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On semi-commuting pair of automorphism of rings

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Abstract

The concept of commuting pair of automorphism of Rings is generalised as semi commuting couple of automorphism of rings and more general outcomes are attained.

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Automorphism, Rings.

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Contents

1. Introduction

Consider $\mathfrak{R}$ as an associative ring. Then the automorphism $\mathbb{J}$ is said to be a commuting automorphism if $\mathbb{J}(\varphi_1)\varphi_1 = \varphi_1\mathbb{J}(\varphi_1)\ \forall\varphi_1 \in \mathfrak{R}$. In [1], Divinsky proved that a if the semisimpleartirian ring possesses a non-trivial commutative automorphism, next it should be commutative. Later, this result was extended by Luh [2] proving that whenever a prime $\mathfrak{R}$ possesses a commutative non-trivial automorphism next it must be a commutative integral domain.

Further the generalization of this result is provided in Mayne [3] by showing that, if $\mathfrak{R}$ is a prime possesses a non-trivial automorphism $\mathbb{J}$ such that, $\forall\varphi_1 \in \mathfrak{R}$, $\mathbb{J}(\varphi_1)\varphi_1 - \varphi_1\mathbb{J}(\varphi_1)$ is in the center of $\mathfrak{R}$ next the prime ring $\mathfrak{R}$ should be commutative.

In [4], an automorphism $\mathbb{J}$ is stated as a semi-commuting automorphism if $\forall\varphi_1 \in \mathfrak{R}$, we have $\mathbb{J}(\varphi_1)\varphi_1 = \pm\ \varphi_1\mathbb{J}(\varphi_1)$. It is also proved that if a prime ring with characteristic $\neq 2, 3$ possesses a semi commuting non-trivial automorphism, then it should be a commutative integral domain.

This work we generalize the concept of semi-commuting automorphism as semi-commuting couple of automorphism and more general outcomes are to be derived.

2. Preliminaries

In this section, we recall some basic definitions and results which will be utilized throughout this paper.

Definition 2.1. *Consider $\mathfrak{R}$ denotes associative ring & $\mathbb{J}$ denotes an automorphism.*

- when $\rho.\mathbb{J}(\rho) = \mathbb{J}(\rho).\rho$, $\forall\rho \in \mathfrak{R}$, next $\mathbb{J}$ is stated that a commuting automorphism
- when $\rho.\mathbb{J}(\rho) = -\mathbb{J}(\rho).\rho$, $\forall\rho \in \mathfrak{R}$, next $\mathbb{J}$ is stated that an anti commuting automorphism
- if either $\rho.\mathbb{J}(\rho) = \mathbb{J}(\rho).\rho$ (or) $\rho.\mathbb{J}(\rho) = -\mathbb{J}(\rho).\rho$, $\forall\rho \in \mathfrak{R}$, then $\mathbb{J}$ is stated that a semi commuting automorphism
- if $[\rho,\mathbb{J}(\tau)] = [\mathbb{J}(\rho),\tau]$, $\forall\rho,\tau \in \mathfrak{R}$, then $\mathbb{J}$ is stated that a strong commuting automorphism
- if $[\omega,\mathbb{J}(\tau)] = \pm[\mathbb{J}(\omega),\tau]$, $\forall\omega,\tau \in \mathfrak{R}$, then $\mathbb{J}$ is stated that a strong semi commuting automorphism
- if $[\varphi_1,\mathbb{J}(\varphi_1)] \in \mathscr{Z}$, $\forall\varphi_1 \in \mathfrak{R}$, then $\mathbb{J}$ is stated that a centralising automorphism
- if $\varphi_1\mathbb{J}(\varphi_1) + \mathbb{J}(\varphi_1)\varphi_1 \in \mathscr{Z}$, $\forall\varphi_1 \in \mathfrak{R}$, then $\mathbb{J}$ is stated that an anti-centralising automorphism
- if either $\varphi_1\mathbb{J}(\varphi_1) - \mathbb{J}(\varphi_1)\varphi_1 \in \mathscr{Z}$ (or) $\varphi_1\mathbb{J}(\varphi_1) + \mathbb{J}(\varphi_1)\varphi_1 \in \mathscr{Z}$, $\forall\varphi_1 \in \mathfrak{R}$, then $\mathbb{J}$ is stated that a semi centralising automorphism

- if $[\omega, \mathbb{J}(\tau)] - [\mathbb{J}(\omega), \tau] \in \mathscr{Z}, \forall \omega, \tau \in \mathfrak{R}$, then $\mathbb{J}$ is stated that a strong centralising automorphism
- if $[\varphi_1, v\mathbb{J}(\varphi_2)] \pm [\mathbb{J}(\varphi_1), v\varphi_2] \in \mathscr{Z}, \forall \varphi_1, \varphi_2 \in \mathfrak{R}$, then $\mathbb{J}$ is stated that a strong semi centralising automorphism

We generalize all the above definitions as follows

Definition 2.2. *Consider $\mathfrak{R}$ denotes associative ring. Assume $\mathbb{H}$ & $\mathbb{J}$ be a couple of non trivial automorphisms.*

- if $\mathbb{H}(\varphi_1)\mathbb{J}(\varphi_1) = \mathbb{J}(\varphi_1)\mathbb{H}(\varphi_1), \forall \varphi_1 \in \mathfrak{R}$ i.e, $[\mathbb{H}(\varphi_1), \mathbb{J}(\varphi_1)] = 0, \forall \varphi_1 \in \mathfrak{R}$, then $\mathbb{H}$ and $\mathbb{J}$ are known as a commuting couple of automorphisms
- if $\mathbb{H}(\rho)\mathbb{J}(\rho) = -\mathbb{J}(\rho)\mathbb{H}(\rho), \forall \rho \in \mathfrak{R}$, then $\mathbb{H}$ and $\mathbb{J}$ are known as an anti commuting couple of automorphisms.
- if either $\mathbb{H}(\rho)\mathbb{J}(\rho) = \mathbb{J}(\rho)\mathbb{H}(\rho)$ (or) $\mathbb{H}(\rho)\mathbb{J}(\rho) = -\mathbb{J}(\rho)\mathbb{H}(\rho), \forall \rho \in \mathfrak{R}$ holds, then $\mathbb{H}$ and $\mathbb{J}$ are known as a semi commuting couple of automorphisms.
- if $[\mathbb{H}(\rho), \mathbb{J}(\tau)] = [\mathbb{J}(\rho), \mathbb{H}(\tau)], \forall \rho, \tau \in \mathfrak{R}$, then $\mathbb{H}$ and $\mathbb{J}$ are known as a strong commuting couple of automorphisms.
- if $[\mathbb{H}(\omega), \mathbb{J}(\tau)] = \pm[\mathbb{J}(\omega), \mathbb{H}(\tau)], \forall \omega, \tau \in \mathfrak{R}$, then $\mathbb{H}$ and $\mathbb{J}$ are known as a strong semi commuting couple of automorphisms.
- if $[\mathbb{H}(\varphi_1), \mathbb{J}(\varphi_1)] \in \mathscr{Z}, \forall \varphi_1 \in \mathfrak{R}$

  i.e, $\mathbb{H}(\varphi_1)\mathbb{J}(\varphi_1) - \mathbb{J}(\varphi_1)\mathbb{H}(\varphi_1) \in \mathscr{Z}, \forall \varphi_1 \in \mathfrak{R}$,

then $\mathbb{H}$ and $\mathbb{J}$ are known as a centralizing couple of automorphisms.

- if $\mathbb{H}(\rho)\mathbb{J}(\rho) + \mathbb{J}(\rho)\mathbb{H}(\rho) \in \mathscr{Z}, \forall \rho \in \mathfrak{R}$, then $\mathbb{H}$ and $\mathbb{J}$ are known as an anti centralizing couple of automorphisms.
- if either $\mathbb{H}(\omega)\mathbb{J}(\omega) - \mathbb{J}(\omega)\mathbb{H}(\omega) \in \mathscr{Z}, \forall \omega \in \mathfrak{R}$ (or) $\mathbb{H}(\omega)\mathbb{J}(\omega) + \mathbb{J}(\omega)\mathbb{H}(\omega) \in \mathscr{Z}, \forall \omega \in \mathfrak{R}$ holds, then $\mathbb{H}$ and $\mathbb{J}$ are known as a semi centralizing couple of automorphisms
- if $[\mathbb{H}(\omega), \mathbb{J}(\tau)] - [\mathbb{J}(\omega), \mathbb{H}(\tau)] \in \mathscr{Z}, \forall \omega, \tau \in \mathfrak{R}$, then $\mathbb{H}$ and $\mathbb{J}$ are known as a strong centralizing couple of automorphisms
- if $[\mathbb{H}(\omega), \mathbb{J}(\tau)] \pm [\mathbb{J}(\omega), \mathbb{H}(\tau)] \in \mathscr{Z}, \forall \omega, \tau \in \mathfrak{R}$, then $\mathbb{H}$ and $\mathbb{J}$ are known as a strong semi centralizing couple of automorphisms.

