, we obtain

$$\begin{aligned}
&\partial_{t_1}(h_1 \mathbf{u}_1^0) + \text{div}(h_1 \mathbf{u}_1^0 \otimes \mathbf{u}_1^0) \\
&\qquad = -h_1 \nabla(p_1^0(\xi^0)) - p_1^0(\xi^0)\nabla h_1 \\
&\qquad\qquad -\frac{1}{2}\frac{1}{Fr_1^2}\nabla(h_1)^2 - p_1^0 \nabla \eta_{|_{z=\eta}} + p_1^0 \nabla \xi|_{z=\xi^0} + \text{fric}_0.
\end{aligned} \tag{2.33}$$

Therefore, the final equation reads

$$\begin{aligned}
&\partial_{t_1}(h_1^0 \mathbf{u}_1^0) + \text{div}(h_1^0 \mathbf{u}_1^0 \otimes \mathbf{u}_1^0) = \\
&-h_1^0 \nabla(p_1^0(\xi^0)) - \frac{1}{2}\frac{1}{Fr_1^2}\nabla(h_1^0)^2 - \frac{1}{Fr_1^2} h_1^0 \nabla \eta + \text{fric}_0,
\end{aligned} \tag{2.34}$$

where the friction term fric_0 (see [7]) is given by

$$\text{fric}_0 = \frac{1}{r}\frac{1}{Fr_1^2} h_2^0 \Big((1-r)\text{sgn}(\mathbf{u}_2)\tan\delta_0 + \big(r\nabla_x h_1^0 + \nabla_x \eta^0\big)\Big). \tag{2.35}$$

Remark 2.3. Notice that the equation (2.34) does not contain the viscous effect. To recover it, we will derive the second-order approximation. To do so, we must take into account the terms of order ε ignored before and perform a parabolic correction of the velocity.

(b) Approximation de Saint-Venant au second ordre

Let us define the average of the velocity $\mathbf{u}_1$ as $\overline{\mathbf{u}}_1 = \frac{1}{h_1}\int_\eta^\xi \mathbf{u}_1 dz$.
We go back to (2.24) to write

$$\begin{aligned}&\partial_{t_1}(h_1\overline{\mathbf{u}}_1) + \text{div}(h_1\overline{\mathbf{u}}_1\otimes\overline{\mathbf{u}}_1)\\ &= \frac{2}{Re_1}\text{div}h_1 D(\overline{\mathbf{u}}_1) - \int_\eta^\xi \nabla_x p_1 - \frac{1}{Re_1}(\partial_z\mathbf{u}_1\nabla\eta - 2\partial_z w_1)|_{z=\eta}.\nabla\eta\\ &- r\mathbf{u}_1\frac{1}{\varepsilon}\text{fric} - \frac{1}{Re_1}(\partial_z\mathbf{u}_1\nabla\xi - 2\partial_z w_1)|_{z=\xi}.\nabla\xi.\end{aligned} \tag{2.36a}$$

We have $\overline{\mathbf{u}_1^2} = \overline{\mathbf{u}}_1^2 + O(\varepsilon^2)$, and $\overline{\mathbf{u}_1\otimes\mathbf{u}_1} = \overline{\mathbf{u}}_1\otimes\overline{\mathbf{u}}_1 + O(\varepsilon^2)$. See [22] for details.
Now we consider the approximation up to order 2 for unknowns

$$\tilde{\mathbf{u}}_1 = \mathbf{u}_1^0 + \varepsilon\mathbf{u}_1^1, \qquad \tilde{p}_1 = p_1^0 + \varepsilon p_1^1, \qquad \tilde{\xi}_1 = \xi_1^0 + \varepsilon\xi_1^1, \qquad \tilde{h}_1 = h_1^0 + \varepsilon h_1^1, \tag{2.37}$$

We consider equations defined in (2.14a)-(2.14c) and write them up to second order. For (2.14a), we get

$$\partial_{t_1}\tilde{h}_1 + \text{div}(\tilde{h}_1\tilde{\mathbf{u}}_1) = O(\varepsilon^2). \tag{2.38}$$

Now, we use the asymptotic hypothesis (2.26) and previous calculations to simplify (2.36a). Using the pressure expression (2.32), gives

$$\nabla\int_\eta^\xi p_1 dz - p_{1|_{z=\xi}}\nabla\xi + p_{1|_{z=\eta}}\nabla\eta = \frac{1}{2}\frac{1}{Fr_1^2}\nabla(h_1^2) + \frac{1}{Fr_1^2}h_1\nabla\eta + h_1\nabla p_{1|_{z=\xi}}. \tag{2.39}$$

Thanks to condition (2.16c), we can write:

$$h_1\nabla p_{1|_{z=\xi}} = -2\varepsilon\mu_{01}\nabla(h_1\text{div}(\mathbf{u}_1^0)) + O(\varepsilon^2). \tag{2.40}$$

Finally, we insert (2.39) and (2.40) into (2.36a) and simplify the terms on the bottom and on the interface ξ. Thus, we get the second-order approximation of the momentum equation for layer 1 as follows:

$$\begin{aligned}&\partial_{t_1}(h_1\overline{\mathbf{u}}_1) + \text{div}(h_1\overline{\mathbf{u}}_1\otimes\overline{\mathbf{u}}_1)\\ &\qquad = 2\varepsilon\mu_{01}\text{div}[h_1 D(\overline{\mathbf{u}}_1)] - \frac{1}{2}\frac{1}{Fr_1^2}\nabla(h_1^2)\\ &\qquad\qquad - \frac{1}{Fr_1^2}h_1\nabla\eta + 2\varepsilon\mu_{01}\nabla(h_1\text{div}(\overline{\mathbf{u}}_1)).\end{aligned} \tag{2.41}$$

2.4. Layer Ω_2: Reynolds

As for the first layer, we look for a second-order approximation, so we develop each unknown at the first order. We set $\tilde{h}_2 = h_2^0 + \varepsilon h_2^1$, $\tilde{\mathbf{u}_2} = \mathbf{u}_2^0 + \varepsilon\mathbf{u}_2^1$, $\tilde{p}_2 = p_2^0 + \varepsilon p_2^1$. The asymptotic regime for layer 2 affects the viscosity and capillary constants. When the surface tension effects are strong, it is essential to have them at the leading order, thus we assume

$$\nu_2 = O(\varepsilon), \qquad \delta = O(\varepsilon^{-2}). \tag{2.42}$$

Consequently, $Re_2 = \frac{\varepsilon UH}{\nu_2} = O(1)$ and $C^{-1} = \frac{\delta}{\varepsilon^2 U\rho_2\nu_2} = O(\varepsilon^{-5})$ and for simplicity we write $C^{-1} = \varepsilon^{-5}C_0^{-1}$.
Now, we study the velocity equation in (2.15a)-(2.15c), which can be written as follows:

$$\partial_z^2\mathbf{u}_2 - \nabla p_2 = O(\varepsilon^4), \tag{2.43}$$

$$\partial_z p_2 = -\varepsilon^4 \frac{Re_2}{Fr_2^2} + O(\varepsilon^4). \tag{2.44}$$

From the definitions of Re_2 and Fr_2, we have $\varepsilon^2 \dfrac{Re_2}{Fr_2^2} = \dfrac{gLH}{U\nu_2} = O(\varepsilon)$, so for the simplicity we introduce

$$\beta_0 = \varepsilon \frac{Re_2}{Fr_2^2} = \varepsilon \frac{1}{\nu_{02} Fr_1^2}. \tag{2.45}$$

.

