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Opial and Ostrowski type inequalities based on trigonometric and hyperbolic type Taylor formulae

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This paper is dedicated to the occasion of Professor Gaston M. N'Guérékata's 70th birthday

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Abstract. Based on [4], we produce a collection of interesting trigonometric and hyperbolic Taylor formulae with integral remainders. Using these we derive Opial and Ostrowski type corresponding inequalities of various kinds and norms.

AMS Subject Classifications: 26A24, 26D10, 26D15.

Keywords: Trigonometric and hyperbolic Taylor formula, Opial inequality, Ostrowski inequality.

Contents

1. Taylor formulae based on linear differential operators

This section is based entirely on [4]. Here K denotes $\mathbb{R}$ or $\mathbb{C}$.

Let I be an interval subset of $\mathbb{R}$, and $c = (c_0, ..., c_n) \in \mathbb{R}^{n+1}$, $c_n = 1$, $n \in \mathbb{N}$, and the n-th order linear differential operator D_c from $C_K^n(I)$ (n-times continuously differentiable K-valued functions defined on I) into $C_K(I)$ (continuous functions from I to K), where $D_c(f) := c_n f^{(n)} + ... + c_1 f' + c_0 f$, with $f \in C_K^n(I)$. Let $\omega_c \in C_{\mathbb{C}}^n(\mathbb{R})$ be the unique solution of initial value problem:

$$D_c(\omega_c) = 0, \ \ \omega_c^{(i)}(0) = \delta_{l,n-1} \ \ (l \in \{0, 1, ..., n-1\}).$$

By Theorem 3.2 of [4], p. 1131, we have that

$$f(x) = (T_{a,c} f)(x) + \int_a^x D_c(f)(t)\,\omega_c(x-t)\,dt, \tag{1.1}$$

where

$$(T_{a,c} f)(x) := \sum_{j=0}^{n-1} \left(f^{(j)}(a) \sum_{i=0}^{n-1-j} c_{i+j+1} \omega_c^{(i)}(x-a) \right), \tag{1.2}$$

for all $x, a \in I$.

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Next, let $k \in \mathbb{N}_0$, $k < n$, $\gamma \in \mathbb{R}$, and we consider $\mathcal{J}_{n,k,\gamma}$ from $\mathbb{R}$ into $\mathbb{R}$, given by

$$\mathcal{J}_{n,k,\gamma}(t) := \sum_{i=0}^{\infty} \frac{\gamma^i t^{i(n-k)+n}}{(i(n-k)+n)!}, \tag{1.3}$$

which converges for all t.

A further interpretation of (1.1), see Theorem 3.3, p. 1132 of [4], given us

$$f(x) = \sum_{j=0}^{k-1} f^{(j)}(a) \frac{(x-a)^j}{j!} + \sum_{j=k}^{n-1} f^{(j)}(a) \mathcal{J}_{n,k,\gamma}^{(n-j)}(x-a)$$
$$+ \int_a^x \left(f^{(n)}(t) - \gamma f^{(k)}(t)\right) \mathcal{J}'_{n,k,\gamma}(x-t)\, dt, \tag{1.4}$$

for all $f \in C_K^n(I)$, $n \in \mathbb{N}$ and $x, a \in I$.

Applications of (1.4) provide us the following interesting Taylor formulae of trigonometric and hyperbolic types:

Theorem 1.1. *([4]) For all $f \in C_K^2(I)$ and $a, x \in I$, we have*

$$f(x) = f(a)\cos(x-a) + f'(a)\sin(x-a) + \int_a^x (f''(t) + f(t))\sin(x-t)\, dt. \tag{1.5}$$

Theorem 1.2. *([4]) For all $f \in C_K^2(I)$ and $a, x \in I$, we have*

$$f(x) = f(a)\cosh(x-a) + f'(a)\sinh(x-a) + \int_a^x (f''(t) - f(t))\sinh(x-t)\, dt. \tag{1.6}$$

Theorem 1.3. *([4]) For all $f \in C_K^4(I)$ and $a, x \in I$, we have*

$$f(x) = f(a)\left(\frac{\cosh(x-a) + \cos(x-a)}{2}\right) + f'(a)\left(\frac{\sinh(x-a) + \sin(x-a)}{2}\right)$$
$$+ f''(a)\left(\frac{\cosh(x-a) - \cos(x-a)}{2}\right) + f'''(a)\left(\frac{\sinh(x-a) - \sin(x-a)}{2}\right)$$
$$+ \int_a^x (f''''(t) - f(t))\left(\frac{\sinh(x-t) - \sin(x-t)}{2}\right) dt. \tag{1.7}$$

Theorem 1.4. *([4]) Let $\alpha, \beta \in \mathbb{R}$ with $\alpha\beta\left(\alpha^2 - \beta^2\right) \neq 0$. Then, for all $f \in C_K^4(I)$ and $a, x \in I$, we have*

$$f(x) = f(a)\left(\frac{\beta^2\cos(\alpha(x-a)) - \alpha^2\cos(\beta(x-a))}{\beta^2 - \alpha^2}\right) +$$
$$f'(a)\left(\frac{\beta^3\sin(\alpha(x-a)) - \alpha^3\sin(\beta(x-a))}{\alpha\beta(\beta^2 - \alpha^2)}\right) +$$
$$f''(a)\left(\frac{\cos(\alpha(x-a)) - \cos(\beta(x-a))}{\beta^2 - \alpha^2}\right) +$$
$$f'''(a)\left(\frac{\beta\sin(\alpha(x-a)) - \alpha\sin(\beta(x-a))}{\alpha\beta(\beta^2 - \alpha^2)}\right) +$$
$$\int_a^x \left(f''''(t) + \left(\alpha^2 + \beta^2\right) f''(t) + \alpha^2\beta^2 f(t)\right)\left(\frac{\beta\sin(\alpha(x-t)) - \alpha\sin(\beta(x-t))}{\alpha\beta(\beta^2 - \alpha^2)}\right) dt. \tag{1.8}$$

Finally, we include the following result.

Theorem 1.5. *([4]) Let $\alpha,\beta\in\mathbb{R}$ with $\alpha\beta\left(\alpha^2-\beta^2\right)\neq 0$. Then, for all $f\in C_K^4(I)$ and $a,x\in I$, we have*

$$f(x)=f(a)\left(\frac{\beta^2\cosh(\alpha(x-a))-\alpha^2\cosh(\beta(x-a))}{\beta^2-\alpha^2}\right)+$$
$$f'(a)\left(\frac{\beta^3\sinh(\alpha(x-a))-\alpha^3\sinh(\beta(x-a))}{\alpha\beta(\beta^2-\alpha^2)}\right)+$$
$$f''(a)\left(\frac{\cosh(\beta(x-a))-\cosh(\alpha(x-a))}{\beta^2-\alpha^2}\right)+$$
$$f'''(a)\left(\frac{\alpha\sinh(\beta(x-a))-\beta\sinh(\alpha(x-a))}{\alpha\beta(\beta^2-\alpha^2)}\right)+$$
$$\int_a^x\left(f''''(t)-\left(\alpha^2+\beta^2\right)f''(t)+\alpha^2\beta^2f(t)\right)\left(\frac{\alpha\sinh(\beta(x-t))-\beta\sinh(\alpha(x-t))}{\alpha\beta(\beta^2-\alpha^2)}\right)dt. \tag{1.9}$$

In this article first we give the needed variants to Theorems 1.1-1.5, then based on these modified Taylor formulae we derive several Ostrowski type inequalities. In between we deal with Opial type inequalities.

2. Main Results

We are inspired by [1]-[3].

We give the following Taylor formula results of trigonometric and hyperbolic types.

Theorem 2.1. *For all $f\in C_K^2(I)$ and $a,x\in I$, we have*

$$f(x)-f(a)=f'(a)\sin(x-a)+2f''(a)\sin^2\left(\frac{x-a}{2}\right)+$$
$$\int_a^x\left[\left(f''(t)+f(t)\right)-\left(f''(a)+f(a)\right)\right]\sin(x-t)\,dt=$$
$$f'(a)\sin(x-a)+2f''(a)\sin^2\left(\frac{x-a}{2}\right)+ \tag{2.1}$$
$$\int_a^x\left[\left(f''(t)-f''(a)\right)+\left(f(t)-f(a)\right)\right]\sin(x-t)\,dt.$$

Proof. Here we use Theorem 1.1.

We have by (1.5) that

$$f(x)-f(a)=f(a)\left(\cos(x-a)-1\right)+f'(a)\sin(x-a)$$
$$+\int_a^x\left(f''(t)+f(t)\right)\sin(x-t)\,dt. \tag{2.2}$$

(By $\cos 2x=1-2\sin^2 x$, $\cos 2x-1=-2\sin^2 x$, and $\cos(x-a)-1=\cos 2\left(\frac{x-a}{2}\right)-1=-2\sin^2\left(\frac{x-a}{2}\right)$.)

Therefore it holds

$$f(x)-f(a)=-2f(a)\sin^2\left(\frac{x-a}{2}\right)+f'(a)\sin(x-a)$$
$$+\int_a^x\left(f''(t)+f(t)\right)\sin(x-t)\,dt= \tag{2.3}$$

$$-2f(a)\sin^2\left(\frac{x-a}{2}\right)+f'(a)\sin(x-a)+(f''(a)+f(a))(1-\cos(x-a))$$

$$+\int_a^x [(f''(t)+f(t))-(f''(a)+f(a))]\sin(x-t)\,dt$$

$$=-2f(a)\sin^2\left(\frac{x-a}{2}\right)+f'(a)\sin(x-a)+(f''(a)+f(a))\,2\sin^2\left(\frac{x-a}{2}\right)+ \tag{2.4}$$

$$\int_a^x [(f''(t)+f(t))-(f''(a)+f(a))]\sin(x-t)\,dt$$

$$=f'(a)\sin(x-a)+2f''(a)\sin^2\left(\frac{x-a}{2}\right)+$$

$$\int_a^x [(f''(t)+f(t))-(f''(a)+f(a))]\sin(x-t)\,dt.$$

Above notice that $(\cos(x-t))'=\sin(x-t)$.

The claim is proved. ■

Theorem 2.2. *For all $f\in C_K^2(I)$ and $a,x\in I$, we have*

$$f(x)-f(a)=f'(a)\sinh(x-a)+2f''(a)\sinh^2\left(\frac{x-a}{2}\right)+$$

$$\int_a^x [(f''(t)-f(t))-(f''(a)-f(a))]\sinh(x-t)\,dt=$$

$$f'(a)\sinh(x-a)+2f''(a)\sinh^2\left(\frac{x-a}{2}\right)+ \tag{2.5}$$

$$\int_a^x [(f''(t)-f''(a))-(f(t)-f(a))]\sinh(x-t)\,dt.$$

Proof. Here we use Theorem 1.2 (1.6).

Notice that $(-\cosh(x-t))'=\sinh(x-t)$.

