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Square-mean pseudo almost automorphic solutions of infinite class under the light of measure theory

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This paper is dedicated to the occasion of Professor Gaston M. N'Guérékata's 70th birthday

Abstract. The aim of this work is to present new concept of square-mean pseudo almost automorphic of infinite class using the measure theory. We use the (μ,ν)-ergodic process to define the spaces of (μ,ν)-pseudo almost automorphic processes of infinite class in the square-mean sense. We present many interesting results on those spaces like completeness and composition theorems and we study the existence and the uniqueness of the square-mean (μ,ν)-pseudo almost automorphic solutions of infinite class for of the stochastic evolution equation. We provide an example to illustrate ours results.

AMS Subject Classifications: 47D09, 34G10, 60J65.

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Contents

1. Introduction

In this work, we study the basic properties of the square-mean (μ,ν)-pseudo almost automorphic process using the measure theory and used those results to study the following stochastic evolution equations in a Hilbert space H,

$$dx(t) = [Ax(t) + L(x_t) + f(t)]dt + g(t)dW(t), \tag{1.1}$$

where $A : D(A) \subset H$ is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geqslant 0}$ on H, $f,g : \mathbb{R} \to L^2(P,H)$ are two stochastic processes, $W(t)$ is a two-sided and standard one-dimensional Brownian notion defined on the filtered probability space $(\Omega,\mathcal{F},P,\mathcal{F}_t)$ with $\mathcal{F}_t = \sigma\{W(u) - W(v) \mid u,v \leqslant t\}$ and L is a bounded linear operator from $\mathcal{B}$ into $Ł^2(P,H)$. The phase space $\mathcal{B}$ is a linear space of functions mapping $]-\infty,0]$ into X for

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every $t \geqslant 0$, x_t denotes the history function of $\mathcal{B}$ defined by $x_t(\theta) = x(t+\theta)$ for $\theta \in]-\infty, 0]$
We assume $(H, ||\cdot||)$ is real separable Hilbert space and $L^2(P, H)$ is the space of all H-valued random variables x such that

$$\mathbb{E}||x||^2 = \int_{\Omega} ||x||^2 dP < +\infty.$$

This work is an extension of [10] whose authors have studied equation (4.1) in the deterministic case. Some recent contributions concerning square-mean pseudo almost automorphic solutions for abstract differrential equations similar to equation (4.1) have been made. For example in [7] the authors studied equation(4.1) without the operator L. They showed that the equation has a unique square-mean μ-pseudo almost automorphic mild solution on $\mathbb{R}$ when f and g are square mean pseudo almost automorphic functions.
In [4] the authors studied the square-mean almost automorphic solutions to a class of nonautonomous stochastic differential equations without our operator L and without delay on a separable real Hilbert space. They established the existence and uniqueness of a square-mean almost automorphic mild solution to those nonautonomous stochastic differential equations with the 'Acquistapace-Terreni' conditions.
In [8] The authors established the existence, uniqueness and stability of square-mean μ-pseudo almost periodic(resp. automorphic) mild solution to a linear and semilinear case of the stochastic evolution equations in case when the functions forcing are both continuous and $S^2-\mu-$pseudo almost periodic (resp. automorphic) and verify some suitable assumptions.
This work is organized as follow, in section 2, we study spectral decomposition of phase space then in section 3 we study square-mean (μ,ν)-Pseudo almost automorphic process, in section 4 we study square-mean pseudo almost automorphic solutions of infinite class and we finish with application of our theory.

2. Variation of constants formula and spectral decomposition

In this work, the state space $(\mathcal{B}, |.|_{\mathcal{B}})$ is a normed linear space of functions mapping $]-\infty, 0]$ into $L^2(P, H)$ and satisfying the following fundamental axioms.

$(\mathbf{A}_1)$ There exist a positive constant H and functions $K(.), M(.) : \mathbb{R}^+ \to \mathbb{R}^+$, with K continuous and M locally bounded, such that for any $\sigma \in \mathbb{R}$ and $a > 0$, if $u :]-\infty, a] \to L^2(P, H)$, $u_\sigma \in \mathcal{B}$, and $u(.)$ is continuous on $[\sigma, \sigma + a]$, then for every $t \in [\sigma, \sigma + a]$ the following conditions hold

(i) $u_t \in \mathcal{B}$,

(ii) $|u(t)| \leq H|u_t|_{\mathcal{B}}$, which is equivalent to $|\varphi(0)| \leq H|\varphi|_{\mathcal{B}}$ for every $\varphi \in \mathcal{B}$

(iii) $|u_t|_{\mathcal{B}} \leq K(t-\sigma) \sup_{\sigma \leq s \leq t} |u(s)| + M(t-\sigma)|u_\sigma|_{\mathcal{B}}$.

$(\mathbf{A}_2)$ For the function $u(.)$ in $(\mathbf{A}_1)$, $t \mapsto u_t$ is a $\mathcal{B}$-valued continuous function for $t \in [\sigma, \sigma + a]$.

(B) The space $\mathcal{B}$ is a Banach space.