Definition 2.3. *Consider $\mathfrak{R}$ denotes associative ring. Let $\mathbb{H}$ & $\mathbb{J}$ be a couple of non trivial automorphisms. We set*

$$\begin{aligned}
\Omega_+ &= \{\varphi_1 \in \mathfrak{R} / \mathbb{H}(\varphi_1)\mathbb{J}(\varphi_1) = \mathbb{J}(\varphi_1)\mathbb{H}(\varphi_1)\} \\
\Omega_- &= \{\varphi_1 \in \mathfrak{R} / \mathbb{H}(\varphi_1)\mathbb{J}(\varphi_1) = -\mathbb{J}(\varphi_1)\mathbb{H}(\varphi_1)\} \\
\Omega_0 &= \{\varphi_1 \in \mathfrak{R} / \ \mathbb{H}(\varphi_1)\mathbb{J}(\varphi_1) = 0\}
\end{aligned}$$

Remark 2.4. *Assume $\mathbb{H}$ and $\mathbb{J}$ are semi-commuting couple of automorphisms, then*

- $\Omega_0 = \Omega_+ \cap \Omega_-$
- $\mathfrak{R} = \Omega_+ \cup \Omega_-$
- $\varphi_1 \in \Omega_0 \implies m\varphi_1 \in \Omega_0$ for any integer m.
- $\varphi_1 \in \Omega_+ \implies m\varphi_1 \in \Omega_+$ for any integer m.
- $\varphi_1 \in \Omega_- \implies m\varphi_1 \in \Omega_-$ for any integer m.

Remark 2.5. *Consider $\mathfrak{R}$ denotes a prime ring of characteristic 2. $\mathbb{H}$ and $\mathbb{J}$ are semi commuting couple of automorphisms. Then $\Omega_+ = \Omega_- = \mathfrak{R}$*

For, $\mathfrak{R}$ is a characteristics 2. Implies $\omega = -\omega, \forall \omega \in \mathfrak{R}$.

Now Ω iff $\mathbb{H}(\omega)\mathbb{J}(\omega) = \mathbb{J}(\omega)\mathbb{H}(\omega)$
iff $\mathbb{H}(\omega)\mathbb{J}(\omega) = -\mathbb{J}(\omega)\mathbb{H}(\omega)$
iff $\omega \in \Omega_-$
i.e, $\Omega_+ = \Omega_-$.

Since, $\mathbb{H}$ and $\mathbb{J}$ are semi commuting couple of automorphisms, $\mathfrak{R} = \Omega_+ \cup \Omega_-$.

Hence, $\mathfrak{R} = \Omega_+ = \Omega_-$.

Remark 2.6. *Consider $\mathfrak{R}$ denotes any ring. Here*

- $[\omega, \varphi + \mu] = [\omega, \varphi] + [\omega, \mu], \forall \omega, \varphi, \mu \in \mathfrak{R}$
- $[\omega + \varphi, \mu] = [\omega, \mu] + [\varphi, \mu], \forall \omega, \varphi, \mu \in \mathfrak{R}$
- $[\omega, \varphi] = -[\varphi, \omega], \forall \omega, \varphi \in \mathfrak{R}$
- $[\omega\varphi\mu] = \omega[\varphi, \mu] + [\omega, \mu]\varphi, \forall \omega, \varphi, \mu \in \mathfrak{R}$
- $[\omega, \varphi] = 0 \; if \; \omega = \varphi$.

3. Main Results

Lemma 3.1. *Consider $\mathfrak{R}$ denotes a prime ring acquiring a commuting non-trivial couple of automorphisms. Then, commutative integral domain $\mathfrak{R}$.*

Proof. Consider $\mathbb{H}$ and $\mathbb{J}$ refers commuting couple of automorphisms $\ni$: $\mathbb{H} \neq \mathbb{J}$. Then

$$[\mathbb{H}(\varphi_1), \mathbb{J}(\varphi_1)] = 0, \forall \varphi_1 \in \mathfrak{R} \tag{3.1}$$

Replace φ_1 by $\varphi_1 + \varphi_2$ in (3.1), one has

$$[\mathbb{H}(\varphi_1 + \varphi_2), \mathbb{J}(\varphi_1 + \varphi_2)] = 0, \forall \varphi_1, \varphi_2 \in \mathfrak{R}$$

i.e, $[\mathbb{H}(\varphi_1), \mathbb{J}(\varphi_1)] + [\mathbb{H}(\varphi_1), \mathbb{J}(\varphi_2)] + [\mathbb{H}(\varphi_2), \mathbb{J}(\varphi_1)] + [\mathbb{H}(\varphi_2), \mathbb{J}(\varphi_2)] = 0$.

By applying (3.1), we have

$$[\mathbb{H}(\varphi_1), \mathbb{J}(\varphi_2)] + [\mathbb{H}(\varphi_2), \mathbb{J}(\varphi_1)] = 0.$$

For all $\varphi_1, \varphi_2 \in \mathfrak{R}$, we have

$$[\mathbb{H}(\varphi_1), \mathbb{J}(\varphi_2)] = -[\mathbb{H}(\varphi_2), \mathbb{J}(\varphi_1)] = [\mathbb{J}(\varphi_1), \mathbb{H}(\varphi_2)]. \tag{3.2}$$

Replace φ_2 by $\varphi_1\varphi_2$ in (3.2), one has
$[\mathbb{H}(\varphi_1),\mathbb{J}(\varphi_2)] = [\mathbb{J}(\varphi_1),\mathbb{H}(\varphi_1\varphi_2)], \forall \varphi_1,\varphi_2 \in \mathfrak{R}$.
i,e, $\mathbb{J}(\varphi_1)[\mathbb{H}(\varphi_1),\mathbb{J}(\varphi_2)] + [\mathbb{H}(\varphi_1),\mathbb{J}(\varphi_1)]\mathbb{J}(\varphi_2) = \mathbb{H}(\varphi_1)[\mathbb{J}(\varphi_1),\mathbb{H}(\varphi_2)] + [\mathbb{J}(\varphi_1),\mathbb{H}(\varphi_1)]\mathbb{H}(\varphi_2)$.
By applying (3.1), we have
$\mathbb{J}(\varphi_1)[\mathbb{H}(\varphi_1),\mathbb{J}(\varphi_2)] = \mathbb{H}(\varphi_1)[\mathbb{J}(\varphi_1),\mathbb{H}(\varphi_2)], \forall \varphi_1,\varphi_2 \in \mathfrak{R}$.
By applying (3.2), one has

$$\begin{aligned} \mathbb{J}(\varphi_1)[\mathbb{H}(\varphi_1),\mathbb{J}(\varphi_2)] &= \mathbb{H}(\varphi_1)[\mathbb{H}(\varphi_1),\mathbb{J}(\varphi_2)], \\ (\mathbb{H}(\varphi_1)-\mathbb{J}(\varphi_1))[\mathbb{H}(\varphi_1),\mathbb{J}(\varphi_2)] &= 0. \end{aligned} \tag{3.3}$$

Since $\mathbb{J}$ is an automorphism, we have

$$(\mathbb{H}(\varphi_1)-\mathbb{J}(\varphi_1))[\mathbb{H}(\varphi_1),\varphi_3] = 0, \forall \varphi_1,\varphi_3 \in \mathfrak{R}. \tag{3.4}$$

Now,

$$\begin{aligned} &\varphi_2[\mathbb{H}(\varphi_1),\varphi_3] \\ &= [\mathbb{H}(\varphi_1),\varphi_2\varphi_3] - [\mathbb{H}(\varphi_1),\varphi_2]\varphi_3 \\ &(\mathbb{H}(\varphi_1)-\mathbb{J}(\varphi_1))\varphi_2[\mathbb{H}(\varphi_1),\varphi_3] \\ &= (\mathbb{H}(\varphi_1)-\mathbb{J}(\varphi_1))[\mathbb{H}(\varphi_1),\varphi_2\varphi_3] \\ &-(\mathbb{H}(\varphi_1)-\mathbb{J}(\varphi_1))[\mathbb{H}(\varphi_1),\varphi_2]\varphi_3. \end{aligned}$$