By $\cosh 2x-1=2\sinh^2 x$, we get that $\cosh(x-a)-1=\cosh 2\left(\frac{x-a}{2}\right)-1=2\sinh^2\left(\frac{x-a}{2}\right)$.

By (1.6) we obtain

$$f(x)-f(a)=f(a)(\cosh(x-a)-1)+f'(a)\sinh(x-a)$$

$$+\int_a^x (f''(t)-f(t))\sinh(x-t)\,dt=$$

$$2f(a)\sinh^2\left(\frac{x-a}{2}\right)+f'(a)\sinh(x-a) \tag{2.6}$$

$$+\int_a^x (f''(t)-f(t))\sinh(x-t)\,dt=$$

$$2f(a)\sinh^2\left(\frac{x-a}{2}\right)+f'(a)\sinh(x-a)+(f''(a)-f(a))(\cosh(x-a)-1)$$

$$+\int_a^x [(f''(t)-f(t))-(f''(a)-f(a))]\sinh(x-t)\,dt$$

$$=2f(a)\sinh^2\left(\frac{x-a}{2}\right)+f'(a)\sinh(x-a)+(f''(a)-f(a))\left(2\sinh^2\left(\frac{x-a}{2}\right)\right)+ \tag{2.7}$$

$$\int_a^x \left[\left(f''(t)-f(t)\right)-\left(f''(a)-f(a)\right)\right]\sinh(x-t)\,dt$$

$$=f'(a)\sinh(x-a)+2f''(a)\sinh^2\left(\frac{x-a}{2}\right)+$$

$$\int_a^x \left[\left(f''(t)-f(t)\right)-\left(f''(a)-f(a)\right)\right]\sinh(x-t)\,dt.$$

The claim is proved. ■

Theorem 2.3. *For all $f\in C_K^4(I)$ and $a,x\in I$, we have*

$$f(x)-f(a)=f'(a)\left(\frac{\sinh(x-a)+\sin(x-a)}{2}\right)+$$

$$f''(a)\left(\frac{\cosh(x-a)-\cos(x-a)}{2}\right)+$$

$$f'''(a)\left(\frac{\sinh(x-a)-\sin(x-a)}{2}\right)+$$

$$f''''(a)\left(\sinh^2\left(\frac{x-a}{2}\right)-\sin^2\left(\frac{x-a}{2}\right)\right)+ \tag{2.8}$$

$$\int_a^x \left[\left(f''''(t)-f''''(a)\right)-\left(f(t)-f(a)\right)\right]\left(\frac{\sinh(x-t)-\sin(x-t)}{2}\right)dt.$$

Proof. Here we use Theorem 1.3 (1.7).

We have that

$$f(x)-f(a)=f(a)\left(\frac{(\cosh(x-a)-1)+(\cos(x-a)-1)}{2}\right)+$$

$$f'(a)\left(\frac{\sinh(x-a)+\sin(x-a)}{2}\right)+$$

$$f''(a)\left(\frac{\cosh(x-a)-\cos(x-a)}{2}\right)+f'''(a)\left(\frac{\sinh(x-a)-\sin(x-a)}{2}\right) \tag{2.9}$$

$$+\int_a^x \left(f''''(t)-f(t)\right)\left(\frac{\sinh(x-t)-\sin(x-t)}{2}\right)dt=$$

$$f(a)\left(\sinh^2\left(\frac{x-a}{2}\right)-\sin^2\left(\frac{x-a}{2}\right)\right)+$$

$$f'(a)\left(\frac{\sinh(x-a)+\sin(x-a)}{2}\right)+ \tag{2.10}$$

$$f''(a)\left(\frac{\cosh(x-a)-\cos(x-a)}{2}\right)+f'''(a)\left(\frac{\sinh(x-a)-\sin(x-a)}{2}\right)$$

$$+\left(f''''(a)-f(a)\right)\left(\sinh^2\left(\frac{x-a}{2}\right)-\sin^2\left(\frac{x-a}{2}\right)\right)+$$

$$\int_a^x \left[\left(f''''(t)-f(t)\right)-\left(f''''(a)-f(a)\right)\right]\left(\frac{\sinh(x-t)-\sin(x-t)}{2}\right)dt=$$

$$f'(a)\left(\frac{\sinh(x-a)+\sin(x-a)}{2}\right)+$$

$$f''(a)\left(\frac{\cosh(x-a)-\cos(x-a)}{2}\right)+f'''(a)\left(\frac{\sinh(x-a)-\sin(x-a)}{2}\right)$$
$$+f''''(a)\left(\sinh^2\left(\frac{x-a}{2}\right)-\sin^2\left(\frac{x-a}{2}\right)\right)+ \tag{2.11}$$
$$\int_a^x\left[\left(f''''(t)-f''''(a)\right)-\left(f(t)-f(a)\right)\right]\left(\frac{\sinh(x-t)-\sin(x-t)}{2}\right)dt.$$

The claim is proved. ■

We continue with

Theorem 2.4. *Let $\alpha,\beta\in\mathbb{R}$ with $\alpha\beta\left(\alpha^2-\beta^2\right)\neq 0$. Then, for all $f\in C_K^4(I)$ and $a,x\in I$, we have*

$$f(x)-f(a)=f'(a)\left(\frac{\beta^3\sin(\alpha(x-a))-\alpha^3\sin(\beta(x-a))}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)+$$
$$f''(a)\left(\frac{\cos(\alpha(x-a))-\cos(\beta(x-a))}{\beta^2-\alpha^2}\right)+$$
$$f'''(a)\left(\frac{\beta\sin(\alpha(x-a))-\alpha\sin(\beta(x-a))}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)+$$
$$\frac{2\left(f''''(a)+\left(\alpha^2+\beta^2\right)f''(a)\right)}{(\alpha\beta)^2\left(\beta^2-\alpha^2\right)}\left(\beta^2\sin^2\left(\frac{\alpha(x-a)}{2}\right)-\alpha^2\sin^2\left(\frac{\beta(x-a)}{2}\right)\right)+ \tag{2.12}$$
$$\frac{1}{\alpha\beta\left(\beta^2-\alpha^2\right)}\int_a^x\left[\left(f''''(t)+\left(\alpha^2+\beta^2\right)f''(t)+\alpha^2\beta^2f(t)\right)-\right.$$
$$\left.\left(f''''(a)+\left(\alpha^2+\beta^2\right)f''(a)+\alpha^2\beta^2f(a)\right)\right]\left[\beta\sin(\alpha(x-t))-\alpha\sin(\beta(x-t))\right]dt.$$

Proof. We see that

$$I^*:=f(a)\left(\frac{\beta^2\left(\cos(\alpha(x-a))-1\right)-\alpha^2\left(\cos(\beta(x-a))-1\right)}{\beta^2-\alpha^2}\right)=$$
$$f(a)\left(\frac{\beta^2\left(\cos(\alpha(x-a))\right)-\beta^2-\alpha^2\left(\cos(\beta(x-a))\right)+\alpha^2}{\beta^2-\alpha^2}\right)=$$
$$f(a)\left[\left(\frac{\beta^2\left(\cos(\alpha(x-a))\right)-\alpha^2\left(\cos(\beta(x-a))\right)}{\beta^2-\alpha^2}\right)-\left(\frac{\beta^2-\alpha^2}{\beta^2-\alpha^2}\right)\right]=$$
$$f(a)\left[\left(\frac{\beta^2\left(\cos(\alpha(x-a))\right)-\alpha^2\left(\cos(\beta(x-a))\right)}{\beta^2-\alpha^2}\right)-1\right]= \tag{2.13}$$
$$f(a)\left(\frac{\beta^2\left(\cos(\alpha(x-a))\right)-\alpha^2\left(\cos(\beta(x-a))\right)}{\beta^2-\alpha^2}\right)-f(a).$$

That is

$$I^*=f(a)\left(\frac{-2\beta^2\sin^2\left(\frac{\alpha(x-a)}{2}\right)+2\alpha^2\sin^2\left(\frac{\beta(x-a)}{2}\right)}{\beta^2-\alpha^2}\right)= \tag{2.14}$$
$$f(a)\left(\frac{\beta^2\left(\cos(\alpha(x-a))\right)-\alpha^2\left(\cos(\beta(x-a))\right)}{\beta^2-\alpha^2}\right)-f(a).$$

By Theorem 1.4, we obtain

$$f(x)-f(a)=f(a)\left(\frac{-2\beta^2\sin^2\left(\frac{\alpha(x-a)}{2}\right)+2\alpha^2\sin^2\left(\frac{\beta(x-a)}{2}\right)}{\beta^2-\alpha^2}\right)+$$

$$f'(a)\left(\frac{\beta^3\sin(\alpha(x-a))-\alpha^3\sin(\beta(x-a))}{\alpha\beta(\beta^2-\alpha^2)}\right)+$$

$$f''(a)\left(\frac{\cos(\alpha(x-a))-\cos(\beta(x-a))}{\beta^2-\alpha^2}\right)+$$

$$f'''(a)\left(\frac{\beta\sin(\alpha(x-a))-\alpha\sin(\beta(x-a))}{\alpha\beta(\beta^2-\alpha^2)}\right)+ \tag{2.15}$$

$$\frac{\left(f''''(a)+\left(\alpha^2+\beta^2\right)f''(a)+\alpha^2\beta^2 f(a)\right)}{(\beta^2-\alpha^2)}$$

$$\left(\frac{2\beta}{\alpha(\alpha\beta)}\sin^2\left(\frac{\alpha(x-a)}{2}\right)-\frac{2\alpha}{\beta(\alpha\beta)}\sin^2\left(\frac{\beta(x-a)}{2}\right)\right)+$$

$$\frac{1}{\alpha\beta(\beta^2-\alpha^2)}\int_a^x\left[\left(f''''(t)+\left(\alpha^2+\beta^2\right)f''(t)+\alpha^2\beta^2 f(t)\right)-\right.$$

$$\left.\left(f''''(a)+\left(\alpha^2+\beta^2\right)f''(a)+\alpha^2\beta^2 f(a)\right)\right]\left[\beta\sin(\alpha(x-t))-\alpha\sin(\beta(x-t))\right]dt.$$

Notice that

$$(\cos(\alpha(x-t)))'=-\sin(\alpha(x-t))(-\alpha)=\alpha\sin(\alpha(x-t)), \tag{2.16}$$

and

$$\left(\frac{1}{\alpha}\cos(\alpha(x-t))\right)'=\sin(\alpha(x-t)), \tag{2.17}$$

and

$$\int_a^x\beta(\sin(\alpha(x-t)))\,dt=\frac{\beta}{\alpha}(\cos(\alpha(x-t)))|_a^x=\frac{\beta}{\alpha}(1-\cos(\alpha(x-a)))$$

$$=\frac{2\beta}{\alpha}\left(\sin^2\left(\frac{\alpha(x-a)}{2}\right)\right). \tag{2.18}$$