Assume that:

$(\mathbf{C_1})$ If $(\varphi_n)_{n\geq 0}$ is a sequence in $\mathcal{B}$ such that $\varphi_n \to 0$ in $\mathcal{B}$ as $n \to +\infty$, then $(\varphi_n(\theta))_{n\geq 0}$ converges to 0 in $L^2(P, H)$.

Let $C(]-\infty, 0], L^2(P, H))$ be the space of continuous functions from $]-\infty, 0]$ to $L^2(P, H)$. Suppose the following assumptions:

($\mathbf{C_2}$) $\mathcal{B} \subset C(]-\infty, 0], L^2(P, H))$.

($\mathbf{C_3}$) there exists $\lambda_0 \in \mathbb{R}$ such that, for all $\lambda \in \mathbb{C}$ with Re$\lambda > \lambda_0$ and $x \in L^2(P, H)$, $e^{\lambda \cdot} x \in \mathcal{B}$ and

$$K_0 = \sup_{\substack{\text{Re}\lambda > \lambda_0, x \in L^2(P,H) \\ x \neq 0}} \frac{|e^{\lambda \cdot} x|_{\mathcal{B}}}{|x|} < \infty,$$

where $(e^{\lambda \cdot} x)(\theta) = e^{\lambda \theta} x \quad \text{for} \quad \theta \in]-\infty, 0]$ and $x \in L^2(P, H)$.

To equation (4.1), associate the following initial value problem

$$\begin{cases} du_t = [Au(t) + L(u_t) + f(t)]dt + g(t)dW(t) \text{ for } t \geq 0 \\ \\ u_0 = \varphi \in \mathcal{B}, \end{cases} \tag{2.1}$$

where $f : \mathbb{R}^+ \to L^2(P, H)$ is a continuous function.

Let us introduce the part A_0 of the operator A in $\overline{D(A)}$ which defined by

$$\begin{cases} D(A_0) = \{x \in D(A) : \ Ax \in \overline{D(A)}\} \\ A_0 x = Ax \text{ for } \ x \in D(A_0) \end{cases}$$

The following assumption is supposed:

($\mathbf{H_0}$) A satisfies the Hille-Yosida condition.

Lemma 2.1. *[2] A_0 generates a strongly continuous semigroup $(T_0(t))_{t \geq 0}$ on $\overline{D(A)}$.*

The phase space $\mathcal{B}_A$ of equation (2.1) is defined by

$$\mathcal{B}_A = \{\varphi \in \mathcal{B} : \ \varphi(0) \in \overline{D(A)}\}.$$

For each $t \geq 0$, the linear operator $\mathcal{U}(t)$ on $\mathcal{B}_A$ is defined by

$$\mathcal{U}(t) = v_t(., \varphi)$$

where $v(., \varphi)$ is the solution of the following homogeneous equation

$$\begin{cases} \dfrac{d}{dt} v_t = Av(t) + L(v_t) \text{ for } t \geq 0 \\ \\ v_0 = \varphi \in \mathcal{B}. \end{cases}$$

Proposition 2.2. *[3] $(\mathcal{U}(t))_{t \geq 0}$ is a strongly continuous semigroup of linear operators on $\mathcal{B}_A$. Moreover, $(\mathcal{U}(t))_{t \geq 0}$ satisfies, for $t \geq 0$ and $\theta \in]-\infty, 0]$, the following translation property*

$$(\mathcal{U}(t)\varphi)(\theta) = \begin{cases} (\mathcal{U}(t+\theta)\varphi)(0) \text{ for } t + \theta \geq 0 \\ \\ \varphi(t+\theta) \text{ for } t + \theta \leq 0. \end{cases}$$

Theorem 2.3. *[3] Assume that $\mathcal{B}$ satisfies ($\boldsymbol{A_1}$), ($\boldsymbol{A_2}$), ($\boldsymbol{B}$), ($\boldsymbol{C_1}$) and ($\boldsymbol{C_2}$). Then $\mathcal{A}_{\mathcal{U}}$ defined on $\mathcal{B}_A$ by*

$$\begin{cases} D(\mathcal{A}_{\mathcal{U}}) = \Big\{\varphi \in C^1(]-\infty,0];X)\cap\mathcal{B}_A;\ \varphi'\in\mathcal{B}_A,\ \varphi(0)\in\overline{D(A)}\ \ and\ \ \varphi'(0)=A\varphi(0)+L(\varphi)\Big\} \\ \\ \mathcal{A}_{\mathcal{U}}\varphi=\varphi'\ \ for\ \ \varphi\in D(\mathcal{A}_{\mathcal{U}}). \end{cases}$$

is the infinitesimal generator of the semigroup $(\mathcal{U}(t))_{t\geq 0}$ on $\mathcal{B}_A$.