The equation for the pressure reads

$$\partial_z p_2 = -\varepsilon \beta_0 = o(\varepsilon^4). \tag{2.46}$$

The next step is to find the transport equation for the sediment. To do so, we start to look for $\tilde{\mathbf{u}}_2$ in (2.43), after we compute $\tilde{p_2}$ and $\tilde{\mathbf{u}}_2|_{z=\eta}$ that appear into the expression of $\tilde{\mathbf{u}}_2$.

Integrating the divergence-free equation, we obtain

$$\nabla \cdot \int_b^\eta \tilde{\mathbf{u}}_2 \mathbf{d}z - \tilde{\mathbf{u}}_{2|_{z=\eta}} \nabla \eta + \tilde{\mathbf{u}}_{2|_{z=b}} \nabla b + \tilde{w}_{2|_{z=\eta}} - \tilde{w}_{2|_{z=b}} = 0.$$

If we take into account the conditions (2.17c), (2.18), the mass equation for the second layer is

$$\partial_{t_2} \tilde{h}_2 + \nabla . \int_b^\eta \tilde{\mathbf{u}}_2 dz = 0. \tag{2.47}$$

We integrate (2.46) from z to η to obtain

$$\tilde{p}_2(z) = \tilde{p}_2(\eta) - \varepsilon \beta_0 (z - \eta)$$

We use the condition at the interface (2.12h) and the condition (2.45) to write

$$\tilde{p}_2|_\eta = \varepsilon \frac{r}{\nu_{02} Fr_1^2} h_1^0.$$

Thus, $\tilde{p}_2(z) = \varepsilon \dfrac{r}{\nu_{02} Fr_1^2} h_1^0 - \varepsilon \beta_0 (z - \eta)$ and $\nabla_x \tilde{p}_2 = \varepsilon \dfrac{r}{\nu_{02} Fr_1^2} \nabla h_1^0 + \varepsilon \beta_0 \nabla \eta$ does not depend on z.

Integrating now (2.43) from z to η, we get

$$\partial_z \tilde{\mathbf{u}}_2 = \partial_z \tilde{u}_{2|_{z=\eta}} + \nabla \tilde{p}_2 (z - \eta) = \partial_z \tilde{u}_{2|_{z=\eta}} + O(\varepsilon).$$

We use a generalized law based on the work [15], that reads

$$\text{fric} = C(\mathbf{u}_1 - \mathbf{u}_2)|_{z=\eta} \tag{2.48}$$

We must also take into account the adimensionalization for this friction term. Thus we assume the following dimension and asymptotic to the coefficient C:

$$C = U\bar{C}; \qquad \bar{C} = \varepsilon C^0.$$

Then, we have

$$\text{fric}_0 = C^0(\mathbf{u}_1^0 - \varepsilon^2 \mathbf{u}_2^0|_{z=\eta}) \tag{2.49}$$

From this expression, we get the value of $\mathbf{u}_2^0$

$$\begin{aligned} \mathbf{u}_2^0 = \mathbf{u}_2|_{z=\eta} &= \frac{1}{\varepsilon^2} \mathbf{u}_1^0 - \frac{1}{\varepsilon^2 C^0} \text{fric}_0 \\ &= \frac{1}{\varepsilon^2} \mathbf{u}_1^0 - \frac{1}{r\varepsilon^2 C^0} \frac{h_2^0}{Fr_1^2} \Big((1-r) \text{sgn}(\mathbf{u}_2) \tan\delta_0 + \big(r \nabla_x h_1^0 + \nabla_x \eta^0 \big) \Big). \end{aligned}$$

Considering the equation (2.47) we have

$$\partial_{t_2} h_2^0 + \text{div}_x \bigg(\frac{1}{\varepsilon^2} h_2^0 \mathbf{u}_1^0 - \frac{1}{r\varepsilon^2 C^0} \frac{(h_2^0)^2}{Fr_1^2} \Big((1-r) \text{sgn}(\mathbf{u}_2) \tan\delta_0 + \big(r \nabla_x h_1^0 + \nabla_x \eta^0 \big) \Big) \bigg) = 0 \tag{2.50}$$

2.5. Final model

In this section, we expose the final model obtained in the previous section as a formal second-order approximation of the initial problem (2.2a)-(2.8). For that, we write this system in dimensional variables.
The final model is given in non-dimensional variables by (2.35), (2.38), (2.41) and (2.50). The model is composed of three equations, mass and momentum for the shallow water flow and lubrification Reynolds equation for the sediment layer. We recover the system in dimensional variables

$$\begin{cases} \partial_t h_1 + \text{div}(h_1 \mathbf{u}_1) = 0, \\ \partial_t(h_1\mathbf{u}_1) + \text{div}(h_1\mathbf{u}_1 \otimes \mathbf{u}_1) + \frac{1}{2}g\nabla(h_1^2) + gh_1\nabla(b + h_2) - 2\nu_1\text{div}[h_1 D(\mathbf{u}_1)] \\ -2\nu_1\nabla(h_1\text{div}(\mathbf{u}_1)) + \dfrac{gh_2}{r}\mathcal{P} = 0, \\ \partial_t h_2 + \text{div}_x(h_2 v_b\sqrt{(\frac{1}{r} - 1)gd_s}) = 0, \end{cases} \tag{2.51}$$

with $\mathcal{P} = \nabla_x(rh_1 + h_2 + b) + (1 - r)\text{sgn}(\mathbf{u}_2^0)\tan\delta_0$
and
$v_b = \dfrac{1}{\sqrt{(\frac{1}{r} - 1)gd_s}}\mathbf{u}_1 - \dfrac{v}{1 - r}\mathcal{P}.$
We note that we were inspired by [6] for the expression of v_b. Note that in this paper, we do not decompose the sediment layer into two entities. We suppose it one. We refer the readers to [6] for the meaning of d_s, v and v_b.