By applying (3.4), one has

$$(\mathbb{H}(\varphi_1)-\mathbb{J}(\varphi_1))\varphi_2[\mathbb{H}(\varphi_1),\varphi_3] = 0, \forall \varphi_1,\varphi_2,\varphi_3 \in \mathfrak{R}.$$

$$(\mathbb{H}(\varphi_1)-\mathbb{J}(\varphi_1))\mathfrak{R}[\mathbb{H}(\varphi_1),c] = 0, \forall \varphi_1,\varphi_3 \in \mathfrak{R}. \tag{3.5}$$

Therefore $\mathbb{H} \neq \mathbb{J}$, must exist at least one $\varphi_{10} \in \mathfrak{R}$ such that $\mathbb{H}(\varphi_{10}) \neq \mathbb{J}(\varphi_{10})$.
Therefore $\mathfrak{R}$ be a prime, $[\mathbb{H}(\varphi_{10}),\varphi_3] = 0, \forall \varphi_3 \in \mathfrak{R}$
i.e, $\mathbb{H}(\varphi_{10}) \in \mathscr{Z}$, the center of $\mathfrak{R}$.
Suppose $\mathbb{H}(\varphi_2) \notin \mathscr{Z}$ for some $\varphi_2 \in \mathfrak{R}$.
Then $\mathbb{H}(\varphi_{10}) + \mathbb{H}(\varphi_2) \notin \mathscr{Z}$.
i.e, $\mathbb{H}(\varphi_{10}+\varphi_2) \notin \mathscr{Z}$.
By applying (3.5), we get
$(\mathbb{H}(\varphi_2)-\mathbb{J}(\varphi_2))\mathfrak{R}[\mathbb{H}(\varphi_2),\varphi_3] = 0, \forall \varphi_3 \in \mathfrak{R}$.
Since $\mathbb{H}(\varphi_2) \notin \mathscr{Z}$
$[\mathbb{H}(\varphi_2),\varphi_3] \neq 0$.
Since $\mathfrak{R}$ ba a prime, one has
$\mathbb{H}(\varphi_2)-\mathbb{J}(\varphi_2) = 0$

$$i.e, \mathbb{H}(\varphi_2) = \mathbb{J}(\varphi_2) \tag{3.6}$$

Also from (3.5), one has
$(\mathbb{H}(\varphi_{10}+\varphi_2)-\mathbb{J}(\varphi_{10}+\varphi_2))\mathfrak{R}[\mathbb{H}(\varphi_{10}+\varphi_2),\varphi_3] = 0, \forall \varphi_3 \in \mathfrak{R}$.
Since $\mathbb{H}(\varphi_{10}+\varphi_2) \notin \mathscr{Z}$
$[\mathbb{H}(\varphi_{10}+\varphi_2),\varphi_3] \neq 0$
So,

$$\mathbb{H}(\varphi_{10}+\varphi_2) = \mathbb{J}(\varphi_{10}+\varphi_2) \tag{3.7}$$

From (3.6) and (3.7), we obtain,
$\mathbb{H}(\varphi_{10}) = \mathbb{J}(\varphi_{10})$, a contradiction.
This paradox demonstrates that
$\mathbb{H}(\varphi_2) \in \mathscr{Z}, \forall \varphi_2 \in \mathfrak{R}$.
Already $\mathbb{H}$ denotes an automorphism.
$\varphi_1 \in \mathscr{Z}, \forall \varphi_1 \in \mathfrak{R}$.
i.e, $\mathfrak{R}$ be a commutative. □

Remark 3.2. *Considering $\mathbb{H} = \mathscr{I}$ the identity automorphism. We attain the result of Luh [2].*

Lemma 3.3. *Consider $\mathfrak{R}$ a prime ring acquiring a semi commuting non-trivial coupled of automorphisms $\mathbb{H}$ & $\mathbb{J}$. If $\mathfrak{R}$ is of char 2, then a commutative integral domain $\mathfrak{R}$.*

Proof. Consider $\mathfrak{R}$ is of char 2, then $\Omega_+ = \Omega_- = \mathfrak{R}$, then $\mathbb{H}$ and $\mathbb{J}$ are commuting couple of automorphisms. By Lemma 3.1, commutative integral domain in $\mathfrak{R}$. □

Lemma 3.4. *Let us take $\mathfrak{R}$ is a ring of char $\neq 2$ acquiring a semi commuting non-trivial couple of automorphisms $\mathbb{H}$ and $\mathbb{J}$. Let $\varphi_1,\varphi_2 \in \Omega_+$, then $\varphi_1+\varphi_2 \in \Omega_+, \varphi_1-\varphi_2 \in \Omega_+$.*

Proof. We assume $\varphi_1,\varphi_2 \in \Omega_+$.
Then $\mathbb{H}(\varphi_1)\mathbb{J}(\varphi_1) = \mathbb{J}(\varphi_1)\mathbb{H}(\varphi_1)$

$$i.e, [\mathbb{H}(\varphi_1),\mathbb{J}(\varphi_1)] = 0 \tag{3.8}$$

and $\mathbb{H}(\varphi_2)\mathbb{J}(\varphi_2) = \mathbb{J}(\varphi_2)\mathbb{H}(\varphi_2)$

$$i.e, [\mathbb{H}(\varphi_2),\mathbb{J}(\varphi_2)] = 0. \tag{3.9}$$

Assume $\varphi_1+\varphi_2 \in \Omega_+$.
Then $[\mathbb{H}(\varphi_1+\varphi_2),\mathbb{J}(\varphi_1+\varphi_2)] = 0$
i.e, $[\mathbb{H}(\varphi_1),\mathbb{J}(\varphi_1)] + [\mathbb{H}(\varphi_1),\mathbb{J}(\varphi_2)] + [\mathbb{H}(\varphi_2),\mathbb{J}(\varphi_1)] + [\mathbb{H}(\varphi_2),\mathbb{J}(\varphi_2)] = 0$.
Using (3.8) and (3.9), we have

$$[\mathbb{H}(\varphi_1),\mathbb{J}(\varphi_2)] + [\mathbb{H}(\varphi_2),\mathbb{J}(\varphi_1)] = 0.$$

$\mathbb{H}(\varphi_1)\mathbb{J}(\varphi_2) - \mathbb{J}(\varphi_2)\mathbb{H}(\varphi_1)_+\mathbb{H}(\varphi_2)\mathbb{J}(\varphi_1) - \mathbb{J}(\varphi_1)\mathbb{H}(\varphi_2) = 0$.

$$\mathbb{H}(\varphi_1)\mathbb{J}(\varphi_2) + \mathbb{H}(\varphi_2)\mathbb{J}(\varphi_1) = \mathbb{J}(\varphi_1)\mathbb{H}(\varphi_2) + \mathbb{J}(\varphi_2)\mathbb{H}(\varphi_1) \tag{3.10}$$

Suppose $\varphi_1-\varphi_2 \in \Omega_+$, then $\varphi_1-\varphi_2 \in \Omega_-$.
For $\mathfrak{R} = \Omega_+ \cup \Omega_-$.
So, $\mathbb{H}(\varphi_1-\varphi_2)\mathbb{J}(\varphi_1-\varphi_2) = -\mathbb{J}(\varphi_1-\varphi_2)\mathbb{H}(\varphi_1-\varphi_2)$
$\mathbb{H}(\varphi_1)\mathbb{J}(\varphi_1) - \mathbb{H}(\varphi_1)\mathbb{J}(\varphi_2) - \mathbb{H}(\varphi_2)\mathbb{J}(\varphi_1) + \mathbb{H}(\varphi_2)\mathbb{J}(\varphi_2) = -\mathbb{J}(\varphi_1)\mathbb{H}(\varphi_1)_+\mathbb{J}(\varphi_1)\mathbb{H}(\varphi_2) + \mathbb{J}(\varphi_2)\mathbb{H}(\varphi_1) - \mathbb{J}(\varphi_2)\mathbb{H}(\varphi_2)$.
Using (3.8)-(3.10), we have,
$\mathbb{H}(\varphi_1)\mathbb{J}(\varphi_1) - \mathbb{H}(\varphi_1)\mathbb{J}(\varphi_2) - \mathbb{H}(\varphi_2)\mathbb{J}(\varphi_1) + \mathbb{H}(\varphi_2)\mathbb{J}(\varphi_2) = -\mathbb{H}(\varphi_1)\mathbb{J}(\varphi_1) + \mathbb{H}(\varphi_1)\mathbb{J}(\varphi_2) + \mathbb{H}(\varphi_2)\mathbb{J}(\varphi_1) - \mathbb{H}(\varphi_2)\mathbb{J}(\varphi_2)$.