Let $\langle X_0\rangle$ be the space defined by

$$\langle X_0\rangle=\{X_0x:\ x\in X\}$$

where the function X_0x is defined by

$$(X_0x)(\theta)=\begin{cases} 0 & \text{if} \quad \theta\in]-\infty,0[, \\ \\ x & \text{if} \quad \theta=0. \end{cases}$$

The space $\mathcal{B}_A\oplus\langle X_0\rangle$ equipped with the norm $|\phi+X_0c|_{\mathcal{B}}=|\phi|_{\mathcal{B}}+|c|$ for $(\phi,c)\in\mathcal{B}_A\times X$ is a Banach space and consider the extension $\mathcal{A}_{\mathcal{U}}$ defined on $\mathcal{B}_A\oplus\langle X_0\rangle$ by

$$\begin{cases} D(\widetilde{\mathcal{A}_{\mathcal{U}}})=\Big\{\varphi\in C^1(]-\infty,0];X):\ \varphi\in D(A)\ \ \text{and}\ \varphi'\in\overline{D(A)}\Big\} \\ \widetilde{\mathcal{A}_{\mathcal{U}}}\varphi=\varphi'+X_0(A\varphi+L(\varphi)-\varphi'). \end{cases}$$

Lemma 2.4. *[3] Assume that $\mathcal{B}$ satisfies ($\boldsymbol{A_1}$), ($\boldsymbol{A_2}$), ($\boldsymbol{B}$), ($\boldsymbol{C_1}$), ($\boldsymbol{C_2}$) and ($\boldsymbol{C_3}$). Then, $\widetilde{\mathcal{A}_{\mathcal{U}}}$ satisfies the Hille-Yosida condition on $\mathcal{B}_A\oplus\langle X_0\rangle$.*

Now, start the variation of constants formula associated to equation (2.1).

Let C_{00} be the space of X-valued continuous function on $]-\infty,0]$ with compact support. Assume that:

(**D**) If $(\varphi_n)_{n\geq 0}$ is a Cauchy sequence in $\mathcal{B}$ and converges compactly to φ on $]-\infty,0]$, then $\varphi\in\mathcal{B}$ and $|\varphi_n-\varphi|\to 0$.

Definition 2.5. *The semigroup $(\mathcal{U}(t))_{t\geq 0}$ is hyperbolic if*

$$\sigma(\mathcal{A}_{\mathcal{U}})\cap i\mathbb{R}=\emptyset$$

Let $(S_0(t))_{t\geq 0}$ be the strongly continuous semigroup defined on the subspace

$$\mathcal{B}_0=\{\varphi\in\mathcal{B}:\ \varphi(0)=0\}$$

by

$$(S_0(t)\phi)(\theta)=\begin{cases} \phi(t+\theta) & \text{if} \ \ t+\theta\leq 0 \\ \\ 0 & \text{if} \ \ t+\theta\geq 0 \end{cases}$$

Definition 2.6. *Assume that the space $\mathcal{B}$ satisfies Axioms **(B)** and **(D)**, $\mathcal{B}$ is said to be a fading memory space, if for all $\varphi\in\mathcal{B}_0$,*

$$|S_0(t)|\to 0\ \ as\ \ t\to+\infty\ in\ \mathcal{B}_0.$$

Moreover, $\mathcal{B}$ is said to be a uniform fading memory space, if

$$|S_0(t)|\to 0\ \ as\ \ t\to+\infty.$$

Lemma 2.7. *If $\mathcal{B}$ is a uniform fading memory space, then the function K can be chosen to be constant and the function M such that $M(t) \to 0$ as $t \to +\infty$.*

Proposition 2.8. *If the phase space $\mathcal{B}$ is a fading memory space, then the space $BC(]-\infty, 0], X)$ of bounded continuous X-valued functions on $]-\infty, 0]$ endowed with the uniform norm topology, is continuous embedding in $\mathcal{B}$. In particular $\mathcal{B}$ satisfies (**C_3**), for $\lambda_0 > 0$.*

For the sequel, make the following assumption:

(**H_1**) $T_0(t)$ is compact on $\overline{D(A)}$ for every $t > 0$.

(**H_2**) $\mathcal{B}$ is a uniform fading memory space.

Theorem 2.9. *[3] Assume that $\mathcal{B}$ satisfies (**A_1**), (**A_2**), (**B**), (**C_1**) and (**H_0**), (**H_1**), (**H_2**) hold. Then the semigroup $(\mathcal{U}(t))_{t\geq 0}$ is decomposed on $\mathcal{B}_A$ as follows*

$$\mathcal{U}(t) = \mathcal{U}_1(t) + \mathcal{U}_2(t) \;\; for \;\; t \geq 0$$

where $(\mathcal{U}_1(t))_{t\geq 0}$ is an exponentially stable semigroup on $\mathcal{B}_A$, which means that there are positive constants α_0 and N_0 such that

$$|\mathcal{U}_1(t)| \leq N_0 e^{-\alpha_0 t}|\varphi| \;\; for \;\; t \geq 0 \;\; and \;\; \varphi \in \mathcal{B}_A$$

and $(\mathcal{U}_2(t))_{t\geq 0}$ is compact for for every $t > 0$.

The following result on the spectral decomposition of the phase space $\mathcal{B}_A$ is obtained.