3. Existence of weak solutions

In this section we assume that bottom vanish in the model (i.e $b(x, y) = 0$)and that the velocities of the sediment and the water are identical. We also needed a regularizing term of the form $-\kappa\nabla \cdot (1 + \dfrac{h_2}{rh_1})\nabla(h_1 + \dfrac{1}{r}h_2)$ on the transport equation. The model studied is as follow:

$$\partial_t h_1 + \text{div}(h_1 u_1) = 0, \tag{3.1}$$

$$\partial_t(h_1u_1) + \text{div}(h_1u_1 \otimes u_1) + gh_1\nabla h_1 + gh_1\nabla h_2 - 2\nu_1\text{div}(h_1 D(u_1)) + gh_2\nabla(h_1 + \frac{1}{r}h_2)$$
$$-\beta h_1\nabla\Delta h_1 + \delta h_1\nabla h_1^{-\alpha} + \bar{\kappa}h_1\nabla\Delta^{2s+1}h_1 = 0, \tag{3.2}$$

$$\partial_t h_2 + \text{div}(h_2u_1) - \kappa\nabla \cdot \left[(1 + \frac{h_2}{rh_1})\nabla(h_1 + \frac{1}{r}h_2)\right] = 0, \tag{3.3}$$

where $\alpha, \kappa, \bar{\kappa}$ are a positive constants $\alpha \neq 0$.The term $\delta h_1\nabla h_1^{-\alpha}$ represente the cold presure, while $\bar{\kappa}h_1\nabla\Delta^{2s+1}h_1$ is the interface tension.
The initial data are

$$h_1(0, x) = h_{1_0}, \quad h_2(0, x) = h_{2_0}, \quad (h_1u_1)(0, x) = \mathbf{m}_0(x) \quad \text{in } \Omega, \tag{3.4}$$

and we assume that h_{1_0}, h_{2_0} and $\mathbf{m}_0$ are such that

$$h_{1_0} \in L^2(\Omega), \quad h_{2_0} \in L^2(\Omega), \quad 0 < h_{1_0}, \quad 0 \le h_{2_0}, \quad \nabla(\sqrt{h_{1_0}}) \in L^2(\Omega),$$
$$\nabla\Delta^s h_{1_0} \in L^2(\Omega), \quad h_{1_0}^{\frac{1-\alpha}{2}} \in L^2(\Omega), \quad \nabla\mathbf{m}_0 \in L^2(\Omega), \quad \mathbf{m}_0 = 0 \quad \text{if} \quad h_{1_0} = 0, \tag{3.5}$$
$$\frac{\mathbf{m}_0}{h_{1_0}} \in L^2(\Omega)$$

3.1. Mains results

Definition 3.1. We say that (h_1, h_2, u_1) is weak solutions of $(3.1) - (3.3)$, with the initial condition (3.4) satisfying (3.5), if
• the initial condition (3.4) hold in $D'((0,T)\times\Omega)$,
• the energy inequalities defined in the **Proposition** 3.2 and **Proposition** 3.4 are satisfied, and the regularities properties obtained in **Corollary** 3.3 and **Corollary** 3.5 hold,
• for all smooth test functions $\varphi = \varphi(t,x)$ with $\varphi(T,\cdot) = 0$, we have:

$$-h_{1_0}\varphi(0,\cdot) - \int_0^T\int_\Omega h_1\partial_t\varphi - \mathbf{m}_0(x)\varphi(0,\cdot) - \int_0^T\int_\Omega h_1u_1 \text{ div }(\varphi) = 0, \tag{3.6}$$

$$h_{2_0}\varphi(0,\cdot) - \int_0^T\int_\Omega h_2u_1\nabla\varphi + \kappa\int_0^T\int_\Omega\Big(1+\frac{h_1}{rh_2}\Big)\nabla(h_1+h_2/r)\nabla\varphi = 0, \tag{3.7}$$

$$\begin{aligned}
-h_{1_0}u_{1_0}\varphi(0,\cdot) &- \int_0^T\int_\Omega(h_1u_1)\partial_t\varphi - \int_0^T\int_\Omega\sqrt{h_1}u_1\otimes\sqrt{h_1}u_1 : D(\varphi) + 2\nu_1\int_0^T\int_\Omega h_1[D(u_1):D(\varphi)]\\
&-g\int_0^T\int_\Omega h_1^2\text{div}(\varphi) - g\int_0^T\int_\Omega h_1h_2 \text{ div}(\varphi) - \frac{g}{2r}\int_0^T\int_\Omega h_2^2\text{div}(\varphi) + \delta\int_0^T\int_\Omega h_1\nabla h_1^{-\alpha}\varphi\\
&-\beta\int_0^T\int_\Omega\Big[h_1\Delta h_1\Big]\text{div}(\varphi) - \beta\int_0^T\int_\Omega\Big[\Delta h_1\nabla h_1\Big]\varphi + \bar{\kappa}\int_0^T\int_\Omega\Big[h_1\nabla\Delta^{2s+1}h_1\Big]\varphi = 0.
\end{aligned}\tag{3.8}$$

3.2. Estimates.

Proposition 3.2. *Let* (h_1, h_2, u_1) *be a smooth solution of* $(3.1) - (3.3)$. *then the following energy inequality holds*

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega\Big[h_1|u_1|^2 + g|h_1+h_2|^2 + g(\frac{1-r}{r})|h_2|^2 + \frac{1}{2}\beta|\nabla h_1|^2 + \frac{\delta}{\alpha-1}|h_1^{\frac{1-\alpha}{2}}|^2 + \frac{\bar{\kappa}}{2}|\nabla\Delta^s h_1|^2\Big]\\
+\frac{\nu_1}{2}\int_\Omega h_1|\nabla u_1 +^t\nabla u_1|^2 + g\kappa\int_\Omega(1+\frac{h_2}{rh_1})|\nabla(h_1+r^{-1}h_2)|^2 = 0
\end{aligned}\tag{3.9}$$

Proof. First, we multiply the momentum equation (3.2) by u_1 and we integrate on Ω. We use the mass conservation equation for simplification. Then, we obtain

$$\begin{aligned}
\bullet \int_\Omega(\partial_t h_1u_1)u_1 + \int_\Omega\text{div}(h_1u_1\otimes u_1)u_1 &= -\int_\Omega\text{div}(h_1u_1)u_1^2 + \int_\Omega h_1u_1\partial_tu_1 + \int_\Omega(h_1u_1\cdot\nabla)u_1\cdot u_1\\
&= \frac{1}{2}\frac{d}{dt}\int_\Omega h_1|u_1|^2,
\end{aligned}$$

$$\begin{aligned}
\bullet\quad g\int_\Omega(h_1u_1\nabla(h_1+h_2) + g\int_\Omega h_2u_1\nabla(h_1+\frac{1}{r}h_2) &= g\int_\Omega(h_1+h_2)\partial_th_1 - g\int_\Omega(h_1+\frac{1}{r}h_2)\text{div}(h_2u_1)\\
&= \frac{1}{2}g\frac{d}{dt}\int_\Omega h_1^2 + g\int_\Omega h_2\partial_th_1 - g\int_\Omega(h_1+\frac{1}{r}h_2)\text{div}(h_2u_1)
\end{aligned}$$

$$\bullet -\int_\Omega 2\nu_1\text{div}(h_1D(u_1)u_1 = 2\nu_1\int_\Omega h_1D(u_1):\nabla u_1 = \frac{\nu_1}{2}\int_\Omega h_1|\nabla u_1 +^t\nabla u_1|^2$$

$$\bullet -\delta\int_\Omega(h_1\nabla h_1^{-\alpha})u_1 = \frac{\delta}{\alpha-1}\frac{d}{dt}\int_\Omega|h_1^{\frac{1-\alpha}{2}}|^2$$

$$\bullet - \int_\Omega h_1 u_1 \nabla \Delta^{2s+1} h_1 = \int_\Omega \partial_t \Delta^{2s+1} h_1 = \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla \Delta^s h_1|^2$$

$$\bullet \quad \beta \int_\Omega h_1 u_1 \nabla \Delta h_1 = \beta \int_\Omega \partial_t h_1 \Delta h_1 = -\frac{1}{2}\beta \frac{d}{dt}\int_\Omega |\nabla h_1|^2$$