$2(\mathbb{H}(\varphi_1)\mathbb{J}(\varphi_1)) - \mathbb{H}(\varphi_1)\mathbb{J}(\varphi_2) - \mathbb{H}(\varphi_2)\mathbb{J}(\varphi_1) + \mathbb{H}(\varphi_2)\mathbb{J}(\varphi_2) = 0$.
Already char $R \neq 2$, we have,
$\mathbb{H}(\varphi_1)\mathbb{J}(\varphi_1) - \mathbb{H}(\varphi_1)\mathbb{J}(\varphi_2) - \mathbb{H}(\varphi_2)\mathbb{J}(\varphi_1) + \mathbb{H}(\varphi_2)\mathbb{J}(\varphi_2) = 0$.

i.e, $\mathbb{H}(\varphi_1)(\mathbb{J}(\varphi_1)-\mathbb{J}(\varphi_2))-\mathbb{H}(\varphi_2)(\mathbb{J}(\varphi_1)-\mathbb{J}(\varphi_2))=0.$

$(\mathbb{H}(\varphi_1)-\mathbb{H}(\varphi_2))(\mathbb{J}(\varphi_1)-\mathbb{J}(\varphi_2))=0.$
i.e, $\mathbb{H}(\varphi_1-\varphi_2)\mathbb{J}(\varphi_1-\varphi_2)=0.$
$\implies \varphi_1-\varphi_2\in\Omega_0\subset\Omega_+\cap\Omega_-.$
$\implies \varphi_1-\varphi_2\in\Omega_0$, a contradiction.
This paradox demonstrates that $\varphi_1-\varphi_2\in\Omega_+$.
Similarly $\varphi_1-\varphi_2\in\Omega_+$ implies $\varphi_1+\varphi_2\in\Omega_+$.
$\varphi_1+\varphi_2\in\Omega_+$ iff $\varphi_1-\varphi_2\in\Omega_+$. □

Remark 3.5. *Let us assume $\mathbb{H}=\mathscr{I}$, the identity automorphism, we obtain Lemma 3.3 [2].*

Lemma 3.6. *Let us take $\mathfrak{R}$ denotes a ring of char ≠2, possessing a semi-commuting non-trivial couple of automorphisms $\mathbb{H}$ and $\mathbb{J}$. Let $\varphi_1,\varphi_2\in\Omega_-$, then $\varphi_1+\varphi_2\in\Omega_-$ iff $\varphi_1-\varphi_2\in\Omega_-$.*

Proof. The proof is same as Lemma 3.4. □

Remark 3.7. *Let us assume $\mathbb{H}=\mathscr{I}$, the identity automorphism, we obtain Lemma 2[5].*

Lemma 3.8. *Let $\mathfrak{R}$ denotes a ring of char ≠2 acquiring a semi commuting non-trivial couple of automorphism, if $\varphi_1\in\Omega_-\backslash\Omega_+$, then $(\mathbb{H}(\varphi_1)\mathbb{J}(\varphi_1))^2=0$.*

Let $\varphi_1\in\Omega_-\backslash\Omega_+$
Then

$$\mathbb{H}(\varphi_1)\mathbb{J}(\varphi_1)=-\mathbb{J}(\varphi_1)\mathbb{H}(\varphi_1) \tag{3.11}$$

Now

$$\begin{aligned}\mathbb{H}(\varphi_1^2)\mathbb{J}(\varphi_1^2) &= \mathbb{H}(\varphi_1)\mathbb{H}(\varphi_1)\mathbb{J}(\varphi_1)\mathbb{J}(\varphi_1)\\ &= \mathbb{H}(\varphi_1)(-\mathbb{J}(\varphi_1)\mathbb{H}(\varphi_1))\mathbb{J}(\varphi_1)\\ &= -(\mathbb{H}(\varphi_1)\mathbb{J}(\varphi_1))(\mathbb{H}(\varphi_1)\mathbb{J}(\varphi_1))\\ &= -(-\mathbb{J}(\varphi_1)\mathbb{H}(\varphi_1))(-\mathbb{J}(\varphi_1)\mathbb{H}(\varphi_1))\\ &= -\mathbb{J}(\varphi_1)(\mathbb{H}(\varphi_1)\mathbb{J}(\varphi_1))\mathbb{H}(\varphi_1)\\ &= -\mathbb{J}(\varphi_1)(-\mathbb{J}(\varphi_1)\mathbb{H}(\varphi_1))\mathbb{H}(\varphi_1)\\ &= \mathbb{J}(\varphi_1^2)\mathbb{H}(\varphi_1^2). \end{aligned} \tag{3.12}$$

So,

$$\varphi_1^2\in\Omega_+. \tag{3.13}$$

So, $[\mathbb{H}(\varphi_1^2)\mathbb{J}(\varphi_1^2)]=0$.
Suppose that, $\varphi_1+\varphi_1^2\notin\Omega_-$, then $\varphi_1+\varphi_1^2\notin\Omega_+$ ($\mathfrak{R}=\Omega_+\cup\Omega_-$).
$[\mathbb{H}(\varphi_1+\varphi_1^2),\mathbb{J}(\varphi_1+\varphi_1^2)]=0$
i.e, $[\mathbb{H}(\varphi_1),\mathbb{J}(\varphi_1)]+[\mathbb{H}(\varphi_1)\mathbb{J}(\varphi_1^2)]+[\mathbb{H}(\varphi_1^2),\mathbb{J}(\varphi_1)]$
$+[\mathbb{H}(\varphi_1^2)\mathbb{J}(\varphi_1^2)]=0$.

$$i.e, [\mathbb{H}(\varphi_1),\mathbb{J}(\varphi_1)]+[\mathbb{H}(\varphi_1)\mathbb{J}(\varphi_1^2)]+[\mathbb{H}(\varphi_1^2),\mathbb{J}(\varphi_1)]=0. \tag{3.14}$$

Using (3.12), then

$$\begin{aligned}\mathbb{H}(\varphi_1)\mathbb{J}(\varphi_1^2) &= \mathbb{H}(\varphi_1)\mathbb{J}(\varphi_1)\mathbb{J}(\varphi_1)\\ &= (-\mathbb{J}(\varphi_1)\mathbb{H}(\varphi_1))\mathbb{J}(\varphi_1), \text{ using (3.11)}\\ &= -\mathbb{J}(\varphi_1)(-\mathbb{J}(\varphi_1)\mathbb{H}(\varphi_1)), \text{ using (3.11)}\\ &= \mathbb{J}(\varphi_1^2)\mathbb{H}(\varphi_1)\end{aligned}$$

$$\begin{aligned}\mathbb{H}(\varphi_1)\mathbb{J}(\varphi_1^2)-\mathbb{J}(\varphi_1^2)\mathbb{H}(\varphi_1) &= 0\\ i.e, [\mathbb{H}(\varphi_1)\mathbb{J}(\varphi_1^2)] &= 0.\end{aligned} \tag{3.15}$$

Similarly

$$\mathbb{H}(\varphi_1^2)\mathbb{J}(\varphi_1)=0. \tag{3.16}$$

From (3.13)-(3.15), we have,

$$[\mathbb{H}(\varphi_1),\mathbb{J}(\varphi_1)]=0.$$

i.e, $\varphi_1\in\Omega_+$ a contradiction.
This contradiction proves that

$$\varphi_1+\varphi_1^2\in\Omega_- \tag{3.17}$$

By Lemma 3.6, $\varphi_1-\varphi_1^2\in\Omega_-$.
Now, $(\varphi_1+\varphi_1^2)+(\varphi_1-\varphi_1^2)=2\varphi_1\in\Omega_-$
From Lemma 3.5, we have $(\varphi_1+\varphi_1^2)+(\varphi_1-\varphi_1^2)=2\varphi_1^2\in\Omega_-$

$$\begin{aligned} So\ \mathbb{H}(2\varphi_1^2)\mathbb{J}(2\varphi_1^2) &= -\mathbb{J}(2\varphi_1^2)\mathbb{H}(2\varphi_1^2),\\ 4\mathbb{H}(\varphi_1^2)\mathbb{J}(\varphi_1^2) &= -4\mathbb{J}(\varphi_1^2)\mathbb{H}(\varphi_1^2)\\ -4(\mathbb{H}(\varphi_1^2)\mathbb{J}(\varphi_1^2))+\mathbb{J}(\varphi_1^2)\mathbb{H}(\varphi_1^2) &= 0\end{aligned} \tag{3.18}$$