Theorem 2.10. *[3] Assume that $\mathcal{B}$ satisfies (**A_1**), (**A_2**), (**B**), (**C_1**), and (**H_0**), (**H_1**), (**H_2**) hold. Then the space $\mathcal{B}_A$ is decomposed as a direct sum*

$$\mathcal{B}_A = S \oplus U$$

of two $\mathcal{U}(t)$ invariant closed subspaces S and U such that the restricted semigroup on $\mathcal{U}$ is a group and there exist positive constants $\overline{M}$ and ω such that

$$|\mathcal{U}(t)\varphi| \leq \overline{M}e^{-\omega t}|\varphi| \;\; for \;\; t \geq 0 \;\; and \;\; \varphi \in S$$

$$|\mathcal{U}(t)\varphi| \leq \overline{M}e^{\omega t}|\varphi| \quad for \;\; t \leq 0 \;\; and \;\; \varphi \in U,$$

where S and U are called respectively the stable and unstable space.

Let $\mathcal{N}$ the Lebesgue σ-field of $\mathbb{R}$ and by $\mathcal{M}$ the set of all positive measures μ on $\mathcal{N}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a, b]) < \infty$, for all $a, b \in \mathbb{R}$ $(a \leq b)$.

Definition 2.11. *Let $x : \mathbb{R} \to L^2(P, H)$ be a stochastic process.*

1. *x said to be stochastically bounded if there exists $C > 0$ such that*

$$\mathbb{E}||x(t)||^2 \leqslant C \; \forall \; t \in \mathbb{R}.$$

2. *x is said to be stochastically continuous if*

$$\lim_{t\to s} \mathbb{E}||x(t) - x(s)||^2 = 0 \; \forall \; s \in \mathbb{R}.$$

Denote by $SBC(\mathbb{R}, L^2(P,H))$, the space of all stochastically bounded and continuous process. Otherwise, this space endowed the following norm

$$||x||_\infty = \sup_{t\in\mathbb{R}} \left(\mathbb{E}||x(t)||^2\right)^{\frac{1}{2}}$$

is a Banach space.

Definition 2.12. *Let $\mu, \nu \in \mathcal{M}$. A stochastic process f is said to be square-mean $(\mu,\nu)-$ ergodic if $f \in SBC(\mathbb{R}, L^2(P,H))$ and satisfied*

$$\lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau} \mathbb{E}||f(\theta)||^2 d\mu(t) = 0.$$

We denote by $\mathcal{E}(\mathbb{R}, L^2(P,H), \mu, \nu)$, the space of all such process.

Definition 2.13. *Let $\mu, \nu \in \mathcal{M}$. A stochastic process f is said to be square-mean $(\mu,\nu)-$ ergodic of infinite class if $f \in SBC(\mathbb{R}, L^2(P,H))$ and satisfied*

$$\lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau} \sup_{\theta\in]-\infty,t]} \mathbb{E}||f(\theta)||^2 d\mu(t) = 0.$$

We denote by $\mathcal{E}(\mathbb{R}, L^2(P,H), \mu, \nu, \infty)$, the space of all such process.
For $\mu, \nu \in \mathcal{M}$ and $a \in \mathbb{R}$, we denote by μ_a and ν_a positives measures on $(\mathbb{R}, \mathcal{N})$ respectively defined by

$$\mu_a(A) = \mu(a+b : b \in A) \text{ and } \nu_a(A) = \nu(a+b : b\in A) \text{ for } A \in \mathcal{N}. \tag{2.2}$$

From $\mu, \nu \in \mathcal{M}$, we formulate the following hypothesis.
(**H$_2$**): For all $a \in \mathbb{R}$, there exists $\beta > 0$ and a bounded intervall I such that $\mu_a(A) \leqslant \beta\mu(A)$ when $A \in \mathcal{N}$ satisfies $A \cap I = \emptyset$.
(**H$_3$**) For all a, b and $c \in \mathbb{R}$, such that $0 \leqslant a < b \leq c$, there exist δ_0 and $\alpha_0 > 0$ such that

$$|\delta| \geqslant \delta_0 \implies \mu(a+\delta, b+\delta) \geqslant \alpha_0 \mu(\delta, c+\delta).$$

(**H$_4$**) Let $\mu, \nu \in \mathcal{M}$ be such that $\limsup_{\tau\to+\infty} \dfrac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])} = \alpha < \infty.$

Proposition 2.14. *Assume that (**H$_4$**) holds. Then $\mathcal{E}(\mathbb{R}, L^2(P,H), \mu, \nu, \infty)$ is a Banach space with the norm $||\cdot||_\infty$.*