Similarly, we get

$$-\int_a^x\alpha\sin(\beta(x-t))\,dt=-\frac{2\alpha}{\beta}\left(\sin^2\left(\frac{\beta(x-a)}{2}\right)\right). \tag{2.19}$$

Furthermore, we see that

$$\frac{\alpha^2\beta^2 f(a)}{(\beta^2-\alpha^2)}\left(\frac{2}{\alpha^2}\sin^2\left(\frac{\alpha(x-a)}{2}\right)-\frac{2}{\beta^2}\sin^2\left(\frac{\beta(x-a)}{2}\right)\right)=$$

$$\frac{f(a)}{(\beta^2-\alpha^2)}\left(2\beta^2\sin^2\left(\frac{\alpha(x-a)}{2}\right)-2\alpha^2\sin^2\left(\frac{\beta(x-a)}{2}\right)\right). \tag{2.20}$$

The claim is proved. ■

We give

Theorem 2.5. *Let $\alpha, \beta \in \mathbb{R}$ and $\alpha\beta\left(\alpha^2-\beta^2\right) \neq 0$. Then, for all $f \in C_K^4(I)$ and $a, x \in I$, we have*

$$
\begin{gathered}
f(x)-f(a)=f'(a)\left(\frac{\beta^3 \sinh(\alpha(x-a))-\alpha^3 \sinh(\beta(x-a))}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)+ \\
f''(a)\left(\frac{\cosh(\beta(x-a))-\cosh(\alpha(x-a))}{\beta^2-\alpha^2}\right)+ \\
f'''(a)\left(\frac{\alpha \sinh(\beta(x-a))-\beta \sinh(\alpha(x-a))}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)+ \\
\frac{2\left(f''''(a)-\left(\alpha^2+\beta^2\right) f''(a)\right)}{(\alpha\beta)^2\left(\beta^2-\alpha^2\right)}\left(\alpha^2 \sinh^2\left(\frac{\beta(x-a)}{2}\right)-\beta^2 \sin^2\left(\frac{\alpha(x-a)}{2}\right)\right)+ \\
\frac{1}{\alpha\beta\left(\beta^2-\alpha^2\right)} \int_a^x \left[\left(f''''(t)-\left(\alpha^2+\beta^2\right) f''(t)+\alpha^2\beta^2 f(t)\right)-\right. \\
\left.\left(f''''(a)-\left(\alpha^2+\beta^2\right) f''(a)+\alpha^2\beta^2 f(a)\right)\right][\alpha \sinh(\beta(x-t))-\beta \sinh(\alpha(x-t))]\, dt.
\end{gathered} \tag{2.21}
$$

Proof. We see that

$$
\begin{gathered}
J:=f(a)\left(\frac{\beta^2(\cosh(\alpha(x-a))-1)-\alpha^2(\cosh(\beta(x-a))-1)}{\beta^2-\alpha^2}\right)= \\
f(a)\left(\frac{\beta^2(\cosh(\alpha(x-a)))-\alpha^2(\cosh(\beta(x-a)))}{\beta^2-\alpha^2}\right)-f(a).
\end{gathered} \tag{2.22}
$$

Hence

$$
\begin{gathered}
J=f(a)\left(\frac{2\beta^2 \sinh^2\left(\frac{\alpha(x-a)}{2}\right)-2\alpha^2 \sinh^2\left(\frac{\beta(x-a)}{2}\right)}{\beta^2-\alpha^2}\right)= \\
f(a)\left(\frac{\beta^2 \cosh(\alpha(x-a))-\alpha^2 \cosh(\beta(x-a))}{\beta^2-\alpha^2}\right)-f(a).
\end{gathered} \tag{2.23}
$$

By Theorem 1.5, we obtain

$$
\begin{gathered}
f(x)-f(a)=f(a)\left(\frac{2\beta^2 \sinh^2\left(\frac{\alpha(x-a)}{2}\right)-2\alpha^2 \sinh^2\left(\frac{\beta(x-a)}{2}\right)}{\beta^2-\alpha^2}\right)+ \\
f'(a)\left(\frac{\beta^3 \sinh(\alpha(x-a))-\alpha^3 \sinh(\beta(x-a))}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)+ \\
f''(a)\left(\frac{\cosh(\beta(x-a))-\cosh(\alpha(x-a))}{\beta^2-\alpha^2}\right)+ \\
f'''(a)\left(\frac{\alpha \sinh(\beta(x-a))-\beta \sinh(\alpha(x-a))}{\alpha\beta\left(\beta^2-\alpha^2\right)}\right)+ \\
\frac{\left(f''''(a)-\left(\alpha^2+\beta^2\right) f''(a)+\alpha^2\beta^2 f(a)\right)}{\left(\beta^2-\alpha^2\right)} \\
\left(\frac{2\alpha}{\beta(\alpha\beta)} \sinh^2\left(\frac{\beta(x-a)}{2}\right)-\frac{2\beta}{\alpha(\alpha\beta)} \sinh^2\left(\frac{\alpha(x-a)}{2}\right)\right)+ \\
\frac{1}{\alpha\beta\left(\beta^2-\alpha^2\right)} \int_a^x \left[\left(f''''(t)-\left(\alpha^2+\beta^2\right) f''(t)+\alpha^2\beta^2 f(t)\right)-\right.
\end{gathered} \tag{2.24}
$$

$$\left(f''''(a)-\left(\alpha^2+\beta^2\right)f''(a)+\alpha^2\beta^2 f(a)\right)\right](\alpha\sinh(\beta(x-t))-\beta\sinh(\alpha(x-t)))\,dt.$$

Notice that

$$(\cosh(\beta(x-t)))'=\sinh(\beta(x-t))(\beta x-\beta t)'= \sinh(\beta(x-t))(-\beta)=-\beta\sinh(\beta(x-t)), \tag{2.25}$$

that is

$$\left(-\frac{1}{\beta}\cosh(\beta(x-t))\right)'=\sinh(\beta(x-t)), \tag{2.26}$$

and

$$\left(-\frac{\alpha}{\beta}\cosh(\beta(x-t))\right)'=\alpha\sinh(\beta(x-t)). \tag{2.27}$$

Thus, it holds

$$\int_a^x \alpha(\sinh(\beta(x-t)))\,dt=-\frac{\alpha}{\beta}(\cosh(\beta(x-t)))|_a^x= -\frac{\alpha}{\beta}(1-\cosh(\beta(x-a)))=-\frac{\alpha}{\beta}(-2)\sinh^2\left(\frac{\beta(x-a)}{2}\right)=\frac{2\alpha}{\beta}\sinh^2\left(\frac{\beta(x-a)}{2}\right). \tag{2.28}$$

I.e.

$$\int_a^x \alpha\sinh(\beta(x-t))\,dt=\frac{2\alpha}{\beta}\sinh^2\left(\frac{\beta(x-a)}{2}\right), \tag{2.29}$$

and

$$\int_a^x (-\beta)\sinh(\alpha(x-t))\,dt=-\frac{2\beta}{\alpha}\sinh^2\left(\frac{\alpha(x-a)}{2}\right). \tag{2.30}$$

Furthermore

$$\frac{\alpha^2\beta^2 f(a)}{(\beta^2-\alpha^2)}\left(\frac{2}{\beta^2}\sinh^2\left(\frac{\beta(x-a)}{2}\right)-\frac{2}{\alpha^2}\sinh^2\left(\frac{\alpha(x-a)}{2}\right)\right)= \frac{f(a)}{(\beta^2-\alpha^2)}\left(2\alpha^2\sinh^2\left(\frac{\beta(x-a)}{2}\right)-2\beta^2\sinh^2\left(\frac{\alpha(x-a)}{2}\right)\right). \tag{2.31}$$

The claim is proved. ■

Next come Opial type inequalilies, for basics see [2].

Theorem 2.6. *Let $f\in C_K^2(I)$, with interval $I\subset\mathbb{R}$, $a,x\in I$, $a<x$, and $f(a)=f'(a)=0$, with $p,q>1$: $\frac{1}{p}+\frac{1}{q}=1$. Then*

$$\int_a^x |f(w)|\,|f''(w)+f(w)|\,dw\le 2^{-\frac{1}{q}}\left(\int_a^x\left(\int_a^w|\sin(w-t)|^p\,dt\right)dw\right)^{\frac{1}{p}}\left(\int_a^x|f''(w)+f(w)|^q\,dw\right)^{\frac{2}{q}}. \tag{2.32}$$

Proof. By Theorem 1.1 we have

$$f(x)=\int_a^x (f''(t)+f(t))\sin(x-t)\,dt, \text{ and } f(w)=\int_a^w (f''(t)+f(t))\sin(w-t)\,dt, \tag{2.33}$$

for $a\le w\le x$.

By Hölder's inequality we have

$$|f(w)|\le\int_a^w |f''(t)+f(t)|\,|\sin(w-t)|\,dt\le$$

$$\left(\int_a^w |\sin(w-t)|^p\, dt\right)^{\frac{1}{p}} \left(\int_a^w |f''(t)+f(t)|^q\, dt\right)^{\frac{1}{q}}. \tag{2.34}$$

Call

$$z(w) := \int_a^w |f''(t)+f(t)|^q\, dt, \ z(a)=0, \ a \le w \le x. \tag{2.35}$$

Then

$$z'(w) = |f''(w)+f(w)|^q, \tag{2.36}$$

and

$$|f''(w)+f(w)| = (z'(w))^{\frac{1}{q}}, \ \text{ all } a \le w \le x.$$

Therefore we have (all $a \le w \le x$)

$$|f(w)|\,|f''(w)+f(w)| \le$$

$$\left(\int_a^w |\sin(w-t)|^p\, dt\right)^{\frac{1}{p}} (z(w)\, z'(w))^{\frac{1}{q}}. \tag{2.37}$$

hence it holds

$$\int_a^x |f(w)|\,|f''(w)+f(w)|\, dw \le$$

$$\int_a^x \left(\int_a^w |\sin(w-t)|^p\, dt\right)^{\frac{1}{p}} (z(w)\, z'(w))^{\frac{1}{q}}\, dw \le$$

$$\left(\int_a^x \left(\int_a^w |\sin(w-t)|^p\, dt\right) dw\right)^{\frac{1}{p}} \left(\int_a^x z(w)\, z'(w)\, dw\right)^{\frac{1}{q}} = \tag{2.38}$$

$$\left(\int_a^x \left(\int_a^w |\sin(w-t)|^p\, dt\right) dw\right)^{\frac{1}{p}} \left(\int_a^x z(w)\, dz(w)\right)^{\frac{1}{q}} =$$

$$\left(\int_a^x \left(\int_a^w |\sin(w-t)|^p\, dt\right) dw\right)^{\frac{1}{p}} \frac{z(x)^{\frac{2}{q}}}{2^{\frac{1}{q}}} =$$

$$\left(\int_a^x \left(\int_a^w |\sin(w-t)|^p\, dt\right) dw\right)^{\frac{1}{p}} \frac{\left(\int_a^x |f''(t)+f(t)|^q\, dt\right)^{\frac{2}{q}}}{2^{\frac{1}{q}}}. \tag{2.39}$$

The claim is proved. ■

Next come several similar results.