Since char $\mathfrak{R}\neq$ 2

$$\begin{aligned}\mathbb{H}(\varphi_1^2)\mathbb{J}(\varphi_1^2)+\mathbb{J}(\varphi_1^2)\mathbb{H}(\varphi_1^2) &= 0.\\ \text{Hence, } \mathbb{H}(\varphi_1^2)\mathbb{J}(\varphi_1^2) &= -\mathbb{J}(\varphi_1^2)\mathbb{H}(\varphi_1^2)\end{aligned}$$

This implies

$$\varphi_1^2\in\Omega_-. \tag{3.19}$$

From (3.12) and (3.18), we have,

$$\mathbb{H}(\varphi_1^2)\mathbb{J}(\varphi_1^2)=-\mathbb{H}(\varphi_1^2)\mathbb{J}(\varphi_1^2)$$

$$2\mathbb{H}(\varphi_1^2)\mathbb{J}(\varphi_1^2)=0$$

Since char $\mathfrak{R}\neq 2$

$$\mathbb{H}(\varphi_1^2)\mathbb{J}(\varphi_1^2)=0. \tag{3.20}$$

Now

$$\begin{aligned}(\mathbb{H}(\varphi_1)\mathbb{J}(\varphi_1))^2 &= \mathbb{H}(\varphi_1)\mathbb{J}(\varphi_1)\mathbb{H}(\varphi_1)\mathbb{J}(\varphi_1)\\ &= \mathbb{H}(\varphi_1)(-\mathbb{H}(\varphi_1)\mathbb{J}(\varphi_1))\mathbb{J}(\varphi_1)\\ &= -\mathbb{H}(\varphi_1^2)\mathbb{J}(\varphi_1^2) \text{ (using (3.11))}\\ &= 0 \text{ (using (3.20))}.\end{aligned} \tag{3.21}$$

Remark 3.9. *When $\mathbb{H} = \mathscr{I}$, the identity transformation we get Lemma 3 [5].*

Lemma 3.10. *Let us assume $\mathfrak{R}$ denotes a ring of characteristic $\neq 2,3$ free prime ring acquiring a non-trivial semi commuting couple of automorphisms $\mathbb{H}$ and $\mathbb{J}$. If $\Omega_- = \mathfrak{R}$, then $\mathfrak{R}$ is commutative.*

Proof. Assume $\omega \in \mathfrak{R}$. Since $\mathfrak{R} = \Omega_-$, we have $\omega \in \Omega_-$. Then

$$\mathbb{H}(\omega)\mathbb{J}(\omega) = -\mathbb{J}(\omega)\mathbb{H}(\omega), \forall \omega \in \mathfrak{R}. \tag{3.22}$$

By using Lemma 3.8, we have

$$(\mathbb{H}(\omega)\mathbb{J}(\omega))^2 = 0, \forall \omega \in \mathfrak{R}. \tag{3.23}$$

Consequently, $(-\mathbb{J}(\omega)\mathbb{H}(\omega))^2 = (\mathbb{H}(\omega)\mathbb{J}(\omega))^2 = 0$

$$i.e, (\mathbb{J}(\omega)\mathbb{H}(\omega))^2 = 0, \forall \omega \in \mathfrak{R}. \tag{3.24}$$

Then, $(\mathbb{H}(\omega+\tau)\mathbb{J}(\omega+\tau))^2 = 0$, $\forall \omega, \tau \in \mathfrak{R}$.

$$(\mathbb{H}(\omega)+\mathbb{H}(\tau))(\mathbb{J}(\omega)+\mathbb{J}(\tau))(\mathbb{H}(\omega) + \mathbb{H}(\tau))(\mathbb{J}(\omega)+\mathbb{J}(\tau)) = 0$$

$$(\mathbb{H}(\omega)\mathbb{J}(\omega)+\mathbb{H}(\omega)\mathbb{J}(\tau)+\mathbb{H}(\tau)\mathbb{J}(\omega) + \mathbb{H}(\tau)\mathbb{J}(\tau))(\mathbb{H}(\omega)\mathbb{J}(\omega) + \mathbb{H}(\omega)\mathbb{J}(\tau)+\mathbb{H}(\tau)\mathbb{J}(\omega)+\mathbb{H}(\tau)\mathbb{J}(\tau)) = 0$$

$$\begin{aligned}
&(\mathbb{H}(\omega)\mathbb{J}(\omega))^2+\mathbb{H}(\omega)\mathbb{J}(\omega)\mathbb{H}(\omega)\mathbb{J}(\tau)\\
&+\mathbb{H}(\omega)\mathbb{J}(\omega).\mathbb{H}(\tau)\mathbb{J}(\omega)+\mathbb{H}(\omega)\mathbb{J}(\omega).\mathbb{H}(\tau)\mathbb{J}(\tau)\\
&+\mathbb{H}(\omega)\mathbb{J}(\tau).\mathbb{H}(\omega)\mathbb{J}(\omega)+(\mathbb{H}(\omega)\mathbb{J}(\tau))^2\\
&+\mathbb{H}(\omega)\mathbb{J}(\tau)\mathbb{H}(\tau)\mathbb{J}(\omega)+\mathbb{H}(\omega)\mathbb{J}(\tau)\mathbb{H}(\tau)\mathbb{J}(\tau)\\
&+\mathbb{H}(\tau)\mathbb{J}(\omega)\mathbb{H}(\omega)\mathbb{J}(\omega)+\mathbb{H}(\tau)\mathbb{J}(\omega)\mathbb{H}(\omega)\mathbb{J}(\tau)\\
&+(\mathbb{H}(\tau)\mathbb{J}(\omega))^2+\mathbb{H}(\tau)\mathbb{J}(\omega)\mathbb{H}(\tau)\mathbb{J}(\tau)\\
&+\mathbb{H}(\tau)\mathbb{J}(\tau)\mathbb{H}(\omega)\mathbb{J}(\omega)+\mathbb{H}(\tau)\mathbb{J}(\tau)\mathbb{H}(\omega)\mathbb{J}(\tau)\\
&+\mathbb{H}(\tau)\mathbb{J}(\tau)\mathbb{H}(\tau)\mathbb{J}(\omega)=0.
\end{aligned} \tag{3.25}$$

Replacing τ by 2τ in (3.25), we have

$$\begin{aligned}
&2\mathbb{H}(\omega)\mathbb{J}(\omega)\mathbb{H}(\omega)\mathbb{J}(\tau)+2\mathbb{H}(\omega)\mathbb{J}(\omega).\mathbb{H}(\tau)\mathbb{J}(\omega)\\
&+4\mathbb{H}(\omega)\mathbb{J}(\omega).\mathbb{H}(\tau)\mathbb{J}(\tau)\\
&+2\mathbb{H}(\omega)\mathbb{J}(\tau).\mathbb{H}(\omega)\mathbb{J}(\omega)+4(\mathbb{H}(\omega)\mathbb{J}(\tau))^2\\
&+4\mathbb{H}(\omega)\mathbb{J}(\tau)\mathbb{H}(\tau)\mathbb{J}(\omega)+8\mathbb{H}(\omega)\mathbb{J}(\tau)\mathbb{H}(\tau)\mathbb{J}(\tau)\\
&+2\mathbb{H}(\tau)\mathbb{H}(\omega)\mathbb{H}(\omega)\mathbb{J}(\omega)+4\mathbb{H}(\tau)\mathbb{J}(\omega)\mathbb{H}(\omega)\mathbb{J}(\tau)\\
&+4(\mathbb{H}(\tau)\mathbb{J}(\omega))^2+8\mathbb{H}(\tau)\mathbb{J}(\omega)\mathbb{H}(\tau)\mathbb{J}(\tau)\\
&+4\mathbb{H}(\tau)\mathbb{J}(\tau)\mathbb{H}(\omega)\mathbb{J}(\omega)+8\mathbb{H}(\tau)\mathbb{J}(\tau)\mathbb{H}(\omega)\mathbb{J}(\tau)\\
&+8\mathbb{H}(\tau)\mathbb{J}(\tau)\mathbb{H}(\tau)\mathbb{J}(\omega)=0.
\end{aligned} \tag{3.26}$$