Proof. It is easy to see that $\mathcal{E}(\mathbb{R}; L^2(P,H), \mu, \nu, \infty)$ is a vector subspace of $SBC(\mathbb{R}, L^2(P,H)))$. To complete the proof, it is enough to prove that $\mathcal{E}(\mathbb{R}; L^2(P,H), \mu, \nu, \infty)$ is closed in $SBC(\mathbb{R}; L^2(P,H))$. Let $(f_n)_n$ be a sequence in $\mathcal{E}(\mathbb{R}; L^2(P,H), \mu, \nu, \infty)$ such that $\lim_{n\to+\infty} f_n = f$ uniformly in $SBC(\mathbb{R}, L^2(P,H))$. From $\nu(\mathbb{R}) = +\infty$, it follows $\nu([-\tau,\tau]) > 0$ for τ sufficiently large. Let $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\|f_n - f\|_\infty < \varepsilon$. Let $n \geq n_0$, then

$$\begin{aligned}
\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2\Big)d\mu(t) &\leq \frac{2}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||f_n(\theta)-f(\theta)||^2\Big)d\mu(t)\\
&\quad+\frac{2}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||f_n(\theta)||^2\Big)d\mu(t)\\
&\leq \frac{2}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{t\in\mathbb{R}}\mathbb{E}||f_n(t)-f(t)||^2\Big)d\mu(t)\\
&\quad+\frac{2}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||f_n(\theta)||^2\Big)d\mu(t)\\
&\leq 2\|f_n-f\|_\infty^2\frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])}+\frac{2}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}|f_n(\theta)|\Big)d\mu(t).
\end{aligned}$$

Consequently

$$\limsup_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2\Big)d\mu(t)\leq 2\alpha\varepsilon \text{ for any } \varepsilon>0.\blacksquare$$

The following theorem is a characterization of square-mean (μ,ν)-ergodic processes(eventually $I=\emptyset$).

Theorem 2.15. *Assume that $f\in SBC(\mathbb{R},L^2(P,H))$. Then the following assertions are equivalent:*

i) $\mathcal{E}(\mathbb{R},L^2(P,H),\mu,\nu,\infty)$

ii) $\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2d\mu(t)=0$

iii) For any $\varepsilon>0$, $\lim_{\tau\to+\infty}\frac{\mu\Big\{t\in[-\tau,\tau]\setminus I:\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2>\varepsilon\Big\}}{\nu([-\tau,\tau]\setminus I)}=0$

Proof. The proof is made like the proof of Theorem(2.13) in [6].
First, we will show that i) is equivalent to ii).
Denote by $A=\nu(I)$, $B=\int_I\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2\Big)d\mu(t)$. A and B belong to $\mathbb{R}$, since the interval I is bounded and the process f is stochastically bounded and continuous. For $\tau>0$ such that $I\subset[-\tau,\tau]$ and $\nu([-\tau,\tau]\setminus I)>0$, it follows

$$\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2\Big)d\mu(t)=\frac{1}{\nu([-\tau,\tau])-A}\Big[\int_{[-\tau,\tau]}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2\Big)d\mu(t)-B\Big]$$
$$=\frac{\nu([-\tau,\tau])}{\nu([-\tau,\tau])-A}\Big[\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2\Big)d\mu(t)-\frac{B}{\nu([-\tau,\tau])}\Big].$$

From above equalities and the fact that $\nu(\mathbb{R})=+\infty$, ii) is equivalent to

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2\Big)d\mu(t)=0,$$

that is i).

Then, we will show that iii) implies ii).
Denote by A_τ^ε and B_τ^ε the following sets

$$A_\tau^\varepsilon=\Big\{t\in[-\tau,\tau]\setminus I:\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2>\varepsilon\Big\}\text{ and } B_\tau^\varepsilon=\Big\{t\in[-\tau,\tau]\setminus I):\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2\leq\varepsilon\Big\}.$$

Assume that iii) holds, that is

$$\lim_{\tau\to+\infty}\frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)}=0. \tag{2.3}$$

From the equality

$$\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2\Big)d\mu(t)=\int_{A_\tau^\varepsilon}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2\Big)d\mu(t)+\int_{B_\tau^\varepsilon}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2\Big)d\mu(t),$$

Then for τ sufficiently large

$$\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2\Big)d\mu(t)\leq\|f\|_\infty^2\frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)}+\varepsilon\frac{\mu(B_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)}.$$

By using ($\mathbf{H_4}$), it follows that

$$\limsup_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2\Big)d\mu(t)\leq\alpha\varepsilon,\quad\text{for any }\ \varepsilon>0,$$

consequently ii) holds.

Thus, we shall show that ii) implies iii).
Assume that ii) holds. From the following inequality

$$\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2\Big)d\mu(t)\geq\int_{A_\tau^\varepsilon}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2\Big)d\mu(t)$$
$$\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2\Big)d\mu(t)\geq\varepsilon\frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)}$$
$$\frac{1}{\varepsilon\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2\Big)d\mu(t)\geq\frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)},$$

for τ sufficiently large, equation (2.3) is obtained, that is iii).■

Definition 2.16. *Let $f\in SBC(\mathbb{R},L^2(P,H))$ and $\tau\in\mathbb{R}$. We denote by f_τ the function defined by $f_\tau(t)=f(t+\tau)$ for $t\in\mathbb{R}$. A subset $\mathfrak{F}$ of $SBC(\mathbb{R},L^2(P,H))$ is said to translation invariant if for all $f\in\mathfrak{F}$ we have $f_\tau\in\mathfrak{F}$ for all $\tau\in\mathbb{R}$.*