We get the following equality:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega h_1|u_1|^2 + \frac{1}{2}g\frac{d}{dt}\int_\Omega h_1^2 + \frac{\nu_1}{2}\int_\Omega h_1|\nabla u_1 +^t \nabla u_1|^2 + \frac{1}{2}g\frac{d}{dt}\int_\Omega h_1^2$$

$$+g\int_\Omega h_2 \partial_t h_1 + \frac{\delta}{\alpha-1}\frac{d}{dt}\int_\Omega |h_1^{\frac{1-\alpha}{2}}|^2$$

$$+\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla \Delta^s h_1|^2 - \frac{1}{2}\beta\frac{d}{dt}\int_\Omega |\nabla h_1|^2 - g\int_\Omega (h_1 + \frac{1}{r}h_2)\mathrm{div}(h_2 u_1) = 0 \tag{3.10}$$

Now, we multiply the transport equation by $g(h_1 + \frac{1}{r}h_2)$ to have:

$$\frac{1}{2r}\frac{d}{dt}\int_\Omega g h_2^2 + \int_\Omega g h_1 \partial_t h_2 - \int_\Omega h_2 u_1 \nabla (h_1 + r^{-1} h_2)$$

$$+g\kappa \int (1 + \frac{h_2}{r h_1})|\nabla (h_1 + r^{-1}h_2)|^2 = 0. \tag{3.11}$$

To end, we add the equations (3.10) and (3.11) and with a simple calculation, we have the proclamed equality. ■

Corollary 3.3. *For (h_1, h_2, u_1) solution of the system $(3.1) - (3.3)$ the following bound holds:*

$$\sqrt{h_1} u_1$$

$$\textit{is bounded in} \quad L^\infty(0,T;L^2(\Omega)), \quad \sqrt{h_1}|\nabla u_1 +^t \nabla u_1| \quad \textit{is bounded in} \quad L^2(0,T;L^2(\Omega)),$$

$$h_1 \quad \textit{is bounded in} \quad L^\infty(0,T;L^2(\Omega)), \quad h_2 \quad \textit{is bounded in} \quad L^\infty(0,T;L^2(\Omega)),$$

$$\sqrt{1 + h_2/rh_1}|\nabla (h_1 + r^{-1}h_2| \quad \textit{is bounded in} \quad L^2(0,T;L^2(\Omega)),$$

$$\nabla h_1 \quad \textit{is bounded in} \quad L^\infty(0,T;(L^2(\Omega))^2), \quad h_1^{\frac{1-\alpha}{2}} \quad \textit{is bounded in} \quad L^\infty(0,T;L^2(\Omega)),$$

$$\nabla \Delta^s h_1 \quad \textit{is bounded in} \quad L^\infty(0,T;(L^2(\Omega))^3).$$

Proposition 3.4. For (h_1, h_2, u_1) solution of model $(3.1) - (3.3)$, we show the following relation :

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left[h_1|u_1 + 2\nu_1 \nabla \mathrm{log} h_1|^2 + g|h_1 + h_2|^2 + g(r^{-1} - 1)|h_2|^2 + \beta|\nabla h_1|^2 + \frac{2\delta}{\alpha - 1}|h_1^{\frac{1-\alpha}{2}}|^2 \right]$$

$$+2\nu_1 \int_\Omega h_1(A(u_1) : A(u_1)) + \nu_1\beta\int_\Omega |\Delta h_1|^2 + \frac{\bar{\kappa}}{2}\int_\Omega |\nabla \Delta^s h_1|^2 + 2\nu_1\bar{\kappa}\int_\Omega |\Delta^{s+1}h_1|^2$$

$$+\frac{8\nu_1\delta\alpha}{(\alpha-1)^2}\int_\Omega |\nabla h_1^{\frac{1-\alpha}{2}}|^2 + 2\nu_1 g\int_\Omega (1 + h_2/rh_1)|\nabla h_1|^2$$

$$\leq r\nu_1 g \int_\Omega (1 + h_2/rh_1)|\nabla (h_1 + r^{-1}h_2|^2. \tag{3.12}$$

Proof. **Proposition** 3.4
The proof of the **Proposition** 3.4 follows the techniques used in [2, 4, 5, 17, 21].

We consider the mass equation:

$$\partial_t h_1 + \text{div}(h_1 u) = 0.$$

We derive this equation with respect to x, y and we make the sum. We have:

$$\partial_t \nabla h_1 + \text{div}(h_1^t \nabla u_1) + \text{div}(u_1 \otimes \nabla h_1) = 0.$$

By Remplacing ∇h_1 by $h_1 \nabla \log h_1$ and multiply by the viscoity $2\nu_1$,we get:

$$2\nu_1 \partial_t (h_1 \nabla \log h_1) + 2\nu_1 \text{div}(h_1^t \nabla u_1) + 2\nu_1 \text{div}(h_1 u_1 \otimes \nabla \log h_1) = 0.$$

Next, we add this equation to the momentum equation to have:

$$\partial_t [h_1(u_1 + 2\nu_1 \nabla \log h_1)] + \text{div}[h_1 u_1 \otimes (u + 2\nu_1 \nabla \log h_1)] - 2\nu_1 \text{div}(h_1(\mathbf{D}(u_1) - \nabla^t u_1)$$
$$+ g h_1 \nabla(h_1 + h_2) + g h_2 \nabla(h_1 + r^{-1} h_2) = 0.$$

We multiply the above equation by $(u + 2\nu_1 \nabla \log h_1)$ and we integrate the result obtained on Ω. We will now transform each term in the previous equation.
We have :

$$\int_\Omega [\partial_t [h_1(u_1 + 2\nu_1 \nabla \log h_1)] + \text{div}[h_1 u_1 \otimes (u_1 + 2\nu_1 \nabla \log h_1)]](u_1 + 2\nu_1 \nabla \log h_1)$$
$$= \frac{1}{2} \frac{d}{dt} \int_\Omega h_1 |u_1 + 2\nu_1 \nabla \log h_1|^2.$$

Using the definition of the tensor of contraint, we get:

$$-2\nu_1 \int_\Omega \text{div}(h_1(\mathbf{D}(u_1) - \nabla^t u_1)(u + 2\nu_1 \nabla \log h_1) = 2\nu_1 \int_\Omega h_1(\mathbf{A}(u_1) : \mathbf{A}(u_1)),$$

where $A(u_1) = \dfrac{\nabla u_1 -^t \nabla u_1}{2}$.
For the terms pressure, surface tension and friction, we only look at those that do not appear in the **Proposition** 3.2. We modify their expressions essentially using integrations by parts. We have:

- $\frac{1}{2} g \int_\Omega h_1 \nabla(h_1 + h_2)(2\nu_1 \nabla \log h_1) = \nu_1 g \int_\Omega |\nabla h_1|^2 + \nu_1 g \int_\Omega \nabla h_1 \nabla h_2,$
- $g \int_\Omega h_2(\nabla(h_1 + r^{-1} h_2)(2\nu_1 \nabla \log h_1) = 2\nu_1 g \int_\Omega \frac{h_2}{h_1} |\nabla h_1|^2 + \frac{2\nu_1}{r} \int_\Omega \frac{h_2}{h_1} \nabla h_1 \nabla h_2.$

The sum of these two terms gives:

$$\frac{1}{2} g \int_\Omega h_1 \nabla(h_1 + h_2)(2\nu_1 \nabla \log h_1) + g \int_\Omega h_2(\nabla(h_1 + r^{-1} h_2)(2\nu_1 \nabla \log h_1)$$
$$= 2\nu 1 g \int_\Omega (1 + \frac{h_2}{h_1}) |\nabla h_1|^2$$
$$+ 2\nu_1 g \int_\Omega (1 + \frac{h_2}{r h_1}) \nabla h_1 \nabla h_2.$$

Now, we change the tension term as follows:

- $-\beta \int_\Omega h_1 \nabla \Delta h_1 (2\nu_1 \nabla \log h_1) = 2\nu_1 \beta \int_\Omega |\Delta h_1|^2.$

For the cold presure term:

- $, -\delta \int_\Omega h_1 \nabla (h_1^{-\alpha})[2\nu_1 \nabla \log h_1] = \dfrac{8\nu_1\delta\alpha}{(\alpha-1)^2}\int_\Omega |\nabla h_1^{\frac{1-\alpha}{2}}|^2.$

Also we have

- $-\bar{\kappa}\int_\Omega [h_1\nabla\Delta^{2s+1}h_1][2\nu_1\nabla \log h_1] = 2\nu_1\bar{\kappa}\int_\Omega |\Delta^{s+1}h_1|^2.$

By bringing these results together and integrating between 0 and T, we deduce the stated inequality. Which completes the proof. ■

Corollary 3.5. *For* (h_1, h_2, u_1) *solution of the system* $(3.1)-(3.3)$ *the following bound holds:*

$$\nabla\sqrt{h_1} \quad \textit{is bounded in} \quad L^\infty(0,T;L^2(\Omega)), \quad \sqrt{h_1}A(u_1) \quad \textit{is bounded in} \quad L^2(0,T;L^2(\Omega)),$$

$$\Delta h_1 \quad \textit{is bounded in} \quad L^2(0,T;L^2(\Omega)), \quad \nabla h_2 \quad \textit{is bounded in} \quad L^2(0,T;(L^2(\Omega))^2),$$

$$\Delta^{s+1}h_1 \quad \textit{is bounded in} \quad L^2(0,T;L^2(\Omega)), \quad \nabla h_1^{\frac{1-\alpha}{2}} \quad \textit{is bounded in} \quad L^2(0,T;(L^2(\Omega))^2).$$

Proposition 3.6. *If h_1 has the regularities established in **Corollaire** 3.3 and **Corollaire** 3.5, then there exist constants c and $\bar{c}$ dependent on δ, $\bar{\kappa}$ such that*

$$c \le h_1(t,x) \le \bar{c} \tag{3.13}$$

Remark 3.7. This result was first proved in [23] and also used in [11],[16].

Remark 3.8. In this paper, we impose a physical condition that is

$$\frac{h_2}{h_1} \le C, \quad \text{where} \quad C \in [0,1], see[24].$$

It implies that the tickness of the sediment layer is small compared to that of the fluid. Using this physical condition, proposition 3.6 and the results in [16], we can prove the existence of solutions of our model.

Remark 3.9. Sobolev's injections give us thanks to the estimentions of the **Corollary** 3.9 and **corollairy** 3.12 that

$$h_1 \text{ and } u_1 \quad \text{are bounded in} \quad L^\infty(0,T;L^p(\Omega)) \text{ for } p \ge 2. \tag{3.14}$$

Theorem 3.10. *There exists a global weak solutions to system* $(3.1)-(3.3)$ *with initial data* $(3.4)-(3.5)$ *and satisfying the inequalities denined in the **Proposition** 3.2 and **Proposition**3.4.*

3.3. Proof of Theorem 3.10

This section is devoted to the proof of theorem 3.10, in six steps. We can, thanks to the preceding estimations, the convergence of the various terms which intervene in the equation. We exploit the ideas presented in [16].

3.4. Step 1: Convergence of the sequences $(\sqrt{h_{1_n}})_{n\ge1}$, $(h_{1_n})_{n\ge1}$, u_{1_n} and $(h_{2_n})_{n\ge1}$

From the mass equation, we derive:

$$\frac{d}{dt}\int_\Omega \left|\sqrt{h_{1_n}}\right|^2 = -\int_\Omega h_{1_n}\nabla u_{1_n} - \int_\Omega u_{1_n}\nabla h_{1_n},$$

which allows us to have

$$(\sqrt{h_{1_n}})_{n\ge1} \text{ bounded in } L^\infty(0,T;L^2(\Omega)).$$

Corollary 3.5 gives us that $\| \nabla\sqrt{h_{1_n}} \|_{L^{+\infty}}$ $(0,T;(L^2(\Omega))^2) \leq c$, so we obtain:

$$(\sqrt{h_{1_n}})_{n\geq 1} \text{ is bounded in } L^\infty(0,T;H^1(\Omega)). \tag{3.15}$$

We still use the mass equation to have:

$$\partial_t\sqrt{h_{1_n}} = \frac{1}{2}\sqrt{h_{1_n}}\text{div}u_{1_n} - \text{div}(\sqrt{h_{1_n}}u_{1_n}),$$

which gives that $\partial_t\sqrt{h_{1_n}}$ is bounded in $L^2(0,T;H^{-1}(\Omega))$.
Applying Aubin-Simon lemma, we can extract a subsequence, still denoted $(h_{1_n})_{n\geq 1}$, such that

$$\sqrt{h_{1_n}} \text{ converges strongly to } \sqrt{h_1} \text{ in } C^0(0,T;L^2(\Omega)).$$

Thanks to the **Remark** 3.9 and Sobolev embeddings, we know that, for all finite p, $\sqrt{h_{1_n}}$ is bounded in $L^\infty(0,T;L^p(\Omega))$ with $p\geq 4$, and this to ensure that $(h_{1_n})_n$ is in $L^\infty(0,T;L^2(\Omega))$.
Equality $\nabla h_{1_n} = 2\sqrt{h_{1_n}}\nabla\sqrt{h_{1_n}}$ enables us to bound the sequence $(\nabla h_{1_n})_n$ in $L^\infty(0,T;(L^{\frac{2p}{2+P}}(\Omega))^2)$ and consequently, we have:

$$(h_{1_n})_n \text{ is bounded in } L^\infty(0,T;W^{1,\frac{2p}{P+2}}(\Omega)).$$

Let us now look at some properties of the derivative in time of h_{1_n}. The mass equation reads:

$$\partial_t h_n = -\text{div}(h_n u_n) = -\sqrt{h_{1_n}}u_{1_n}\nabla\sqrt{h_{1_n}} - \sqrt{h_{1_n}}\text{div}\sqrt{h_{1_n}}u_{1_n}.$$