Theorem 2.7. *Let $f \in C_K^2(I)$, $a, x \in I$, $a < x$, and $f(a) = f'(a) = 0$, with $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\int_a^x |f(w)|\,|f''(w)-f(w)|\, dw \le$$

$$2^{-\frac{1}{q}} \left(\int_a^x \left(\int_a^w |\sinh(w-t)|^p\, dt\right) dw\right)^{\frac{1}{p}} \left(\int_a^x |f''(w)-f(w)|^q\, dw\right)^{\frac{2}{q}}. \tag{2.40}$$

Proof. By Theorem 2.6, use of (1.6). ■

Theorem 2.8. *Let $f \in C_K^4(I)$, interval $I \subset \mathbb{R}$, let $a, x \in I$, $a < x$, $f(a) = f'(a) = f''(a) = f'''(a) = 0$, with $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\int_a^x |f(w)| \left| f^{(iv)}(w) - f(w) \right| dw \leq$$

$$2^{-\left(1+\frac{1}{q}\right)} \left(\int_a^x \left(\int_a^w |\sinh(w-t) - \sin(w-t)|^p dt \right) dw \right)^{\frac{1}{p}}$$

$$\left(\int_a^x \left| f^{(iv)}(w) - f(w) \right|^q dw \right)^{\frac{2}{q}}. \tag{2.41}$$

Proof. As in Theorem 2.6, use of (1.7). ■

Theorem 2.9. *All as in Theorem 2.8. Let $\alpha, \beta \in \mathbb{R} : \alpha\beta\left(\alpha^2 - \beta^2\right) \neq 0$. Then*

$$\int_a^x |f(w)| \left| f^{(4)}(w) + \left(\alpha^2 + \beta^2\right) f''(w) + \alpha^2\beta^2 f(w) \right| dw \leq$$

$$\frac{1}{2^{\frac{1}{q}} \alpha\beta\left(\beta^2 - \alpha^2\right)} \left(\int_a^x \left(\int_a^w |\beta \sin(\alpha(w-t)) - \alpha \sin(\beta(w-t))|^p dt \right) dw \right)^{\frac{1}{p}}$$

$$\left(\int_a^x \left| f^{(4)}(w) + \left(\alpha^2 + \beta^2\right) f''(w) + \alpha^2\beta^2 f(w) \right|^q dw \right)^{\frac{2}{q}}. \tag{2.42}$$

Proof. As in Theorem 2.6, use of (1.8). ■

Theorem 2.10. *All as in Theorem 2.8. Let $\alpha \in \mathbb{R}$, $\alpha \neq 0$. Then*

$$\int_a^x |f(w)| \left| f^{(4)}(w) + 2\alpha^2 f''(w) + \alpha^4 f(w) \right| dw \leq$$

$$\frac{1}{2^{\frac{1}{q}+1} \alpha^3} \left(\int_a^x \left(\int_a^w |\sin(\alpha(w-t)) - \alpha(w-t)\cos(\alpha(w-t))|^p dt \right) dw \right)^{\frac{1}{p}} \tag{2.43}$$

$$\left(\int_a^x \left| f^{(4)}(w) + 2\alpha^2 f''(w) + \alpha^4 f(w) \right|^q dw \right)^{\frac{2}{q}}.$$

Proof. As in Theorem 2.6, use of Corollary 3.8 of [4], p. 1135. ■

Theorem 2.11. *All as in Theorem 2.9. Then*

$$\int_a^x |f(w)| \left| f^{(4)}(w) - \left(\alpha^2 + \beta^2\right) f''(w) + \alpha^2\beta^2 f(w) \right| dw \leq$$

$$\frac{1}{2^{\frac{1}{q}} \alpha\beta\left(\beta^2 - \alpha^2\right)} \left(\int_a^x \left(\int_a^w |\alpha \sinh(\beta(w-t)) - \beta \sinh(\alpha(w-t))|^p dt \right) dw \right)^{\frac{1}{p}} \tag{2.44}$$

$$\left(\int_a^x \left| f^{(4)}(w) - \left(\alpha^2 + \beta^2\right) f''(w) + \alpha^2\beta^2 f(w) \right|^q dw \right)^{\frac{2}{q}}.$$

Proof. As in Theorem 2.6, use of (1.9). ■

Theorem 2.12. *All as in Theorem 2.10. Then*

$$\int_a^x |f(w)| \left| f^{(4)}(w) - 2\alpha^2 f''(w) + \alpha^4 f(w) \right| dw \leq$$

$$\frac{1}{2^{\frac{1}{q}+1}\alpha^3} \left(\int_a^x \left(\int_a^w |\alpha(w-t)\cosh(\alpha(w-t)) - \sinh(\alpha(w-t))|^p \, dt \right) dw \right)^{\frac{1}{p}} \tag{2.45}$$

$$\left(\int_a^x \left| f^{(4)}(w) - 2\alpha^2 f''(w) + \alpha^4 f(w) \right|^q dw \right)^{\frac{2}{q}}.$$

Proof. As in Theorem 2.6, use of Corollary 3.10 of [4], p. 1135. ■

We finish Opial type inequalities with the following general result.

Theorem 2.13. *Let $n \in \mathbb{N}$, $c = (c_0, ..., c_n) \in K^{n+1}$ with $c_n = 1$, $f \in C_K^n(I)$ and $x, a \in I$, $a < x$, with interval $I \subset \mathbb{R}$. Let, also $p, q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Assume further that $f^{(j)}(a) = 0$, $j = 0, 1, ..., n-1$. Then*

$$\int_a^x |f(w)| |D_c(f)(w)| \, dw \leq \tag{2.46}$$

$$2^{-\frac{1}{q}} \left(\int_a^x \left(\int_a^w |\omega_c(w-t)|^p \, dt \right) dw \right)^{\frac{1}{p}} \left(\int_a^x |D_c(f)(w)|^q \, dw \right)^{\frac{2}{q}}.$$

Proof. By Theorem 3.2 of [4], p. 1131, for $x, a \in I$, we have

$$f(x) = (T_{a,c}f)(x) + \int_a^x D_c(f)(t)\,\omega_c(x-t)\,dt, \tag{2.47}$$

where

$$(T_{a,c}f)(x) := \sum_{j=0}^{n-1} \left(f^{(j)}(a) \sum_{i=0}^{n-1-j} c_{i+j+1}\omega_c^{(i)}(x-a) \right). \tag{2.48}$$

Because $f^{(j)}(a) = 0$, $j = 0, 1, ..., n-1$, we get

$$f(x) = \int_a^x D_c(f)(t)\,\omega_c(x-t)\,dt. \tag{2.49}$$

The rest of the proof as similar to Theorem 2.6 is omitted. ■

Next we present Ostrowski type inequalities involving $\|\cdot\|_\infty$. For basics see [2].

Theorem 2.14. *Let $f \in C_K^3([c,d])$, $a \in [c,d]$, such that $f'(a) = f''(a) = 0$. Then*

1)

$$\left| \frac{1}{d-c} \int_c^d f(x)\,dx - f(a) \right| \leq \|f''' + f'\|_\infty \frac{\left[(d-a)^3 + (a-c)^3 \right]}{6(d-c)}. \tag{2.50}$$

2) When $f'\left(\frac{c+d}{2}\right) = f''\left(\frac{c+d}{2}\right) = 0$, and $a = \frac{c+d}{2}$, we get

$$\left| \frac{1}{d-c} \int_c^d f(x)\,dx - f\left(\frac{c+d}{2}\right) \right| \leq \|f' + f'''\|_\infty \frac{(d-c)^2}{24}. \tag{2.51}$$

Proof. By Theorem 2.1 (2.1) we have

$$f(x) - f(a) = \int_a^x \left[\left(f''(t) + f(t) \right) - \left(f''(a) + f(a) \right) \right] \sin(x-t)\, dt. \tag{2.52}$$

Let $x \geq a$, then

$$\begin{aligned} |f(x) - f(a)| &\leq \int_a^x \left| \left(f''(t) + f(t) \right) - \left(f''(a) + f(a) \right) \right| |\sin(x-t)|\, dt \\ &\leq \int_a^x \left| \left(f''(t) + f(t) \right) - \left(f''(a) + f(a) \right) \right| dt \\ &\leq \|f''' + f'\|_\infty \int_a^x (t-a)\, dt = \|f''' + f'\|_\infty \frac{(x-a)^2}{2}. \end{aligned} \tag{2.53}$$

Let $x < a$, then

$$-\left(f(x) - f(a) \right) = \int_x^a \left[\left(f''(a) + f(a) \right) - \left(f''(t) + f(t) \right) \right] \sin(x-t)\, dt.$$

Hence

$$|f(x) - f(a)| \leq \int_x^a \left| \left(f''(a) + f(a) \right) - \left(f''(t) + f(t) \right) \right| |\sin(x-t)|\, dt$$

$$\leq \|f''' + f'\|_\infty \int_x^a (a-t)\, dt = \|f''' + f'\|_\infty \frac{(a-t)^2}{2} \Big|_a^x = \|f''' + f'\|_\infty \frac{(a-x)^2}{2}. \tag{2.54}$$

Therefore

$$|f(x) - f(a)| \leq \frac{\|f''' + f'\|_\infty}{2} (x-a)^2, \tag{2.55}$$

$\forall\, x \in [c,d]$.

We observe that

$$\begin{aligned} &\left| \frac{1}{d-c} \int_c^d f(x)\, dx - f(a) \right| = \left| \frac{1}{d-c} \int_c^d \left(f(x) - f(a) \right) dx \right| \leq \\ &\frac{1}{d-c} \int_c^d |f(x) - f(a)|\, dx \leq \frac{1}{d-c} \left(\frac{\|f''' + f'\|_\infty}{2} \right) \int_c^d (x-a)^2\, dx = \\ &\frac{\|f''' + f'\|_\infty}{2(d-c)} \left[\int_c^a (a-x)^2\, dx + \int_a^d (x-a)^2\, dx \right] = \\ &\frac{\|f''' + f'\|_\infty}{6(d-c)} \left[(a-x)^3 \Big|_a^c + (x-a)^3 \Big|_a^d \right] = \\ &\frac{\|f''' + f'\|_\infty}{6(d-c)} \left[(a-c)^3 + (d-a)^3 \right] = \\ &\left(\frac{\|f''' + f'\|_\infty}{6(d-c)} \right) \left[(d-a)^3 + (a-c)^3 \right]. \end{aligned} \tag{2.56}$$

The claim is proved. ∎

Using $|\sin x| \leq |x|$, we obtain an alternative Ostrowski type inequality.