Definition 2.17. *Let μ_1 and $\mu_2\in\mathcal{M}$. μ_1 is said to be equivalent to μ_2 ($\mu_1\sim\mu_2$) if there exist constants α and $\beta>0$ and a bounded interval I(eventually $I=\emptyset$) such that $\alpha\mu_1(A)\leqslant\mu_2(A)\leqslant\beta\mu_1(A)$ for $A\in\mathcal{N}$ satisfying $A\cap I=\emptyset$.*

Remark 2.18. *The relation $\sim$ is an equivalence relation on $\mathcal{M}$.*

Theorem 2.19. *Let $\mu_1,\mu_2,\nu_1,\nu_2\in\mathcal{M}$. If $\mu_1\sim\mu_2$ and $\nu_1\sim\nu_2$, then*
$\mathcal{E}(\mathbb{R},L^2(P,H),\mu_1,\nu_1,\infty)=\mathcal{E}(\mathbb{R},L^2(P,H),\mu_2,\nu_2,\infty)$.

Proof. Since $\mu_1\sim\mu_2$ and $\nu_1\sim\nu_2$ there exist some constants $\alpha_1,\alpha_2,\beta_1,\beta_2>0$ and a bounded interval I (eventually $I=\emptyset$) such that $\alpha_1\mu_1(A)\leq\mu_2(A)\leq\beta_1\mu_1(A)$ and $\alpha_2\nu_1(A)\leq\nu_2(A)\leq\beta_2\nu_1(A)$ for each $A\in\mathcal{N}$ satisfies $A\cap I=\emptyset$ i.e

$$\frac{1}{\beta_2\nu_1(A)}\leq\frac{1}{\nu_2(A)}\leq\frac{1}{\alpha_2\nu_1(A)}.$$

Since $\mu_1\sim\mu_2$ and $\mathcal{N}$ is the Lebesgue σ-field, then for τ sufficiently large, it follows that

$$\frac{\alpha_1\mu_1\Big(\Big\{t\in[-\tau,\tau]\setminus I:\sup\limits_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2>\varepsilon\Big\}\Big)}{\beta_2\nu_1([-\tau,\tau]\setminus I)}\leq\frac{\mu_2\Big(\Big\{t\in[-\tau,\tau]\setminus I:\sup\limits_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2>\varepsilon\Big\}\Big)}{\nu_2([-\tau,\tau]\setminus I)}$$
$$\leq\frac{\beta_1\mu_1\Big(\Big\{t\in[-\tau,\tau]\setminus I:\sup\limits_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2>\varepsilon\Big\}\Big)}{\alpha_2\nu_1([-\tau,\tau]\setminus I)}$$

Consequently by Theorem 3.2, $\mathcal{E}(\mathbb{R},X,\mu_1,\nu_1,\infty)=\mathcal{E}(\mathbb{R},X,\mu_2,\nu_2,\infty)$.■
Let μ, $\nu\in\mathcal{M}$ denote by

$$cl(\mu,\nu)=\{\omega_1,\omega_2:\mu\sim\omega_1\text{ and }\nu\sim\omega_2\}.$$

Lemma 2.20. *[5] Let $\mu \in \mathcal{M}$. Then μ satisfies ($\boldsymbol{H_2}$) if and only if the measures μ and μ_τ are equivalent for all $\tau \in \mathbb{R}$.*

Lemma 2.21. *[6] ($\boldsymbol{H_3}$) implies for all σ, $\limsup_{\tau\to\infty} \dfrac{\mu([-\tau-\sigma,\tau+\sigma])}{\mu([-\tau,\tau])} < +\infty$.*

Theorem 2.22. *Let $\mu,\nu \in \mathcal{M}$ satisfy ($\boldsymbol{H_2}$). Then $\mathcal{E}(\mathbb{R}, L^2(P,H),\mu,\nu,\infty)$ is translation invariant.*

Proof. The proof of this theorem is inspired of Theorem (3.5) in [5]. Let $f \in \mathcal{E}(\mathbb{R}, L^2(P,H),\mu,\nu,\infty)$ and $a \in \mathbb{R}$. Since $\nu(\mathbb{R}) = +\infty$. there exists $a_0 > 0$ such that $\nu([-\tau-|a|,\tau+|a|]) > 0$ for $|a| \geqslant a_0$. Let us denote by

$$M_a(\tau) = \frac{1}{\nu_a([-\tau,\tau])}\int_{[-\tau,\tau]} \left(\sup_{\theta\in]-\infty,t]} \mathbb{E}||f(\theta)||^2\right) d\mu_a(t) \ \ \forall \tau > 0 \text{ and } a \in \mathbb{R},$$