So, we get

$$(h_{1_n}u_{1_n})_n \text{ bounded in } L^\infty(0,T;(L^{\frac{2p}{p+2}}(\Omega))^2) \text{ and } (\partial_t h_{1_n})_n \text{ bounded in}$$
$$L^\infty(0,T;W^{-1,\frac{2p}{P+2}}(\Omega))$$

Thanks to Aubin-Simon lemma again, we find:

$$h_{1_n} \longrightarrow h_1 \quad \text{dans} \quad C^0(0,T;L^{\frac{2p}{2+P}}(\Omega)).$$

Last, we consider the bottom term h_{2_n}: with **Corollary** 3.5 and the bound on $(\sqrt{h_{2_n}})_n$ in $L^\infty(0,T;L^2(\Omega))$, we know that the sequence $(\nabla h_{2_n})_n$ is bounded in $L^2(0;T;(L^2(\Omega))^2)$, which gives:

$$(h_{2_n})_n \text{ is bounded in } L^\infty(0,T;H^1(\Omega)).$$

For the time derivative of h_{2_n}, we restart from Equation (3.3). We have:

$$\partial_t h_{2_n} = -\text{div}(h_{2_n}u_{1_n}) + \kappa\nabla\cdot\left[(1+\frac{h_{2_n}}{rh_{1_n}})\nabla(h_{1_n}+\frac{1}{r}h_{2_n})\right]. \tag{3.16}$$

According to the Sobolev embeddings, the first term is in $W^{-1,\frac{2p}{p+2}}(\Omega)$, since h_{2_n} is bounded in $L^2(\Omega)$ and u_{2_n} is bounded in $L^p(\Omega)$. The last term is in $W^{-1,1}(\Omega)$.
We then deduce that

$$\partial_t h_{2_n} \text{ is bounded in } W^{-1,1}(\Omega).$$

Therfore, thanks to the Aubin Simon Lemma, we get

$$h_{2_n} \longrightarrow h_2 \text{ Strongly in } W^{-1,\frac{2p}{p+2}}(\Omega).$$

Now we are interested in the velocity u_{1_n}. Thanks to the **Corollary** 3.3, **Corollary** 3.5 and the **Remark** 3.9 we have

$$u_{1_n} \text{ is bounded in } L^\infty(0,T;H^1(\Omega)).$$

Also we have $\partial_t u_{1_n} = \dfrac{1}{h_{1_n}}\partial_t(h_{1_n}u_{1_n}) + u_{1_n}\nabla u_{1_n} + u_{1_n}^2\dfrac{\nabla h_{1_n}}{h_{1_n}}$, thanks to the **Proposition**3.6 and the **Remark** 3.9, we have

$$\partial_t u_{1_n} \text{ is bounded in } W^{-1,1}(\Omega).$$

The Aubin Simon **Lemma** ensures that

$$u_{1_n} \longrightarrow u_1 \text{ Strongly in } \mathcal{C}^0(0,T;W^{-1,1}(\Omega)).$$

3.5. Step 2: Convergence of the sequences $\frac{h_{2_n}}{h_{1_n}}$ and $(1+\dfrac{h_{2_n}}{rh_{1_n}})\nabla(h_{1_n}+\frac{1}{r}h_{2_n})$

We have

$$\left|\frac{h_{2_n}}{h_{1_n}} - \frac{h_2}{h_1}\right|^2 = \left|\frac{h_{2_n}h_1 - h_2h_1 + h_2h_1 - h_2h_{1_n}}{h_{1_n}h_1}\right|^2 \le \mathbf{K}|h_{2_n}-h_2|^2 + |h_{1_n}-h_1|^2$$

thanks to the **Proposition** 3.6. According to the **Step 1**, we have

$$\left|\frac{h_{2_n}}{h_{1_n}} - \frac{h_2}{h_1}\right|^2 \to 0, \quad \text{then} \quad \frac{h_{2_n}}{h_{1_n}} \longrightarrow \frac{h_2}{h_1} \quad \text{strongly in } L^2(0,T;L^2(\Omega))$$

consequently,

$$(1+\frac{h_{2_n}}{rh_{1_n}})\nabla(h_{1_n}+\frac{1}{r}h_{2_n}) \longrightarrow (1+\frac{h_2}{rh_1})\nabla(h_1+\frac{1}{r}h_2) \text{ weakly in } L^1(0,T;(L^1(\Omega)).$$

3.6. Step 3: Weak convergences of $h_{1_n}\nabla\Delta^{2s+1}h_{1_n}$ and $h_{1_n}\nabla\left[h_{1_n}^{-\alpha}\right]$

Concerning the two terms, we have

$$h_{1_n}\nabla\Delta^{2s+1}h_{1_n} \text{bounded in}$$
$$L^2(0,T;W^{-1,1}(\Omega)) \text{ and } h_{1_n}\nabla\left[h_{1_n}^{-\alpha}\right] \text{ bounded in } L^2(0,T;L^{\frac{2p}{p+2}}(\Omega))$$

So, we have

$$h_{1_n}\nabla\Delta^{2s+1}h_{1_n} \text{ converges weakly to } h_1\nabla\Delta^{2s+1}h_1 \text{ in } L^2(0,T;W^{-1,1}(\Omega)),$$

and

$$h_{1_n}\nabla\left[h_{1_n}^{-\alpha}\right] \text{converges weakly to } h_1\nabla\left[h_1^{-\alpha}\right] \text{ in } L^2(0,T;L^{\frac{2p}{p+2}}(\Omega)).$$

3.7. Step 4: Convergence of ∇h_{1_n} and Δh_{1_n}

As Δh_{1_n} and ∇h_{1_n} are bounded respectively in $L^2(0,T;L^2(\Omega))$ and $L^\infty(0,T;(L^2(\Omega))$, so we have:

$$\nabla h_{1_n} \text{ bounded in } L^2(0,T;H^1(\Omega)).$$

Using the mass equation, one has $\partial_t\nabla h_{1_n} = -\nabla \mathrm{div} h_{1_n}u_{1_n}$, as $h_{1_n}u_{1_n}$ is bounded in $L^2(0,T;L^2(\Omega))$, we have

$$\partial_t\nabla h_{1_n} \text{ is bounded in } L^2(0,T;H^{-2}(\Omega)).$$

Then, applying Aubin-Simon Lemma, it follows that

$$\nabla h_{1_n} \longrightarrow \nabla h_1 \text{ strongly in } L^2(0,T;(L^q(\Omega))^2), \quad q\in[1,+\infty[.$$

But as we have shown that ∇h_{1_n} is bounded in $L^\infty(0,T;(L^2(\Omega))$, hence

$$\nabla h_{1_n} \longrightarrow \nabla h_1 \text{ strongly in } L^2(0,T;(L^2(\Omega))^2).$$

Thanks to the **Corrolary** 3.5, we have finally

$$\Delta h_{1_n} \longrightarrow \Delta h_1 \text{ weakly in } L^2(0,T;L^2(\Omega)).$$