From (3.25)-(3.26), gives

$$\begin{aligned}
&-2\mathbb{H}(\omega)\mathbb{J}(\omega)\mathbb{H}(\omega)\mathbb{J}(\tau)-2\mathbb{H}(\omega)\mathbb{J}(\omega).\mathbb{H}(\tau)\mathbb{J}(\omega)\\
&-2\mathbb{H}(\omega)\mathbb{J}(\tau).\mathbb{H}(\omega)\mathbb{J}(\omega)+4\mathbb{H}(\omega)\mathbb{J}(\tau)\mathbb{H}(\tau)\mathbb{J}(\tau)\\
&-2\mathbb{H}(\tau)\mathbb{J}(\omega)\mathbb{H}(\omega)\mathbb{J}(\omega)+4\mathbb{H}(\tau)\mathbb{J}(\omega)\mathbb{H}(\tau)\mathbb{J}(\tau)\\
&+4\mathbb{H}(\tau)\mathbb{J}(\tau)\mathbb{H}(\omega)\mathbb{J}(\tau)\\
&+4\mathbb{H}(\tau)\mathbb{J}(\tau)\mathbb{H}(\tau)\mathbb{J}(\omega)=0, \ \forall \omega,\tau \in \mathfrak{R}.
\end{aligned} \tag{3.27}$$

Exchanging ω and τ in (3.27), we have

$$\begin{aligned}
&-2\mathbb{H}(\tau)\mathbb{J}(\tau)\mathbb{H}(\tau)\mathbb{J}(\omega)-2\mathbb{H}(\tau)\mathbb{J}(\tau).\mathbb{H}(\omega)\mathbb{J}(\tau)\\
&-2\mathbb{H}(\tau)\mathbb{J}(\omega).\mathbb{H}(\tau)\mathbb{J}(\tau)+4\mathbb{H}(\tau)\mathbb{J}(\omega)\mathbb{H}(\omega)\mathbb{J}(\omega)\\
&-2\mathbb{H}(\omega)\mathbb{J}(\tau)\mathbb{H}(\tau)\mathbb{J}(\tau)+4\mathbb{H}(\omega)\mathbb{J}(\tau)\mathbb{H}(\omega)\mathbb{J}(\omega)\\
&+4\mathbb{H}(\omega)\mathbb{J}(\omega)\mathbb{H}(\tau)\mathbb{J}(\omega)\\
&+4\mathbb{H}(\omega)\mathbb{J}(\omega)\mathbb{H}(\omega)\mathbb{J}(\tau)=0.
\end{aligned} \tag{3.28}$$

From, (3.27)-(3.28) gives

$$6(\mathbb{H}(\tau)\mathbb{J}(\omega)\mathbb{H}(\omega)\mathbb{J}(\omega)+\mathbb{H}(\omega)\mathbb{J}(\tau).\mathbb{H}(\omega)\mathbb{J}(\omega) + \mathbb{H}(\omega)\mathbb{J}(\omega)\mathbb{H}(\tau)\mathbb{J}(\omega)+\mathbb{H}(\omega)\mathbb{J}(\omega)\mathbb{H}(\omega)\mathbb{J}(\tau)) = 0$$

Char $\mathfrak{R} \neq 2(\mathfrak{R} \neq 3)$, one has,

$$\begin{aligned}
&\mathbb{H}(\tau)\mathbb{J}(\omega)\mathbb{H}(\omega)\mathbb{J}(\omega)+\mathbb{H}(\omega)\mathbb{J}(\tau).\mathbb{H}(\omega)\mathbb{J}(\omega)\\
&+\mathbb{H}(\omega)\mathbb{J}(\omega)\mathbb{H}(\tau)\mathbb{J}(\omega)\\
&+\mathbb{H}(\omega)\mathbb{J}(\omega)\mathbb{H}(\omega)\mathbb{J}(\tau)=0
\end{aligned} \tag{3.29}$$

$$\begin{aligned}
&\mathbb{H}(\omega)\mathbb{J}(\omega)\{\mathbb{H}(\tau)\mathbb{J}(\omega)\mathbb{H}(\omega)\mathbb{J}(\omega)\\
&+\mathbb{H}(\omega)\mathbb{J}(\tau).\mathbb{H}(\omega)\mathbb{J}(\omega)\\
&+\mathbb{H}(\omega)\mathbb{J}(\omega)\mathbb{H}(\tau)\mathbb{J}(\omega)\\
&+\mathbb{H}(\omega)\mathbb{J}(\omega)\mathbb{H}(\omega)\mathbb{J}(\tau)\}\mathbb{H}(\omega)=0.
\end{aligned} \tag{3.30}$$

$$\begin{aligned}
&\mathbb{H}(\omega)\mathbb{J}(\omega)\mathbb{H}(\tau)(\mathbb{J}(\omega)\mathbb{H}(\omega)^2\\
&+\mathbb{H}(\omega)\mathbb{J}(\omega)\mathbb{H}(\omega)\mathbb{J}(\tau)\mathbb{H}(\omega)\mathbb{J}(\omega))\mathbb{H}(\omega)\\
&+(\mathbb{H}(\omega)\mathbb{J}(\omega))^2\mathbb{H}(\tau)\mathbb{J}(\omega)\mathbb{H}(\omega)\\
&+(\mathbb{H}(\omega)\mathbb{J}(\omega))^2\mathbb{H}(\omega)\mathbb{J}(\tau)\mathbb{H}(\omega)=0.
\end{aligned} \tag{3.31}$$

Using (3.23) and (3.24), one has,

$$\mathbb{H}(\omega)\mathbb{J}(\omega)\mathbb{H}(\omega)\mathbb{J}(\tau)\mathbb{H}(\omega)\mathbb{J}(\omega)\mathbb{H}(\omega)=0.$$

$$(\mathbb{H}(\omega)\mathbb{J}(\omega)\mathbb{H}(\omega))(\mathbb{J}(\tau))(\mathbb{H}(\omega)\mathbb{J}(\omega)\mathbb{H}(\omega))=0.$$

Since $\mathbb{J}$ be an automorphism

$$(\mathbb{H}(\omega)\mathbb{J}(\omega)\mathbb{H}(\omega))c(\mathbb{H}(\omega)\mathbb{J}(\omega)\mathbb{H}(\omega))=0, \ \forall \omega, c \in \mathfrak{R} \tag{3.32}$$

Since $\mathfrak{R}$ be a prime, we have

$$\mathbb{H}(\omega)\mathbb{J}(\omega)\mathbb{H}(\omega)=0, \ \forall \omega \in \mathfrak{R}. \tag{3.33}$$

Similarly

$$\mathbb{J}(\omega)\mathbb{H}(\omega)\mathbb{J}(\omega)=0, \ \forall \omega \in \mathfrak{R}. \tag{3.34}$$

From (3.29)-(3.31), we have

$$\begin{aligned}
&\mathbb{H}(\omega)\mathbb{J}(\tau)\mathbb{H}(\omega)\mathbb{J}(\omega)\\
&+\mathbb{H}(\omega)\mathbb{J}(\omega)\mathbb{H}(\tau)\mathbb{J}(\omega)=0 \qquad (3.35)\\
\implies\ &(\mathbb{H}(\omega)\mathbb{J}(\tau)\mathbb{H}(\omega)\mathbb{J}(\omega)\\
&+\mathbb{H}(\omega)\mathbb{J}(\omega)\mathbb{H}(\tau)\mathbb{J}(\omega))\mathbb{H}(\omega)=0\\
\implies\ &\mathbb{H}(\omega)\mathbb{J}(\tau)\mathbb{H}(\omega)\mathbb{J}(\omega)\mathbb{H}(\omega)\\
&+\mathbb{H}(\omega)\mathbb{J}(\omega)\mathbb{H}(\tau)\mathbb{J}(\omega)\mathbb{H}(\omega)=0.
\end{aligned}$$

Using (3.33) we have,

$$\begin{aligned} \mathbb{H}(\omega)\mathbb{J}(\omega)\mathbb{H}(\tau)\mathbb{J}(\omega)\mathbb{H}(\omega) &= 0 \\ (\mathbb{H}(\omega)\mathbb{J}(\omega))\mathbb{H}(\tau)(\mathbb{J}(\omega)\mathbb{H}(\omega)) &= 0 \end{aligned}$$

Since $\mathbb{H}$ is an automorphism,

$$(\mathbb{H}(\omega)\mathbb{J}(\omega))c(\mathbb{J}(\omega)\mathbb{H}(\omega)) = 0,\ \forall \omega, c \in \mathfrak{R}$$

Since $\mathfrak{R}$ is prime

$$\mathbb{H}(\omega)\mathbb{J}(\omega) = 0 \& \mathbb{J}(\omega)\mathbb{H}(\omega) = 0.$$

This implies $\omega \in \Omega_0 \subseteq \Omega_+, \forall \omega \in \mathfrak{R}$.
Hence $\mathfrak{R}$ is commutative. □

Remark 3.11. *When $\mathbb{H} = \mathscr{I}$, the identity automorphism Lemma 4 [5] follows.*