where ν_a is the positive measure defined by equation(4.3). By using Lemma (2.20), it follows that ν and ν_a are equivalent, μ and μ_a are equivalent by using Theorem (2.19) we have $\mathcal{E}(\mathbb{R}, L^2(P,H),\mu_a,\nu_a,\infty) = \mathcal{E}(\mathbb{R}, L^2(P,H),\mu,\nu,\infty)$ therefore $f \in \mathcal{E}(\mathbb{R}, L^2(P,H),\mu_a,\nu_a,\infty)$ that is $\lim_{\tau\to+\infty} M_a(\tau) = 0$ for all $a \in \mathbb{R}$.
For all $A \in \mathcal{N}$, we denote by $\mathcal{X}_A$ the characteristic function of A, by using definition of the measure μ_a, we obtain that

$$\int_{[-\tau,\tau]} \mathcal{X}_A(t)d\mu_a(t) = \int_{[-\tau,\tau]} \mathcal{X}_A(t)d\mu(t+a) = \int_{[-\tau+a,\tau+a]} d\mu(t) \text{ for all } A \in \mathcal{N}$$

and since $t \mapsto \sup_{\theta\in]\infty,t]} \mathbb{E}||f(\theta)||^2$ is the pointwise limit of an increasing sequence of linear combinations of functions [[12]; Theorem 1.17 p.15], we deduce that

$$\int_{[-\tau,\tau]} \sup_{\theta\in]-\infty,t]} \mathbb{E}||f(\theta)||^2 d\mu_a(t) = \int_{[-\tau+a,\tau+a]} \sup_{\theta\in]-\infty,t-a]} \mathbb{E}||f(\theta)||^2 d\mu(t).$$

If we denote by $a^+ := \max(a,0)$ and $a^- := \max(-a,0)$ we have $|a|+a = 2a^+$ and $|a|-a = 2a^-$, and then $[-\tau+a-|a|,\tau+a|a|] = [-\tau-2a^-,\tau+2a^+]$. Therefore we obtain

$$M_a(\tau+|a|) = \frac{1}{\nu([-\tau-2a^-,\tau+2a^+])}\int_{[-\tau-2a^-,\tau+2a^+]} \sup_{\theta\in]-\infty,\,t-a]} \mathbb{E}||f(\theta)||^2 d\mu(t). \tag{2.4}$$

From equation (2.4) and the following inequality

$$\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]} \sup_{\theta\in]-\infty,\,t-a]} \mathbb{E}||f(\theta)||^2 d\mu(t) \leqslant \frac{1}{\nu([-\tau,\tau])}\int_{[-\tau-2a^-,\tau+2a^+]} \sup_{\theta\in]-\infty,\,t-a]} \mathbb{E}||f(\theta)||^2 d\mu(t)$$

we obtain

$$\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]} \sup_{\theta\in]-\infty,\,t-a]} \mathbb{E}||f(\theta)||^2 d\mu(t) \leqslant \frac{\nu([-\tau-2a^-,\tau+2a^+]}{\nu([-\tau,\tau])} \times M_a(\tau+|a|).$$

That implies ,
$\dfrac{1}{\nu([-\tau,\tau])}\displaystyle\int_{[-\tau,\tau]} \sup_{\theta\in]-\infty,\,t-a]} \mathbb{E}||f(\theta)||^2 d\mu(t) \leqslant \dfrac{\nu([-\tau-2a^-,\tau+2a^+]}{\nu([-\tau,\tau])} \times M_a(\tau+|a|)$
That implies

$$\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]} \sup_{\theta\in]-\infty,\,t-a]} \mathbb{E}||f(\theta)||^2 d\mu(t) \leqslant \frac{\nu([-\tau-2|a|,\tau+2|a|])}{\nu([-\tau,\tau])} \times M_a(\tau+|a|). \tag{2.5}$$

From equation(2.4) and equation(2.5) and using Lemma (2.21) we deduce that

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\sup_{\theta\in]-\infty,\,t-a]}\mathbb{E}||f(\theta)||^2d\mu(t)=0$$

which equivalent to

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\sup_{\theta\in]-\infty,\,t]}\mathbb{E}||f(\theta-a)||^2d\mu(t)=0.$$

That is $f_{-a}\in\mathcal{E}(\mathbb{R},L^2(P,H),\mu,\nu,\infty)$. We have proved that $f\in\mathcal{E}(\mathbb{R},L^2(P,H),\mu,\nu,\infty)$ then $f_{-a}\in\mathcal{E}(\mathbb{R},L^2(P,H),\mu,\nu,\infty)$ for $a\in\mathbb{R}$. That is $\mathcal{E}(\mathbb{R},L^2(P,H),\mu,\nu,\infty)$ is translation invariant.

Proposition 2.23. *Let ν, $\mu\in\mathcal{M}$ satisfy . Then $SPAA(\mathbb{R},L^2(P,H),\mu,\nu,\infty)$ is translation invariant, that is for all $\alpha\in\mathbb{R}$ and $f\in SPAA(\mathbb{R},L^2(P,H),\mu,\nu,\infty)$, $f_\alpha\in SPAA(\mathbb{R},L^2(P,H),\mu,\nu,\infty)$.*

Lemma 2.24. *(Ito's isometry). [13] Let $W:[0,T]\times\Omega\to\mathbb{R}$ denote the canonical real-valued Wiener process defined up to time $T>0$, and let $X:[0,T]\times\Omega\to\mathbb{R}$ be a stochastic process that is adapted to the natural filtration $\mathcal{F}_*^W$ of the Wiener process. Then*

$$\mathbb{E}\left[\left(\int_0^T X_t\,\mathrm{d}W_t\right)^2\right]=\mathbb{E}\left[\int_0^T X_t^2\,\mathrm{d}t\right],$$

where $\mathbb{E}$ denotes expectation with respect to classical Wiener measure.