3.8. Step 5: Convergence of $(h_{1_n}u_{1_n})_{n\geq 1}$

In the previous part, we proved that the sequence $(h_{1_n}u_{1_n})_n$ is bounded in $L^\infty(0,T;(L^{\frac{2p}{p+2}}(\Omega))^2)$ where p is an integer greater than four.Writing the gradient as follows:

$$\nabla(h_{1_n}u_{1_n}) = 2\sqrt{h_{1_n}}u_{1_n}\nabla\sqrt{h_{1_n}} + \sqrt{h_{1_n}}\sqrt{h_{1_n}}\nabla u_{1_n},$$

since the first term is in $L^\infty(0,T;L^1(\Omega))$ and the second one belongs to $L^2(0,T;L^{\frac{2p}{p+2}}(\Omega))$, we have: $(h_{1_n}u_{1_n})_n$ bounded in $L^2(0,T;W^{1,1}(\Omega))$.
Moreover, the momentum equation (3.2) enables us to write the time derivative of the water discharge:

$$\partial_t(h_{1_n}u_{1_n}) = -\text{div}(h_{1_n}u_{1_n}\otimes u_{1_n}) - gh_{1_n}\nabla(h_{1_n}+h_{2_n}) + 2\nu_1\text{div}(h_{1_n}D(u_{1_n}))$$

$$-gh_{2_n}\nabla(h_{1_n}+\frac{h_{2_n}}{r}) + \beta\nabla\Delta h_{1_n} - \delta h_{1_n}\nabla\left[h_{1_n}^{-\alpha}\right] - \bar{\kappa}h_{1_n}\nabla\Delta^{2s+1}h_{1_n}$$

We then study each term:
- $\text{div}(h_{1_n}u_{1_n}\otimes u_{1_n}) = \text{div}(\sqrt{h_{1_n}}u_{1_n}\otimes\sqrt{h_{1_n}}u_{1_n})$ which is in $L^\infty(0,T;W^{-1,1}(\Omega))$,

- as h_{1_n} is bounded in $L^\infty(0,T;L^p(\Omega))$ and $\nabla(h_{1_n}+h_{2_n})$ is in $L^2(0,T;L^2(\Omega))$, the we have: $h_{1_n}\nabla(h_{1_n}+h_{2_n})$ bounded in $L^2(0,T;L^{\frac{2p}{p+2}}(\Omega))$

- remark that

$$h_{1_n}\nabla u_{1_n} = \nabla(h_{1_n}u_{1_n}) - u_{1_n}\otimes\nabla h_{1_n} = \nabla(\sqrt{h_{1_n}}\sqrt{h_{1_n}}u_{1_n}) - 2\sqrt{h_{1_n}}u_{1_n}\nabla\sqrt{h_{1_n}}, \tag{3.17}$$

we know that the first term is in $L^\infty(0,T;W^{-1,\frac{2p}{p+2}}(\Omega))$ and the second one in $L^\infty(0,T;(L^1\Omega))$. So we have $h_nD(u_n)$ bounded in $L^2(0,T;W^{-1,\frac{2p}{p+2}}(\Omega))$.

- Also, h_{2_n} is bounded in $L^\infty(0,T;L^2(\Omega))$ and $\nabla(h_{1_n}+\frac{h_{2n}}{r})$ is bounded in $L^2(0,T;L^2(\Omega))$, therefore $h_{2_n}\nabla(h_{1_n}+\frac{h_{2n}}{r})$ is bounded in $L^2(0,T;L^1(\Omega))$.

- We have Δh_{1_n} is bounded in $L^2(0,T;L^2(\Omega))$, so that $h_{1_n}\nabla\Delta h_{1_n}$ is bounded in $L^2(0,T;W^{-1,1}(\Omega))$.

- One knowns that $\nabla\Delta^s h_{1_n}$ is bounded in $L^\infty(0,T;L^2(\Omega))$ and $\Delta^{s+1}h_{1_n}$ is bounded in $L^2(0,T;L^2(\Omega))$. Thus $h_{1_n}\nabla\Delta^{2s+1}h_{1_n}$ is bounded in
$L^2(0,T;L^1(\Omega))\subset L^2(0,T;W^{-1,1}(\Omega))$.

- Thanks to the **Proposition** 3.6, h_{1_n} is bounded in $L^\infty(0,T;L^\infty))$, hence $h_{1_n}\nabla\left[h_{1_n}^{-\alpha}\right]$ is bounded in $L^2(0,T;W^{-1,1}(\Omega))$.

Finally, note that this terms are included in $L^2(0,T;W^{-1,1}(\Omega))$, which means that $\partial_t(h_{1_n}u_{1_n})$ is also in this space. Then, applying Aubin-Simon lemma, we obtain:

$$(h_{1_n}u_{1_n})_n\text{strongly converges to } h_1u_1 \text{ in } C^0(0,T;W^{-1,1}(\Omega)).$$

3.9. Step 6: Convergence of $(\sqrt{h_{1_n}}u_{1_n})_{n\geq 1}$.

As we have $\mathbf{m}_n = h_{1_n}u_{1_n}$, so,we have $\sqrt{h_{1_n}u_{1_n}} = \dfrac{\mathbf{m}_n}{\sqrt{h_{1_n}}}$

We will show the convergence of this term. We know that $\dfrac{\mathbf{m}_n}{\sqrt{h_{1_n}}}$ is bounded in $L^\infty(0,T;L^2(\Omega))$. Consequently Fatou lemma reads:

$$\int_\Omega \liminf \frac{\mathbf{m}_n}{h_{1_n}} \leq \liminf \int_\Omega \frac{\mathbf{m}_n^2}{h_{1_n}} < +\infty$$

Then, we can define the limit velocity taking $u_{1_n}(t,x) = \frac{\mathbf{m}_n(t,x)}{h(t,x)}$ ($h_{1_n}(t,x) \neq 0$). So we have a link between the limits $\mathbf{m}_n(t,x) = h_{1_n}(t,x)u_{1_n}(t,x)$ and:

$$\int_\Omega \frac{\mathbf{m}_n^2}{h_{1_n}} = \int_\Omega h_{1_n}|u_{1_n}|^2 < +\infty =$$