Lemma 3.12. *Consider a prime ring $\mathfrak{R}$ of char $\neq$ 2, 3 acquiring a semi commuting non trivial pair of automorphism $\mathbb{H}$ & $\mathbb{J}$, If $\varphi_1, \varphi_2 \in \Omega_+$, then either $\varphi_1 + \in \Omega_+$ (or) $\varphi_1, \varphi_2, \varphi_1 + \varphi_2, \varphi_1 - \varphi_2 \in \Omega_+$.*

Proof. We assume $\varphi_1, \varphi_2 \in \Omega_+$.
If $\varphi_1 + \varphi_2 \in \Omega_+$, there is nothing to prove. So, assume

$$\varphi_1 + \varphi_2 \notin \Omega_+. \tag{3.36}$$

By applying Lemma 3.6,

$$\varphi_1 - \varphi_2 \notin \Omega_+. \tag{3.37}$$

Consider $2\varphi_1 + \varphi_2$, when $2\varphi_1 + \varphi_2 \in \Omega_+$, then by Lemma 3.1, $2\varphi_1 - \varphi_2 \in \Omega_+$.
Now

$$\varphi_1 \in \Omega_+ \textit{ implies } 2\varphi_1 \in \Omega_+. \tag{3.38}$$

By using Lemma 3.1,

$$(2\varphi_1 - \varphi_2) - \varphi_2 = 2(\varphi_1 - \varphi_2) \in \Omega_+. \tag{3.39}$$

$$\begin{aligned} &\mathbb{H}(2(\varphi_1 - \varphi_2))\mathbb{J}(2(\varphi_1 - \varphi_2)) \\ &= \mathbb{J}(2(\varphi_1 - \varphi_2))\mathbb{H}(2(\varphi_1 - \varphi_2)) \\ &4\mathbb{H}(\varphi_1 - \varphi_2)\mathbb{J}(\varphi_1 - \varphi_2) \\ &= 4\mathbb{J}(\varphi_1 - \varphi_2)\mathbb{H}(\varphi_1 - \varphi_2). \end{aligned}$$

Since char of $R \neq 2$, we have,

$$\mathbb{H}(\varphi_1 - \varphi_2)\mathbb{J}(\varphi_1 - \varphi_2) = \mathbb{J}(\varphi_1 - \varphi_2)\mathbb{H}(\varphi_1 - \varphi_2).$$

This implies $\varphi_1 - \varphi_2 \in \Omega_+$ contradicting (3.37).
This paradox demonstrates that $2\varphi_1 + \varphi_2 \in \Omega_+$.
Hence

$$2\varphi_1 + \varphi_2 \in \Omega_- (\mathfrak{R} = \Omega_+ \cap \Omega) \tag{3.40}$$

Using Lemma 3.1

$$2\varphi_1 - \varphi_2 \in \Omega_-. \tag{3.41}$$

Now $2\varphi_1 + \varphi_2 \in \Omega_-$

$$2(2\varphi_1 + \varphi_2) \in \Omega_-.$$

i.e, $3(\varphi_1 + \varphi_2) + (\varphi_1 - \varphi_2) = 2(2\varphi_1 + \varphi_2) \in \Omega_-$.
Using Lemma 3.1, $3(\varphi_1 + \varphi_2) - (\varphi_1 - \varphi_2) \in \Omega_-$

$$i.e, 2\varphi_1 + 4\varphi_2 \in \Omega_-. \tag{3.42}$$

Now,

$$\varphi_1 + \varphi_2 \in \Omega\ (\mathfrak{R} = \Omega_+ \cup \Omega_-). \tag{3.43}$$

$$\implies 3(\varphi_1 + \varphi_2) \in \Omega_-$$

i.e, $(2\varphi_1 + 4\varphi_2) + (\varphi_1 - \varphi_2) = 3(\varphi_1 + \varphi_2) \in \Omega_-$.
Using Lemma 3.1, $(2\varphi_1 + 4\varphi_2) - (\varphi_1 - \varphi_2) \in \Omega_-$.
i.e, $\varphi_1 + 5\varphi_2 \in \Omega_-$.
Also, $(\varphi_1 + 5\varphi_2) + (\varphi_1 - \varphi_2) = 2\varphi_1 + 4\varphi_2 \in \Omega_-$ (using (3.42)).
Further, we have $(\varphi_1 + 5\varphi_2) - (\varphi_1 - \varphi_2) \in \Omega_-$
i.e, $6\varphi_2 \in \Omega_-$.
Hence

$$\varphi_2 \in \Omega_-. \tag{3.44}$$

So $2\varphi_2 \in \Omega_-$

$$(\varphi_1 + \varphi_2) - (\varphi_1 - \varphi_2) = 2\varphi_2 \in \Omega_-.$$

Using Lemma 3.1, $(\varphi_1 + \varphi_2) + (\varphi_1 - \varphi_2) \in \Omega_-$.

$$i.e, 2\varphi_1 \in \Omega_-$$

$$i.e, \varphi_1 \in \Omega_-. \tag{3.45}$$

Thus $\varphi_1, \varphi_2, \varphi_1 + \varphi_2, \varphi_1 - \varphi_2 \in \Omega_-$.
This completes the proof. □

Remark 3.13. *When $\mathbb{H} = \mathscr{I}$, the identity automorphism, we obtain lemma 5 [5].*

Corollary 3.14. *Consider $\mathfrak{R}$ is a prime ring of char $\neq$ 2, 3 possessing a semi-commuting non-trivial couple of automorphism $\mathbb{H}$ & $\mathbb{J}$. If $\varphi_1 \in \Omega_+, \Omega_-$ and $\varphi_2 \in \Omega_+$, then $\varphi_1 + \varphi_2 \in \Omega_+$.*

Proof. If $\varphi_1 + \varphi_2 \notin \Omega_+$.
Using Lemma 3.5, $\varphi_1 \in \Omega_-$, a contradiction.
Thus, $\varphi_1 + \varphi_2 \in \Omega_-$. □

Theorem 3.15. *Consider $\mathfrak{R}$ is a prime ring of char $\neq$ 2,3 possessing a semi-commuting non-trivial couple of automorphism $\mathbb{H}$ & $\mathbb{J}$. Then, a commutative integral domain $\mathfrak{R}$.*

Proof. Let us take $\mathfrak{R}$ is not commutative.
Then, $\mathfrak{R} \notin \Omega_+$ & $\mathfrak{R} \notin \Omega_-$.
Using Lemma 3.1 and Lemma 3.5, $\exists\ \varphi_1 \in \Omega_- / \Omega_+$ and $\varphi_2 \in \Omega_- / \Omega_+$.

Suppose $\varphi_1+\varphi_2 \in \Omega_-$.
Since $(\varphi_1+\varphi_2)-\varphi_2 = \varphi_1 \notin \Omega_-$
We have $(\varphi_1+\varphi_2)+\varphi_2 = \varphi_1 \notin \Omega_-$

$$i.e, \varphi_1+2\varphi_2 \notin \Omega_- \tag{3.46}$$

also $(\varphi_1+\varphi_2)-\varphi_2 = \varphi_1 \in \Omega_+$
implies $(\varphi_1+\varphi_2)+\varphi_2 \in \Omega_+$

$$i.e,, \varphi_1+2\varphi_2 \in \Omega_+. \tag{3.47}$$

Now, $\varphi_1 \in \Omega_+/\Omega_-$ and $\varphi_1+2\varphi_2 \in \Omega_+$ and using previous corollary

$\varphi_1+(\varphi_1+2\varphi_2) \in \Omega_+$ and so $\varphi_1-(\varphi_1+2\varphi_2) = -2\varphi_2 \in \Omega_+$

i.e, $\varphi_2 \in \Omega_+$ a contradiction.
Suppose $\varphi_1+\varphi_2 \notin \Omega_-$, then $\varphi_1+\varphi_2 \in \Omega_+$ $(\mathfrak{R} = \Omega_+ \cup \Omega_-)$.
Now $\varphi_1+\varphi_2 \in \Omega_+/\Omega_-$ and $\varphi_1 \in \Omega_+$
By the previous corollary
$(\varphi_1+\varphi_2)+\varphi_1 \in \Omega_+$ and so
$(\varphi_1+\varphi_2)-\varphi_1 = \varphi_2 \in \Omega_+$ again a contradiction.
Similarly, suppose $\varphi_1+\varphi_2 \in \Omega_+$.
Since $(\varphi_1+\varphi_2)-\varphi_1 = \varphi_2 \notin \Omega_+$, we have $(\varphi_1+\varphi_2)+\varphi_2 \notin \Omega_+$

$$i.e, 2\varphi_1+\varphi_2 \notin \Omega_+. \tag{3.48}$$

Also $(\varphi_1+\varphi_2)-\varphi_1 = \varphi_2 \in \Omega_-$ implies $(\varphi_1+\varphi_2)+\varphi_1 \in \Omega_-$.