Theorem 2.15. *All as in Theorem 2.14. Then*

1)

$$\left|\frac{1}{d-c}\int_c^d f(x)\,dx - f(a)\right| \leq \|f''' + f'\|_\infty \frac{\left[(d-a)^4 + (a-c)^4\right]}{24(d-c)}; \tag{2.57}$$

2) When $f'\left(\frac{c+d}{2}\right) = f''\left(\frac{c+d}{2}\right) = 0$, $a = \frac{c+d}{2}$, *we get that*

$$\left|\frac{1}{d-c}\int_c^d f(x)\,dx - f\left(\frac{c+d}{2}\right)\right| \leq \|f' + f'''\|_\infty \frac{(d-c)^3}{192}. \tag{2.58}$$

Proof. Again by Theorem 2.1 (2.1) we have

$$f(x) - f(a) = \int_a^x \left[(f''(t) + f(t)) - (f''(a) + f(a))\right]\sin(x-t)\,dt. \tag{2.59}$$

Let $x \geq a$, then

$$|f(x) - f(a)| \leq \int_a^x |(f''(t) + f(t)) - (f''(a) + f(a))|\,|\sin(x-t)|\,dt$$

$$\leq \|f''' + f'\|_\infty \int_a^x (t-a)\,|\sin(x-t)|\,dt$$

$$\leq \|f''' + f'\|_\infty \int_a^x (x-t)^{2-1}(t-a)^{2-1}\,dt \tag{2.60}$$

$$\|f''' + f'\|_\infty \frac{\Gamma(2)\Gamma(2)}{\Gamma(4)}(x-a)^3 = \|f''' + f'\|_\infty \frac{(x-a)^3}{3!}.$$

That is

$$|f(x) - f(a)| \leq \|f''' + f'\|_\infty \frac{(x-a)^3}{3!}. \tag{2.61}$$

Let $x < a$, then

$$-(f(x) - f(a)) = \int_x^a \left[(f''(a) + f(a)) - (f''(t) + f(t))\right]\sin(x-t)\,dt. \tag{2.62}$$

Hence

$$|f(x) - f(a)| \leq \int_x^a |(f''(a) + f(a)) - (f''(t) + f(t))|\,|\sin(x-t)|\,dt$$

$$\leq \|f''' + f'\|_\infty \int_x^a (a-t)\,|x-t|\,dt = \|f''' + f'\|_\infty \int_x^a (a-t)^{2-1}(t-x)^{2-1}\,dt \tag{2.63}$$

$$= \|f''' + f'\|_\infty \frac{\Gamma(2)\Gamma(2)}{\Gamma(4)}(a-x)^3 = \|f''' + f'\|_\infty \frac{(a-x)^3}{3!}.$$

That is

$$|f(x) - f(a)| \leq \|f''' + f'\|_\infty \frac{(a-x)^3}{3!}. \tag{2.64}$$

Next we observe that

$$\left|\frac{1}{d-c}\int_c^d f(x)\,dx - f(a)\right| \leq \frac{1}{d-c}\int_c^d |f(x) - f(a)|\,dx$$

$$\leq \frac{\|f''' + f'\|_\infty}{3!\,(d-c)} \left[\int_c^a (a-x)^3\,dx + \int_a^d (x-a)^3\,dx \right]$$

$$= \frac{\|f''' + f'\|_\infty}{4!\,(d-c)} \left[(a-c)^4 + (d-a)^4 \right]. \tag{2.65}$$

The claim is proved. ■

We continue with more involved Ostrowski type inequalities.

Theorem 2.16. *Let $f \in C_K^5([c,d])$, $a, x \in [c,d]$, and assume that $f^{(i)}(a) = 0$, $i = 1,2,3,4$. Here $\alpha, \beta \in \mathbb{R}$: $\alpha\beta\left(\alpha^2 - \beta^2\right) \neq 0$. Then*

1)

$$\left| \frac{1}{d-c} \int_c^d f(x)\,dx - f(a) \right| \leq \tag{2.66}$$

$$\frac{(|\alpha| + |\beta|) \left\| f^{(5)} + \left(\alpha^2 + \beta^2\right) f^{(3)} + \alpha^2\beta^2 f' \right\|_\infty}{6\,|\alpha\beta\,(\beta^2 - \alpha^2)|\,(d-c)} \left[(d-a)^3 + (a-c)^3 \right],$$

2) When $f^{(i)}\left(\frac{c+d}{2}\right) = 0$, $i = 1,2,3,4$, we have:

$$\left| \frac{1}{d-c} \int_c^d f(x)\,dx - f\left(\frac{c+d}{2}\right) \right| \leq$$

$$\frac{(|\alpha| + |\beta|) \left\| f^{(5)} + \left(\alpha^2 + \beta^2\right) f^{(3)} + \alpha^2\beta^2 f' \right\|_\infty}{24\,|\alpha\beta\,(\beta^2 - \alpha^2)|} (d-c)^2. \tag{2.67}$$

Example 2.17. *(to (2.66)) Let $\alpha = 1$, $\beta = 10$, then*

$$\left| \frac{1}{d-c} \int_c^d f(x)\,dx - f(a) \right| \leq \tag{2.68}$$

$$\frac{\left\| f^{(5)} + 101 f^{(3)} + 100 f' \right\|_\infty}{540\,(d-c)} \left[(d-a)^3 + (a-c)^3 \right].$$

Example 2.18. *(to (2.67)) When $\alpha = 2$, $\beta = 1$, we get*

$$\left| \frac{1}{d-c} \int_c^d f(x)\,dx - f\left(\frac{c+d}{2}\right) \right| \leq \frac{\left\| f^{(5)} + 5 f^{(3)} + 4 f' \right\|_\infty}{48} (d-c)^2. \tag{2.69}$$

Proof. (of Theorem 2.16) By Theorem 2.4 (2.12) we have:

$$f(x) - f(a) = \frac{1}{\alpha\beta\,(\beta^2 - \alpha^2)}$$

$$\int_a^x \left[\left(f''''(t) + \left(\alpha^2 + \beta^2\right) f''(t) + \alpha^2\beta^2 f(t) \right) - \right.$$

$$\left. \left(f''''(a) + \left(\alpha^2 + \beta^2\right) f''(a) + \alpha^2\beta^2 f(a) \right) \right] \left[\beta \sin(\alpha(x-t)) - \alpha \sin(\beta(x-t)) \right] dt. \tag{2.70}$$

Let $x \geq a$, then

$$|f(x) - f(a)| \leq$$

$$\frac{(|\alpha| + |\beta|)}{|\alpha\beta\,(\beta^2 - \alpha^2)|} \left\| f^{(5)} + \left(\alpha^2 + \beta^2\right) f^{(3)} + \alpha^2\beta^2 f' \right\|_\infty \int_a^x (t-a)\,dt =$$

$$\frac{(|\alpha|+|\beta|)}{2\left|\alpha\beta\left(\beta^2-\alpha^2\right)\right|}\left\|f^{(5)}+\left(\alpha^2+\beta^2\right)f^{(3)}+\alpha^2\beta^2 f'\right\|_\infty (x-a)^2. \tag{2.71}$$

Let $x<a$, then

$$-(f(x)-f(a))=\frac{1}{\alpha\beta\left(\beta^2-\alpha^2\right)}$$

$$\int_x^a\left[\left(f''''(t)+\left(\alpha^2+\beta^2\right)f''(t)+\alpha^2\beta^2 f(t)\right)-\right.$$

$$\left.\left(f''''(a)+\left(\alpha^2+\beta^2\right)f''(a)+\alpha^2\beta^2 f(a)\right)\right]\left[\beta\sin(\alpha(x-t))-\alpha\sin(\beta(x-t))\right]dt. \tag{2.72}$$

Hence we have

$$|f(x)-f(a)|\leq$$

$$\frac{(|\alpha|+|\beta|)}{\left|\alpha\beta\left(\beta^2-\alpha^2\right)\right|}\left\|f^{(5)}+\left(\alpha^2+\beta^2\right)f^{(3)}+\alpha^2\beta^2 f'\right\|_\infty\int_x^a(a-t)\,dt=$$

$$\frac{(|\alpha|+|\beta|)}{\left|\alpha\beta\left(\beta^2-\alpha^2\right)\right|}\left\|f^{(5)}+\left(\alpha^2+\beta^2\right)f^{(3)}+\alpha^2\beta^2 f'\right\|_\infty\frac{(a-x)^2}{2}. \tag{2.73}$$

Therefore it holds

$$|f(x)-f(a)|\leq\frac{(|\alpha|+|\beta|)\left\|f^{(5)}+\left(\alpha^2+\beta^2\right)f^{(3)}+\alpha^2\beta^2 f'\right\|_\infty}{2\left|\alpha\beta\left(\beta^2-\alpha^2\right)\right|}(x-a)^2, \tag{2.74}$$

$\forall\, x\in[c,d]$.

We observe that

$$\left|\frac{1}{d-c}\int_c^d f(x)\,dx-f(a)\right|\leq\frac{1}{d-c}\int_c^d|f(x)-f(a)|\,dx\leq$$

$$\frac{(|\alpha|+|\beta|)\left\|f^{(5)}+\left(\alpha^2+\beta^2\right)f^{(3)}+\alpha^2\beta^2 f'\right\|_\infty}{2\left|\alpha\beta\left(\beta^2-\alpha^2\right)\right|(d-c)}\int_c^d(x-a)^2\,dx=$$

$$\frac{(|\alpha|+|\beta|)\left\|f^{(5)}+\left(\alpha^2+\beta^2\right)f^{(3)}+\alpha^2\beta^2 f'\right\|_\infty}{6\left|\alpha\beta\left(\beta^2-\alpha^2\right)\right|(d-c)}\left[(a-c)^3+(d-a)^3\right]. \tag{2.75}$$

The claim is proved. ∎

A long alternative Ostrowski type inequality follows.