Thanks to the **Remark**3.9, we have:$\sqrt{h_{1_n}}|u|^2$ in $L^2(0,T;L^2(\Omega))$.
As $(\mathbf{m}_n)_n$ and $(h_{1_n})_n$ converge , the sequence of $\sqrt{h_{1_n}}u_{1_n}$ converges to $\sqrt{h_1}u_1$.
Moreover, for all M positive, $(\sqrt{h_{1_n}}u_{1_n}1_{|u_{1_n}|\leq M})_n$ converges to $\sqrt{h_1}u1_{|u_1|\leq M}$.
Finally, let us consider the following norm:

$$\int_\Omega \left|\sqrt{h_{1_n}}u_{1_n} - \sqrt{h_1}u_1\right|^2 \leq \int_\Omega \left(\left|\sqrt{h_{1_n}}u_{1_n}1_{|u_{1_n}|\leq M} - \sqrt{h_1}u1_{|u_1|\leq M}\right|\right.$$

$$\left.+\left|\sqrt{h_{1_n}}u_{1_n}1_{|u_{1_n}|>M}\right| + \left|\sqrt{h}u_1 1_{|u_1|>M}\right|\right)^2 \leq$$

$$3\int_\Omega \left|\sqrt{h_{1_n}}u_{1_n}1_{|u_{1_n}|\leq M} - \sqrt{h_1}u_1 1_{|u_1|\leq M}\right|^2 + 3\int_\Omega \left|\sqrt{h_{1_n}}u_{1_n}1_{|u_{1_n}|>M}\right|^2$$

$$+3\int_\Omega \left|\sqrt{h_1}u_1 1_{|u_1|>M}\right|^2.$$

Since $(\sqrt{h_n}u_n)_n$ is in $L^\infty(0,T;L^p(\Omega))$, $(\sqrt{h_{1_n}}u_{1_n}1_{|u_{1_n}|\leq M})_n$ is bounded in this space. So, as we have seen previously, the first integral tends to zero. Let us study the other two terms:

$$\int_\Omega \left|\sqrt{h_{1_n}}u_{1_n}1_{|u_{1_n}|>M}\right|^2 \leq \frac{1}{M^2}\int_\Omega h_{1_n}|u_{1_n}|^4 \leq \frac{k}{M^2} \text{ and}$$

$$\int_\Omega \left|\sqrt{h_1}u_1 1_{|u_1|>M}\right|^2 \leq \frac{1}{M^2}\int_\Omega h_1|u_1|^4 \leq \frac{k'}{M^2}$$

for all $M > 0$. When M tends to the infinity, our two integrals tend to zero.
Then

$$(\sqrt{h_{1_n}}u_{1_n})_n \text{converges strongly to } \sqrt{h_1}u_1 \text{ in } L^2(0,T;L^2(\Omega)).$$

This ends the proof of Theorem 3.10.

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References

[1] Y. BRENIER, Homogeneous hydrostatic flows with convex velocity profiles. *Nonlinearity.*, **12(3)**(1999), 495–512, .

[2] D. BRESCH AND B. DESJARDINS, Existence of global weak solution for 2D viscous shallow water equations and convergence to the quasi-geostrophic model, *Comm. Math. Phys.*, **238(1-3)**(2003), 211–223.

[3] D. BRESCH AND B. DESJARDINS, On the construction of approximate solutions for the 2D viscous shallow water model and for compressible Navier-Stokes models, *J. Math. Pures Appl.*, **86**(2006), 362–368.

[4] D. BRESCH, B. DESJARDINS AND D. GÉRARD-VARET, On compressible Navier-Stokes equations with density dependent viscosities in bounded domains, *J. Math. Pures Appl.*, **87(9)**(2007), 227–235.

[5] D.BRESCH D, B. DESJARDINS B AND C.K. LIN, On some compressible fluid models : Korteweg, lubrication and shallow water systems, *Communications in partial differential equations*, **28(3,4)**(2003), 843–868.

[6] E. D. FERNANDEZ-NIETO, G. NARBONA-REINA AND J. D. ZABSONRÉ, Formal derivation of a bilayer model coupling shallow water and Reynolds lubrication equations: evolution of a thin pollutant layer over water, *Euro. Jnl of Applied Mathematics*, **24(6)**(2013), 803–833.

[7] E. D. FERNANDEZ-NIETO, T. MORALES DE LUNA, G. NARBONA-REINA AND J. D. D. ZABSONRÉ, Formal derivation of the saint-venant-exner including arbitrarily sloping sediment beds and associated energy, *Mathematical Modelling and Numerical Analysis*, **51**(2017), 115–145.

[8] F. GERBEAU AND B. PERTHAME, Derivation of viscous Saint-Venant system for laminar shallow-water: Numerical validation, *Discrete and Continuous Dynamical Systems-B*, **1(1)**(2001), 89–102.

[9] E. GRENIER. ON THE DERIVATION OF HOMOGENEOUS HYDROSTATIC EQUATIONS, *ESIAM :Math. Model.Numer. Anal.*, **33(5)**(1999), 965–970.

[10] P.-L.LIONS Mathematical Topics in Fluid Mechanics. Vol.1 :incompressible models, *Oxford University Press, Oxford*, (1996).

[11] G. KITAVTSEV, P. LAUREN¸COT AND B. NIETHAMMER, Weak solutions to lubrication equations in the presence of strong slippage, *Methods and Applications of Analysis*, **18**(2011),183–202.

[12] F. MARCHE, Derivation of a new two-dimensional viscous shallow water model with varying topography, bottom and capillary effects, *European J. Mech. B/Fluids*, **26(1)**(2007), 49–63.

[13] F. MARCHE, Theoretical and Numerical Study of Shallow Water Models. Applications to Nearshore Hydrodynamics. *PhD Thesis, University of Bordeaux*.

[14] A. MELLET, A. VASSEUR, On the barotropic compressible Navier-Stokes equations *Comm. Partial Differential Equations*, **32(3)**(2007), 431–452.

[15] G. NARBONA-REINA, J. D. D. ZABSONRÉ, E.D. FERNÁNDEZ-NIETO AND D. BRESCH, Derivation of bilayer model for Shalow Water equations with viscosity. Numerical validation, *CMES*, **43(1)**(2009), 27–71, .

[16] B. ROAMBA, J. D. D; ZABSONRÉ AND S. TRAORÉ, Formal derivation and existence of global weak solutions of a two-dimensional bilayer model coupling shallow water and Reynolds lubrication equations, **Asymptotic Analysis**, **99**(2016), 207–239.

[17] B. ROAMBA, J. D. D ZABSONRÉ AND Y. ZONGO, Weak solutions to pollutant transport model in a dimensional case, *Annals of the University of Craiova, Mathematics and Computer Science Series*, **44(1)**(2027), 137–148.

[18] J. SIMON Compact set in the space $L^p(0;t;B)$, *Ann. Mat. Pura appl.*, **146(4)**1987, 65–96.

[19] B. TOUMBOU, D. LE ROUX AND A. SENE, An existence theorem for a 2-D coupled sedimentation shallow-water model,*C. R. Math. Acad. Sci. Paris*, **344**(2007), 443-446.

[20] J.-L. LIONS, Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires, *Dunod*, 1969.

[21] J. D. D; ZABSONRÉ, C. LUCAS AND E. FERNÀNDEZ-NIETO, An energetically consistent viscous sedimentation model, *Math. Models Methods Appl. Sci.*, **19(3)**(2009), 477–499.

[22] J.D.D. ZABSONRÉ, Modèle visqueux en sédimentation et stratification. Obtension formelle, stabilité théorique et schémes volumes finis bien équilibrés. *Thesis of the University of Savoie*, (2008), (France).

[23] E. ZATORSKA, Fundamental problems to equations of compressible chemically reacting flows, *Phd Thesis, university of Warsamw*, (2013), (Poland).

[24] ZONGO YACOUBA, ROAMBA BRAHIMA, YIRA BOULAYE AND JEAN DE DIEU ZABSONRÉ, On the existence of global weak solutions of a 2D sediment transport model, *Nonautonomous Dynamical Systems*, **9(1)**(2022), 182–204.