$$i.e, 2\varphi_1+\varphi_2 \in \Omega_-. \tag{3.49}$$

Now $\varphi_2 \in \Omega_-/\Omega_+$ and $2\varphi_1+\varphi_2 \in \Omega_-$ and using a previous Corollary.
$\varphi_2+(2\varphi_1+\varphi_2) \in \Omega_-$ and so

$$\varphi_2-(2\varphi_1+\varphi_2) = -2\varphi_1 \in \Omega_-$$

i.e, $\varphi_1 \in \Omega_-$ a contradiction.
Suppose $\varphi_1+\varphi_2 \notin \Omega_+$. Then $\varphi_1+\varphi_2 \in \Omega_-$ $(\mathfrak{R} = \Omega_+ \cup \Omega_-)$.
Now $\varphi_1+\varphi_2 \in \Omega_- \backslash \Omega_+$ and $\varphi_2 \in \Omega_-$
By previous corollary $(\varphi_1+\varphi_2)+\varphi_2 \in \Omega_-$.
So $(\varphi_1+\varphi_2)-\varphi_2 = \varphi_1 \in \Omega_-$ a contradiction. Thus, $\mathfrak{R}$ is commutative. □

Remark 3.16. *When $\mathbb{H} = \mathscr{I}$, the identity automorphism, we obtain Theorem 3.1 [5].*

Theorem 3.17. *Let us take $\mathfrak{R}$ is a prime ring of char $\neq 2$ acquiring a semi-commuting non-trivial couple of automorphism $\mathbb{H}$ & $\mathbb{J}$. Suppose a non-zero center $\mathfrak{R}$, then, a commutative integral domain $\mathfrak{R}$.*

Proof. Assume $O \neq \varphi_3 \notin \mathscr{Z}$, the center of $\mathfrak{R}$, if $\mathfrak{R}$ is not commutative. Then, $\exists$

$$\varphi_1 \notin \Omega_-/\Omega_+. \tag{3.50}$$

Consider $\varphi_3+\varphi_1$.
If $\varphi_3+\varphi_1 \in \Omega_+$, then

$$[\mathbb{H}(\varphi_3+\varphi_1), \mathbb{J}(\varphi_3+\varphi_1)] = 0 \tag{3.51}$$

$$[\mathbb{H}(\varphi_3), \mathbb{J}(\varphi_3)] + [\mathbb{H}(\varphi_3), \mathbb{J}(\varphi_1)] + [\mathbb{H}(\varphi_1), \mathbb{J}(\varphi_3)] + [\mathbb{H}(\varphi_1), \mathbb{J}(\varphi_1)] = 0.$$

Since $\varphi_3 \in \mathscr{Z}$, and $\mathbb{H}$ and $\mathbb{J}$ are automorphism, we obtain, $\mathbb{H}(\varphi_3) \in \mathscr{Z}$ and $\mathbb{J}(\varphi_3) \in \mathscr{Z}$.
Hence, $[\mathbb{H}(\varphi_1), \mathbb{J}(\varphi_1)] = 0$ contradicting.

$$\varphi_1 \in \Omega_+. \text{ So } \varphi_3+\varphi_1 \in \Omega_+.$$

Since, $\mathfrak{R} = \Omega_+ \cup \Omega_-, \ \varphi_3+\varphi_1 \in \Omega_-$.

$$\mathbb{H}(\varphi_3+\varphi_1). = -\mathbb{J}(\varphi_3+\varphi_1).\mathbb{H}(\varphi_3+\varphi_1).$$

$$\begin{aligned} &\mathbb{H}(\varphi_3)\mathbb{J}(\varphi_3) + \mathbb{H}(\varphi_3)\mathbb{J}(\varphi_1) + \mathbb{H}(\varphi_1)\mathbb{J}(\varphi_3) \\ &+ \mathbb{H}(\varphi_1)\mathbb{J}(\varphi_1) = -\mathbb{J}(\varphi_3)\mathbb{H}(\varphi_3) - \mathbb{J}(\varphi_3)\mathbb{H}(\varphi_1) \\ &- \mathbb{J}(\varphi_1)\mathbb{H}(\varphi_3) - \mathbb{J}(\varphi_1)\mathbb{H}(\varphi_1). \end{aligned} \tag{3.52}$$

Since $\varphi_1 \in \Omega_-/\Omega_+$, we obtain,

$$\mathbb{H}(\varphi_1)\mathbb{J}(\varphi_1) = -\mathbb{J}(\varphi_1)\mathbb{H}(\varphi_1).$$

Then (3.52) becomes

$$\begin{aligned} &\mathbb{H}(\varphi_3)\mathbb{J}(\varphi_3) + \mathbb{H}(\varphi_3)\mathbb{J}(\varphi_1) + \mathbb{H}(\varphi_1)\mathbb{J}(\varphi_3) \\ &= -\mathbb{J}(\varphi_3)\mathbb{H}(\varphi_3) - \mathbb{J}(\varphi_3)\mathbb{H}(\varphi_3) - \mathbb{J}(\varphi_3)\mathbb{H}(\varphi_3). \end{aligned}$$

Since, $\mathbb{H}(\varphi_3) \in \mathscr{Z}$ and $\mathbb{J}(\varphi_3) \in \mathscr{Z}$, we obtain

$$2(\mathbb{J}(\varphi_3)\mathbb{H}(\varphi_3)) + \mathbb{J}(\varphi_1)\mathbb{H}(\varphi_3) + \mathbb{J}(\varphi_3)\mathbb{H}(\varphi_1)) = 0$$

$$i.e, \mathbb{J}(\varphi_3)\mathbb{H}(\varphi_3) + \mathbb{J}(\varphi_1)\mathbb{H}(\varphi_3) + \mathbb{J}(\varphi_3)\mathbb{H}(\varphi_1) = 0$$

$$\mathbb{J}(\varphi_3)\mathbb{H}(\varphi_1) = -(\mathbb{J}(\varphi_3)\mathbb{H}(\varphi_3) + \mathbb{J}(\varphi_1)\mathbb{H}(\varphi_3)). \tag{3.53}$$

$$\begin{aligned} & \mathbb{J}(\varphi_3)\mathbb{H}(\varphi_1)\mathbb{J}(\varphi_1) \\ = \ & -(\mathbb{J}(\varphi_3)\mathbb{H}(\varphi_3) + \mathbb{J}(\varphi_1)\mathbb{H}(\varphi_3)) \\ = \ & -(\mathbb{J}(\varphi_3)\mathbb{H}(\varphi_3)\mathbb{J}(\varphi_1)) + \mathbb{J}(\varphi_1)\mathbb{H}(\varphi_3)\mathbb{J}(\varphi_1) \\ = \ & -(\mathbb{J}(\varphi_1)\mathbb{J}(\varphi_3)\mathbb{H}(\varphi_3) + \mathbb{J}(\varphi_1)\mathbb{J}(\varphi_1)\mathbb{H}(\varphi_3)) \\ = \ & -\mathbb{J}(\varphi_1)(\mathbb{J}(\varphi_3)\mathbb{H}(\varphi_3) + \mathbb{J}(\varphi_1)\mathbb{H}(\varphi_3)) \\ = \ & \mathbb{J}(\varphi_1)\mathbb{J}(\varphi_3)\mathbb{H}(\varphi_1), \\ = \ & \mathbb{J}(\varphi_3)\mathbb{J}(\varphi_1)\mathbb{H}(\varphi_1) \ (\mathbb{J}(\varphi_3) \in \mathscr{Z}) \\ \Longrightarrow \ & \mathbb{J}(\varphi_3)(\mathbb{H}(\varphi_1)\mathbb{J}(\varphi_1) - \mathbb{J}(\varphi_1)\mathbb{H}(\varphi_1)) = 0, \end{aligned}$$

for all $\mathbb{J}(\varphi_3) \in \mathscr{Z}$, $\mathbb{H}(\varphi_3) \in \mathscr{Z}$. Since $0 \neq \varphi_3$ and $\mathbb{J}$ be an automorphism $0 \neq \mathbb{J}(\varphi_3)$, we know $\mathfrak{R}$ is prime $\mathbb{H}(\varphi_1)\mathbb{J}(\varphi_1) = \mathbb{J}(\varphi_1)\mathbb{H}(\varphi_1) = 0$. Hence, $\mathfrak{R}$ is commutative. □

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