Theorem 2.19. *Let $f\in C_K^5([c,d])$, $a,x\in[c,d]$, and assume that $f^{(i)}(a)=0$, $i=1,2,3,4$. Here $\alpha,\beta\in\mathbb{R}$: $\alpha\beta\left(\alpha^2-\beta^2\right)\neq 0$. Then*

1)

$$\left|\frac{1}{d-c}\int_c^d f(x)\,dx-f(a)\right|\leq \tag{2.76}$$

$$\frac{\left\|f^{(5)}+\left(\alpha^2+\beta^2\right)f^{(3)}+\alpha^2\beta^2 f'\right\|_\infty}{12\left|\beta^2-\alpha^2\right|(d-c)}\left[(d-a)^4+(a-c)^4\right].$$

2) Above let $a=\frac{c+d}{2}$, then

$$\left|\frac{1}{d-c}\int_c^d f(x)\,dx-f\left(\frac{c+d}{2}\right)\right|\leq$$

$$\frac{\left\|f^{(5)}+\left(\alpha^2+\beta^2\right)f^{(3)}+\alpha^2\beta^2 f'\right\|_\infty}{96\left|\beta^2-\alpha^2\right|}(d-c)^3. \tag{2.77}$$

Example 2.20. *(to (2.76)) Let* $\alpha = 1$, $\beta = 10$, *then*

$$\left| \frac{1}{d-c} \int_c^d f(x)\,dx - f(a) \right| \leq \tag{2.78}$$

$$\frac{\left\| f^{(5)} + 101 f^{(3)} + 100 f' \right\|_\infty}{1188\,(d-c)} \left[(d-a)^4 + (a-c)^4 \right].$$

Example 2.21. *(to (2.77)) When* $\alpha = 2$, $\beta = 1$, *we get:*

$$\left| \frac{1}{d-c} \int_c^d f(x)\,dx - f\left(\frac{c+d}{2}\right) \right| \leq \frac{\left\| f^{(5)} + 5 f^{(3)} + 4 f' \right\|_\infty}{288} (d-c)^3. \tag{2.79}$$

Proof. (of Theorem 2.19) By Theorem 2.4 (2.12) we have:

$$f(x) - f(a) = \frac{1}{\alpha\beta\left(\beta^2-\alpha^2\right)}$$

$$\int_a^x \left[\left(f''''(t) + \left(\alpha^2+\beta^2\right) f''(t) + \alpha^2\beta^2 f(t) \right) - \right. \tag{2.80}$$

$$\left. \left(f''''(a) + \left(\alpha^2+\beta^2\right) f''(a) + \alpha^2\beta^2 f(a) \right) \right] \left[\beta \sin(\alpha(x-t)) - \alpha \sin(\beta(x-t)) \right] dt.$$

Let $x \geq a$ (by $|\sin x| \leq |x|$), then

$$|f(x) - f(a)| \leq \frac{1}{\left|\alpha\beta\left(\beta^2-\alpha^2\right)\right|} \left\| f^{(5)} + \left(\alpha^2+\beta^2\right) f^{(3)} + \alpha^2\beta^2 f' \right\|_\infty$$

$$\int_a^x (t-a) \left[|\beta| \left|\alpha(x-t)\right| + |\alpha| \left|\beta(x-t)\right| \right] dt \tag{2.81}$$

$$= \frac{2|\alpha\beta|}{\left|\alpha\beta\left(\beta^2-\alpha^2\right)\right|} \left\| f^{(5)} + \left(\alpha^2+\beta^2\right) f^{(3)} + \alpha^2\beta^2 f' \right\|_\infty \int_a^x (x-t)(t-a)\,dt$$

$$= \frac{2}{\left|\beta^2-\alpha^2\right|} \left\| f^{(5)} + \left(\alpha^2+\beta^2\right) f^{(3)} + \alpha^2\beta^2 f' \right\|_\infty \frac{(x-a)^3}{3!} \tag{2.82}$$

$$= \frac{1}{3\left|\beta^2-\alpha^2\right|} \left\| f^{(5)} + \left(\alpha^2+\beta^2\right) f^{(3)} + \alpha^2\beta^2 f' \right\|_\infty (x-a)^3.$$

So, when $x \geq a$, we get that

$$|f(x) - f(a)| \leq \frac{1}{3\left|\beta^2-\alpha^2\right|} \left\| f^{(5)} + \left(\alpha^2+\beta^2\right) f^{(3)} + \alpha^2\beta^2 f' \right\|_\infty (x-a)^3. \tag{2.83}$$

Let $x < a$, then

$$-(f(x) - f(a)) = \frac{1}{\alpha\beta\left(\beta^2-\alpha^2\right)}$$

$$\int_x^a \left[\left(f''''(t) + \left(\alpha^2+\beta^2\right) f''(t) + \alpha^2\beta^2 f(t) \right) - \right. \tag{2.84}$$

$$\left. \left(f''''(a) + \left(\alpha^2+\beta^2\right) f''(a) + \alpha^2\beta^2 f(a) \right) \right] \left[\beta \sin(\alpha(x-t)) - \alpha \sin(\beta(x-t)) \right] dt.$$

Hence (by $|\sin x| \leq |x|$), we get that

$$|f(x) - f(a)| \leq \frac{1}{\left|\alpha\beta\left(\beta^2-\alpha^2\right)\right|} \left\| f^{(5)} + \left(\alpha^2+\beta^2\right) f^{(3)} + \alpha^2\beta^2 f' \right\|_\infty$$

$$\int_x^a (a-t)\left[|\beta|\,|\alpha|\,(t-x)+|\alpha|\,|\beta|\,(t-x)\right]dt$$

$$= \frac{2}{|\beta^2-\alpha^2|}\left\|f^{(5)}+\left(\alpha^2+\beta^2\right)f^{(3)}+\alpha^2\beta^2 f'\right\|_\infty \int_x^a (a-t)(t-x)\,dt \tag{2.85}$$

$$= \frac{2}{|\beta^2-\alpha^2|}\left\|f^{(5)}+\left(\alpha^2+\beta^2\right)f^{(3)}+\alpha^2\beta^2 f'\right\|_\infty \frac{(a-x)^3}{3!}$$

$$= \frac{1}{3\,|\beta^2-\alpha^2|}\left\|f^{(5)}+\left(\alpha^2+\beta^2\right)f^{(3)}+\alpha^2\beta^2 f'\right\|_\infty (a-x)^3 .$$

So, when $x<a$, we got that

$$|f(x)-f(a)| \leq \frac{1}{3\,|\beta^2-\alpha^2|}\left\|f^{(5)}+\left(\alpha^2+\beta^2\right)f^{(3)}+\alpha^2\beta^2 f'\right\|_\infty (a-x)^3 . \tag{2.86}$$

We observe that

$$\left|\frac{1}{d-c}\int_c^d f(x)\,dx - f(a)\right| \leq \frac{1}{d-c}\int_c^d |f(x)-f(a)|\,dx \leq$$

$$\frac{1}{3\,|\beta^2-\alpha^2|\,(d-c)}\left\|f^{(5)}+\left(\alpha^2+\beta^2\right)f^{(3)}+\alpha^2\beta^2 f'\right\|_\infty$$

$$\left[\int_c^a (a-x)^3\,dx + \int_a^d (x-a)^3\,dx\right] = \tag{2.87}$$

$$\frac{\left\|f^{(5)}+\left(\alpha^2+\beta^2\right)f^{(3)}+\alpha^2\beta^2 f'\right\|_\infty}{12\,|\beta^2-\alpha^2|\,(d-c)}\left[(d-a)^4+(a-c)^4\right].$$

The claim is proved. ■

More Ostrowski type inequalities follow regarding $\|\cdot\|_p$, $p\geq 1$.

Theorem 2.22. *Let $f\in C_K^2([c,d])$, $a\in[c,d]$, and assume that $f'(a)=f''(a)=0$. Then*

$$\left|\frac{1}{d-c}\int_c^d f(x)\,dx - f(a)\right| \leq$$

$$\frac{\left[(a-c)\,\|f''+f-f(a)\|_{L_1([c,a])}+(d-a)\,\|f''+f-f(a)\|_{L_1([a,d])}\right]}{d-c}. \tag{2.88}$$

Proof. As in the proof of Theorem 2.14 (2.53):

let $x\geq a$, then

$$|f(x)-f(a)| \leq \int_a^x |f''(t)+f(t)-f(a)|\,dt \leq$$

$$\int_a^d |f''(t)+f(t)-f(a)|\,dt = \|f''+f-f(a)\|_{L_1([a,d])} . \tag{2.89}$$

Next let $x<a$, by (2.54) we get that

$$|f(x)-f(a)| \leq \int_x^a |f(a)-f''(t)-f(t)|\,dt \leq$$

$$\int_c^a |f''(t)+f(t)-f(a)|\,dt = \|f''+f-f(a)\|_{L_1([c,a])} . \tag{2.90}$$

Hence it holds

$$\left|\frac{1}{d-c}\int_c^d f(x)\,dx - f(a)\right| = \frac{1}{d-c}\left|\int_c^d (f(x)-f(a))\,dx\right| \leq$$

$$\frac{1}{d-c}\int_c^d |f(x)-f(a)|\,dx =$$

$$\frac{1}{d-c}\left[\int_c^a |f(x)-f(a)|\,dx + \int_a^d |f(x)-f(a)|\,dx\right] \leq$$

$$\frac{1}{d-c}\left[\|f''+f-f(a)\|_{L_1([c,a])}(a-c) + \|f''+f-f(a)\|_{L_1([a,d])}(d-a)\right], \tag{2.91}$$

proving the claim. ■

Theorem 2.23. *All as in Theorem 2.22. Then*

$$\left|\frac{1}{d-c}\int_c^d f(x)\,dx - f(a)\right| \leq$$

$$\frac{\left[(a-c)^2\|f''+f-f(a)\|_{L_1([c,a])} + (d-a)^2\|f''+f-f(a)\|_{L_1([a,d])}\right]}{2(d-c)}. \tag{2.92}$$

Proof. As in (2.60) for $x \geq a$, we get

$$|f(x)-f(a)| \leq \int_a^x |(f''(t)+f(t))-f(a)|\,|\sin(x-t)|dt \leq$$

$$\int_a^d |(f''(t)+f(t))-f(a)|\,(x-t)\,dt \leq (x-a)\|f''+f-f(a)\|_{L_1([a,d])}. \tag{2.93}$$

Also, as in (2.63) for $x < a$, we have

$$|f(x)-f(a)| \leq \int_x^a |f(a)-f''(t)-f(t)|\,|\sin(x-t)|dt \leq$$

$$\int_x^a |f(a)-f''(t)-f(t)|\,(t-x)\,dt \leq (a-x)\|f''+f-f(a)\|_{L_1([c,a])}. \tag{2.94}$$

Hence it holds (see (2.91))

$$\left|\frac{1}{d-c}\int_c^d f(x)\,dx - f(a)\right| \leq$$

$$\frac{1}{d-c}\left[\int_c^a |f(x)-f(a)|\,dx + \int_a^d |f(x)-f(a)|\,dx\right] \leq$$

$$\frac{1}{d-c}\left[\|f''+f-f(a)\|_{L_1([c,a])}\int_c^a (a-x)\,dx + \right.$$

$$\left.\|f''+f-f(a)\|_{L_1([a,d])}\int_a^d (x-a)\,dx\right] = \tag{2.95}$$

$$\frac{1}{d-c}\left[\|f''+f-f(a)\|_{L_1([c,a])}\frac{(a-c)^2}{2} + \|f''+f-f(a)\|_{L_1([a,d])}\frac{(d-a)^2}{2}\right],$$

proving the claim. ■

Theorem 2.24. *All as in Theorem 2.22. Let also $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\left| \frac{1}{d-c} \int_c^d f(x)\,dx - f(a) \right| \leq$$

$$\frac{\left[\|f'' + f - f(a)\|_{L_p([c,a])} (a-c)^{1+\frac{1}{q}} + \|f'' + f - f(a)\|_{L_p([a,d])} (d-a)^{1+\frac{1}{q}} \right]}{\left(1 + \frac{1}{q}\right)(d-c)}. \tag{2.96}$$

Proof. Let $x \geq a$, as in (2.89), we get

$$|f(x) - f(a)| \leq \int_a^x |(f''(t) + f(t)) - f(a)|\,dt \leq$$

$$\left(\int_a^x |f''(t) + f(t) - f(a)|^p\,dt \right)^{\frac{1}{p}} \left(\int_a^x 1dt \right)^{\frac{1}{q}} \leq$$

$$\|f'' + f - f(a)\|_{L_p([a,d])} (x-a)^{\frac{1}{q}}. \tag{2.97}$$

Let $x < a$, as in (2.90), we get

$$|f(x) - f(a)| \leq \int_x^a |f''(t) + f(t) - f(a)|\,dt \leq$$

$$\left(\int_x^a |f''(t) + f(t) - f(a)|^p\,dt \right)^{\frac{1}{p}} \left(\int_x^a 1dt \right)^{\frac{1}{q}} \leq$$

$$\|f'' + f - f(a)\|_{L_p([c,a])} (a-x)^{\frac{1}{q}}. \tag{2.98}$$

Acting as in (2.91) we obtain

$$\left| \frac{1}{d-c} \int_c^d f(x)\,dx - f(a) \right| \leq$$

$$\frac{1}{d-c} \left[\int_c^a |f(x) - f(a)|\,dx + \int_a^d |f(x) - f(a)|\,dx \right] \leq$$

$$\frac{1}{d-c} \left[\|f'' + f - f(a)\|_{L_p([c,a])} \int_c^a (a-x)^{\frac{1}{q}}\,dx + \right.$$

$$\left. \|f'' + f - f(a)\|_{L_p([a,d])} \int_a^d (x-a)^{\frac{1}{q}}\,dx \right] =$$

$$\frac{1}{d-c} \left[\|f'' + f - f(a)\|_{L_p([c,a])} \frac{(a-c)^{1+\frac{1}{q}}}{\left(1+\frac{1}{q}\right)} + \right.$$

$$\left. \|f'' + f - f(a)\|_{L_p([a,d])} \frac{(d-a)^{1+\frac{1}{q}}}{\left(1+\frac{1}{q}\right)} \right], \tag{2.99}$$

proving the claim. ■

Theorem 2.25. *All as in Theorem 2.24. Then*

$$\left|\frac{1}{d-c}\int_c^d f(x)\,dx - f(a)\right| \leq$$

$$\frac{\left[(a-c)^{2+\frac{1}{q}}\|f''+f-f(a)\|_{L_p([c,a])}+(d-a)^{2+\frac{1}{q}}\|f''+f-f(a)\|_{L_p([a,d])}\right]}{(q+1)^{\frac{1}{q}}\left(2+\frac{1}{q}\right)(d-c)}. \tag{2.100}$$

Proof. Let $x \geq a$, as in (2.93), we get

$$|f(x)-f(a)| \leq \int_a^x |(f''(t)+f(t))-f(a)|(x-t)\,dt \leq$$

$$\left(\int_a^x |f''(t)+f(t)-f(a)|^p\,dt\right)^{\frac{1}{p}}\left(\int_a^x (x-t)^q\,dt\right)^{\frac{1}{q}} \leq$$

$$\|f''+f-f(a)\|_{L_p([a,d])}\frac{(x-a)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}}. \tag{2.101}$$

Also, when $x < a$, by (2.94) we obtain

$$|f(x)-f(a)| \leq \int_x^a |f''(t)+f(t)-f(a)|(t-x)\,dt \leq$$

$$\left(\int_x^a |f''(t)+f(t)-f(a)|^p\,dt\right)^{\frac{1}{p}}\left(\int_x^a (t-x)^q\,dt\right)^{\frac{1}{q}} \leq$$

$$\|f''+f-f(a)\|_{L_p([c,a])}\frac{(a-x)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}}. \tag{2.102}$$

As in (2.95) we derive

$$\left|\frac{1}{d-c}\int_c^d f(x)\,dx - f(a)\right| \leq$$

$$\frac{1}{d-c}\left[\int_c^a |f(x)-f(a)|\,dx + \int_a^d |f(x)-f(a)|\,dx\right] \leq$$

$$\frac{1}{d-c}\left[\|f''+f-f(a)\|_{L_p([c,a])}\frac{\int_c^a (a-x)^{\frac{q+1}{q}}\,dx}{(q+1)^{\frac{1}{q}}}+\right.$$

$$\left.\|f''+f-f(a)\|_{L_p([a,d])}\frac{\int_a^d (x-a)^{\frac{q+1}{q}}\,dx}{(q+1)^{\frac{1}{q}}}\right] = \tag{2.103}$$

$$\frac{1}{d-c}\left[\|f''+f-f(a)\|_{L_p([c,a])}\frac{(a-c)^{\frac{q+1}{q}+1}}{(q+1)^{\frac{1}{q}}\left(\frac{q+1}{q}+1\right)}\right.$$

$$\left.+\|f''+f-f(a)\|_{L_p([a,d])}\frac{(d-a)^{\frac{q+1}{q}+1}}{(q+1)^{\frac{1}{q}}\left(\frac{q+1}{q}+1\right)}\right] =$$

$$\frac{\left[\|f''+f-f(a)\|_{L_p([c,a])}(a-c)^{2+\frac{1}{q}}+\|f''+f-f(a)\|_{L_p([a,d])}(d-a)^{2+\frac{1}{q}}\right]}{(q+1)^{\frac{1}{q}}\left(2+\frac{1}{q}\right)(d-c)}, \tag{2.104}$$

proving the claim. ■

We continue with more involved L_p, $p \geq 1$, Ostrowski type inequalities.

Theorem 2.26. *Let* $f \in C_K^4([c,d])$*;* $a \in [c,d] : f^{(i)}(a) = 0$*,* $i = 1,2,3,4$*. Here* $\alpha, \beta \in \mathbb{R} : \alpha\beta\left(\alpha^2 - \beta^2\right) \neq 0$*. Then*

$$\left|\frac{1}{d-c}\int_c^d f(x)\,dx - f(a)\right| \leq \left(\frac{(|\alpha|+|\beta|)}{|\alpha\beta(\beta^2-\alpha^2)|(d-c)}\right) \tag{2.105}$$

$$\Big[(a-c)\left\|f'''' + \left(\alpha^2+\beta^2\right)f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_1([c,a])} +$$

$$(d-a)\left\|f'''' + \left(\alpha^2+\beta^2\right)f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_1([a,d])}\Big].$$

Proof. By (2.70) we obtain ($x \geq a$)

$$|f(x) - f(a)| \leq \left(\frac{(|\alpha|+|\beta|)}{|\alpha\beta(\beta^2-\alpha^2)|}\right)$$

$$\int_a^x \left|\left[\left(f''''(t) + \left(\alpha^2+\beta^2\right)f''(t) + \alpha^2\beta^2 f(t)\right) - \right.\right.$$

$$\left.\left.\left(f''''(a) + \left(\alpha^2+\beta^2\right)f''(a) + \alpha^2\beta^2 f(a)\right)\right]\right| dt \leq$$

$$\left(\frac{(|\alpha|+|\beta|)}{|\alpha\beta(\beta^2-\alpha^2)|}\right)\int_a^d \left|f''''(t) + \left(\alpha^2+\beta^2\right)f''(t) + \alpha^2\beta^2 f(t) - \alpha^2\beta^2 f(a)\right| dt = \tag{2.106}$$

$$\left(\frac{(|\alpha|+|\beta|)}{|\alpha\beta(\beta^2-\alpha^2)|}\right)\left\|f'''' + \left(\alpha^2+\beta^2\right)f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_1([a,d])}.$$

From (2.72) we get ($x < a$)

$$|f(x) - f(a)| \leq$$

$$\left(\frac{(|\alpha|+|\beta|)}{|\alpha\beta(\beta^2-\alpha^2)|}\right)\int_x^a \left|f''''(t) + \left(\alpha^2+\beta^2\right)f''(t) + \alpha^2\beta^2 f(t) - \alpha^2\beta^2 f(a)\right| dt \leq \tag{2.107}$$

$$\left(\frac{(|\alpha|+|\beta|)}{|\alpha\beta(\beta^2-\alpha^2)|}\right)\int_c^a \left|f''''(t) + \left(\alpha^2+\beta^2\right)f''(t) + \alpha^2\beta^2 f(t) - \alpha^2\beta^2 f(a)\right| dt =$$

$$\left(\frac{(|\alpha|+|\beta|)}{|\alpha\beta(\beta^2-\alpha^2)|}\right)\left\|f'''' + \left(\alpha^2+\beta^2\right)f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_1([c,a])}.$$

By (2.91) we get that

$$\left|\frac{1}{d-c}\int_c^d f(x)\,dx - f(a)\right| \leq$$

$$\frac{1}{d-c}\left[\int_c^a |f(x) - f(a)|\,dx + \int_a^d |f(x) - f(a)|\,dx\right] \leq \tag{2.108}$$

$$\left(\frac{(|\alpha|+|\beta|)}{|\alpha\beta(\beta^2-\alpha^2)|(d-c)}\right)$$

$$\Big[\left\|f'''' + \left(\alpha^2+\beta^2\right)f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_1([c,a])}(a-c)$$

$$+\left\|f'''' + \left(\alpha^2+\beta^2\right)f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_1([a,d])}(d-a)\Big].$$

The claim is proved. ■

The counterpart of (2.105) follows.

Theorem 2.27. *All as in Theorem 2.26. Then*

$$\left|\frac{1}{d-c}\int_a^d f(x)\,dx - f(a)\right| \leq \frac{1}{(d-c)\left|\beta^2-\alpha^2\right|}$$

$$\Big[(a-c)^2\left\|f'''' + \left(\alpha^2+\beta^2\right)f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_1([c,a])} +$$

$$(d-a)^2\left\|f'''' + \left(\alpha^2+\beta^2\right)f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_1([a,d])}\Big]. \tag{2.109}$$

Proof. By (2.80) we have ($x \geq a$)

$$|f(x) - f(a)| \leq \frac{1}{\left|\alpha\beta\left(\beta^2-\alpha^2\right)\right|}$$

$$\int_a^x \left|f''''(t) + \left(\alpha^2+\beta^2\right)f''(t) + \alpha^2\beta^2 f(t) - \alpha^2\beta^2 f(a)\right| 2\left|\alpha\beta\right|(x-t)\,dt \leq \tag{2.110}$$

$$\frac{2}{\left|\beta^2-\alpha^2\right|}\int_a^x \left|f''''(t) + \left(\alpha^2+\beta^2\right)f''(t) + \alpha^2\beta^2 f(t) - \alpha^2\beta^2 f(a)\right|(x-t)\,dt \leq$$

$$\frac{2(x-a)}{\left|\beta^2-\alpha^2\right|}\left\|f'''' + \left(\alpha^2+\beta^2\right)f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_1([a,d])}.$$

When $x < a$, by (2.84), we obtain

$$|f(x) - f(a)| \leq \frac{1}{\left|\alpha\beta\left(\beta^2-\alpha^2\right)\right|}$$

$$\int_x^a \left|f''''(t) + \left(\alpha^2+\beta^2\right)f''(t) + \alpha^2\beta^2 f(t) - \alpha^2\beta^2 f(a)\right| 2\left|\alpha\beta\right|(t-x)\,dt \leq$$

$$\frac{2}{\left|\beta^2-\alpha^2\right|}\int_x^a \left|f''''(t) + \left(\alpha^2+\beta^2\right)f''(t) + \alpha^2\beta^2 f(t) - \alpha^2\beta^2 f(a)\right|(t-x)\,dt \leq \tag{2.111}$$

$$\frac{2(a-x)}{\left|\beta^2-\alpha^2\right|}\left\|f'''' + \left(\alpha^2+\beta^2\right)f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_1([c,a])}.$$

By (2.91) we get that

$$\left|\frac{1}{d-c}\int_c^d f(x)\,dx - f(a)\right| \leq$$

$$\frac{1}{d-c}\left[\int_c^a |f(x)-f(a)|\,dx + \int_a^d |f(x)-f(a)|\,dx\right] \leq$$

$$\frac{2}{(d-c)\left|\beta^2-\alpha^2\right|}\Big[\left\|f'''' + \left(\alpha^2+\beta^2\right)f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_1([c,a])}\int_c^a (a-x)\,dx$$

$$+\left\|f'''' + \left(\alpha^2+\beta^2\right)f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_1([a,d])}\int_a^d (x-a)\,dx\Big] = \tag{2.112}$$

$$\frac{1}{(d-c)\left|\beta^2-\alpha^2\right|}\Big[(a-c)^2\left\|f'''' + \left(\alpha^2+\beta^2\right)f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_1([c,a])}$$

$$+(d-a)^2\left\|f'''' + \left(\alpha^2+\beta^2\right)f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_1([a,d])}\Big].$$

The claim is proved. ■

The counterparts of the last two Theorems 2.26, 2.27, for $p > 1$, follow.

Theorem 2.28. *Let $f \in C_K^4([c,d])$, $a \in [c,d] : f^{(i)}(a) = 0$, $i = 1,2,3,4$. Here $\alpha, \beta \in \mathbb{R} : \alpha\beta\left(\alpha^2 - \beta^2\right) \neq 0$. Let also $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\left|\frac{1}{d-c}\int_c^d f(x)\,dx - f(a)\right| \leq \left(\frac{(|\alpha| + |\beta|)}{|\alpha\beta(\beta^2 - \alpha^2)|(d-c)\left(1 + \frac{1}{q}\right)}\right)$$

$$\left[(a-c)^{1+\frac{1}{q}}\left\|f'''' + \left(\alpha^2 + \beta^2\right)f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_p([c,a])} + \right.$$

$$\left.(d-a)^{1+\frac{1}{q}}\left\|f'''' + \left(\alpha^2 + \beta^2\right)f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_p([a,d])}\right]. \tag{2.113}$$

Proof. Let $x \geq a$, by (2.106), we obtain

$$|f(x) - f(a)| \leq \left(\frac{(|\alpha| + |\beta|)}{|\alpha\beta(\beta^2 - \alpha^2)|}\right)$$

$$\int_a^x \left|\left[\left(f''''(t) + \left(\alpha^2 + \beta^2\right)f''(t) + \alpha^2\beta^2 f(t)\right) - \alpha^2\beta^2 f(a)\right]\right| dt \leq$$

$$\left(\frac{(|\alpha| + |\beta|)}{|\alpha\beta(\beta^2 - \alpha^2)|}\right)$$

$$\left(\int_a^x \left|f''''(t) + \left(\alpha^2 + \beta^2\right)f''(t) + \alpha^2\beta^2 f(t) - \alpha^2\beta^2 f(a)\right|^p dt\right)^{\frac{1}{p}} (x-a)^{\frac{1}{q}} \leq \tag{2.114}$$

$$\left(\frac{(|\alpha| + |\beta|)}{|\alpha\beta(\beta^2 - \alpha^2)|}\right)\left\|f'''' + \left(\alpha^2 + \beta^2\right)f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_p([a,d])} (x-a)^{\frac{1}{q}}.$$

Let $x < a$, by (2.107), we get

$$|f(x) - f(a)| \leq \left(\frac{(|\alpha| + |\beta|)}{|\alpha\beta(\beta^2 - \alpha^2)|}\right)$$

$$\int_x^a \left|\left(f''''(t) + \left(\alpha^2 + \beta^2\right)f''(t) + \alpha^2\beta^2 f(t)\right) - \alpha^2\beta^2 f(a)\right| dt \leq$$

$$\left(\frac{(|\alpha| + |\beta|)}{|\alpha\beta(\beta^2 - \alpha^2)|}\right)$$

$$\left(\int_x^a \left|f''''(t) + \left(\alpha^2 + \beta^2\right)f''(t) + \alpha^2\beta^2 f(t) - \alpha^2\beta^2 f(a)\right|^p dt\right)^{\frac{1}{p}} (a-x)^{\frac{1}{q}} \leq \tag{2.115}$$

$$\left(\frac{(|\alpha| + |\beta|)}{|\alpha\beta(\beta^2 - \alpha^2)|}\right)\left\|f'''' + \left(\alpha^2 + \beta^2\right)f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_p([c,a])} (a-x)^{\frac{1}{q}}.$$

By (2.91) we get that

$$\left|\frac{1}{d-c}\int_c^d f(x)\,dx - f(a)\right| \leq$$

$$\frac{1}{d-c}\left[\int_c^a |f(x) - f(a)|\,dx + \int_a^d |f(x) - f(a)|\,dx\right] \leq$$

$$\left(\frac{(|\alpha| + |\beta|)}{|\alpha\beta(\beta^2 - \alpha^2)|(d-c)}\right)$$

$$\left[\left\|f'''' + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_p([c,a])} \left(\int_c^a (a-x)^{\frac{1}{q}}\, dx\right)\right.$$

$$\left. + \left\|f'''' + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_p([a,d])} \left(\int_a^d (x-a)^{\frac{1}{q}}\, dx\right)\right] = \tag{2.116}$$

$$\left(\frac{(|\alpha|+|\beta|)}{\left|\alpha\beta\left(\beta^2-\alpha^2\right)\right|(d-c)\left(1+\frac{1}{q}\right)}\right)$$

$$\left[(a-c)^{1+\frac{1}{q}} \left\|f'''' + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_p([c,a])} +\right.$$

$$\left.(d-a)^{1+\frac{1}{q}} \left\|f'''' + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_p([a,d])}\right].$$

The claim is proved. ■

At last we give

Theorem 2.29. *All as in Theorem 2.28. Then*

$$\left|\frac{1}{d-c}\int_c^d f(x)\, dx - f(a)\right| \leq \frac{2}{\left|\beta^2-\alpha^2\right|(q+1)^{\frac{1}{q}}\left(2+\frac{1}{q}\right)(d-c)}$$

$$\left[(a-c)^{2+\frac{1}{q}} \left\|f'''' + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_p([c,a])} +\right.$$

$$\left.(d-a)^{2+\frac{1}{q}} \left\|f'''' + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_p([a,d])}\right]. \tag{2.117}$$

Proof. Let $x \geq a$, by (2.110), we get that

$$|f(x) - f(a)| \leq \frac{2}{\left|\beta^2-\alpha^2\right|}$$

$$\int_a^x \left|f''''(t) + \left(\alpha^2+\beta^2\right) f''(t) + \alpha^2\beta^2 f(t) - \alpha^2\beta^2 f(a)\right| (x-t)\, dt \leq$$

$$\frac{2}{\left|\beta^2-\alpha^2\right|}\left(\int_a^x \left|f''''(t) + \left(\alpha^2+\beta^2\right) f''(t) + \alpha^2\beta^2 f(t) - \alpha^2\beta^2 f(a)\right|^p dt\right)^{\frac{1}{p}} \tag{2.118}$$

$$\left(\int_a^x (x-t)^q\, dt\right)^{\frac{1}{q}} \leq$$

$$\frac{2}{\left|\beta^2-\alpha^2\right|}\left\|f'''' + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a)\right\|_{L_p([a,d])} \frac{(x-a)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}}.$$

Let $x < a$, by (2.111), we derive

$$|f(x) - f(a)| \leq \frac{2}{\left|\beta^2-\alpha^2\right|}$$

$$\int_x^a \left|f''''(t) + \left(\alpha^2+\beta^2\right) f''(t) + \alpha^2\beta^2 f(t) - \alpha^2\beta^2 f(a)\right| (t-x)\, dt \leq$$

$$\frac{2}{\left|\beta^2-\alpha^2\right|}\left(\int_x^a \left|f''''(t) + \left(\alpha^2+\beta^2\right) f''(t) + \alpha^2\beta^2 f(t) - \alpha^2\beta^2 f(a)\right|^p dt\right)^{\frac{1}{p}}$$

$$\left(\int_x^a (t-x)^q \, dt\right)^{\frac{1}{q}} \leq$$

$$\frac{2}{|\beta^2-\alpha^2|} \left\| f'''' + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a) \right\|_{L_p([c,a])} \frac{(a-x)^{\frac{q+1}{q}}}{(q+1)^{\frac{1}{q}}}. \tag{2.119}$$

Finally, by (2.91) we get that

$$\left| \frac{1}{d-c} \int_c^d f(x) \, dx - f(a) \right| \leq$$

$$\frac{1}{d-c} \left[\int_c^a |f(x) - f(a)| \, dx + \int_a^d |f(x) - f(a)| \, dx \right] \leq$$

$$\frac{2}{|\beta^2-\alpha^2| (q+1)^{\frac{1}{q}} (d-c)}$$

$$\left[\left(\int_c^a (a-x)^{\frac{q+1}{q}} dx \right) \left\| f'''' + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a) \right\|_{L_9([c,a])} \right.$$

$$\left. + \left(\int_a^d (x-a)^{\frac{q+1}{q}} dx \right) \left\| f'''' + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a) \right\|_{L_p([a,d])} \right] = \tag{2.120}$$

$$\frac{2}{|\beta^2-\alpha^2| (q+1)^{\frac{1}{q}} \left(2+\frac{1}{q}\right) (d-c)}$$

$$\left[(a-c)^{2+\frac{1}{q}} \left\| f'''' + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a) \right\|_{L_p([c,a])} + \right.$$

$$\left. (d-a)^{2+\frac{1}{q}} \left\| f'''' + \left(\alpha^2+\beta^2\right) f'' + \alpha^2\beta^2 f - \alpha^2\beta^2 f(a) \right\|_{L_p([a,d])} \right].$$

The claim is proved. ■

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