3. Square-Mean (μ,ν)-Pseudo Almost automorphic Process

In this section, we define square-mean (μ,ν)-pseudo almost automorphic and we study their basic properties.

Definition 3.1. *Let $f:\mathbb{R}\to L^2(P,H)$ be a continuous stochastic process. f is said be square-mean almost automorphic process if for every sequence of real numbers $(t'_n)_n$, we can extract a subsequence $(t_n)_n$ such that, for some stochastic process $g:\mathbb{R}\to L^2(P,H)$, we have*

$$\lim_{n\to+\infty}\mathbb{E}||f(t+t_n)-g(t)||^2=0 \text{ for all } t\in\mathbb{R}$$

and

$$\lim_{n\to+\infty}\mathbb{E}||g(t-t_n)-f(t)||^2=0 \text{ for all } t\in\mathbb{R}$$

We denote the space of all such stochastic process by $SAA(\mathbb{R},L^2(P,H))$.

Theorem 3.2. *[11] $SAA(\mathbb{R},L^2(P,H))$ equiped with the norm $||\cdot||_\infty$ is a Banach space.*

Definition 3.3. *Let $f:\mathbb{R}\to L^2(P,H)$ be a bounded continuous stochastic process. f is said be square-mean compact almost automorphic process if for every sequence of real numbers $(t'_n)_n$, we can extract a subsequence $(t_n)_n$ such that, for some stochastic process $h:\mathbb{R}\to L^2(P,H)$, we have*

$$\lim_{n\to+\infty}\mathbb{E}||f(t+t_n)-h(t)||^2=0 \text{ for all } t\in\mathbb{R}$$

and

$$\lim_{n\to+\infty}\mathbb{E}||h(t-t_n)-f(t)||^2=0 \text{ for all } t\in\mathbb{R}$$

uniformly on compact subsets of $\mathbb{R}$. We denote the space of all such stochastic process by $SAA_c(\mathbb{R},L^2(P,H))$.

Theorem 3.4. *$SAA_c(\mathbb{R},L^2(P,H))$ equiped with the norm $||\cdot||_\infty$ is a Banach space.*

Definition 3.5. *A function $f : \mathbb{R} \times L^2(P,H) \to L^2(P,H)$, $(t,x) \mapsto f(t,x)$, which is jointly continuous, is said to be square mean almost automorphic in $t \in \mathbb{R}$ for each $x \in L^2(P,H)$ if for every sequence of real numbers $(t'_n)_n$, there exist a subsequence $(t_n)_n$ such that for some function g*
$\lim_{n\to+\infty} \mathbb{E}||f(t+t_n,x) - g(t,x)||^2 = 0$ *and* $\lim_{n\to+\infty} \mathbb{E}||g(t-t_n,x) - f(t,x)||^2 = 0$
for each $t \in \mathbb{R}$ and each $x \in L^2(P,H)$.

We denote the space off all such stochastic processes by $SAA(\mathbb{R} \times L^2(P,H), L^2(P,H))$.

Definition 3.6. *Let $\mu,\nu \in \mathcal{M}$ and $f : \mathbb{R} \to L^2(P,H)$ be a continuous stochastic process.*
f is said be (μ,ν)-square mean pseudo almost automorphic process if it can be decomposed as follows

$$f = g + \phi,$$

where $g \in SAA(\mathbb{R}, L^2(P,H))$ and $\phi \in \mathcal{E}(\mathbb{R}, L^2(P,H), \mu, \nu)$.

We denote the space of all such stochastic processes by $SPAA(\mathbb{R}, L^2(P,H), \mu, \nu)$.

Definition 3.7. *Let $\mu,\nu \in \mathcal{M}$ and $f : \mathbb{R} \to L^2(P,H)$ be a continuous stochastic process.*
f is said be (μ,ν)-square mean compact pseudo almost automorphic process if it can be decomposed as follows

$$f = g + \phi,$$

where $g \in SAA_c(\mathbb{R}, L^2(P,H))$ and $\phi \in \mathcal{E}(\mathbb{R}, L^2(P,H), \mu, \nu)$.

We denote the space of all such stochastic processes by $SPAA_c(\mathbb{R}, L^2(P,H), \mu, \nu)$.
Hence, together with Theorem 2.22 and Definition 3.7, we arrive at the following conclusion.

Theorem 3.8. *Let μ, $\nu \in \mathcal{M}$ and $f \in SPAA(\mathbb{R}, L^2(P,H), \mu, \nu, \infty)$ be such that*

$$f = g + \phi,$$

where $g \in SAA(\mathbb{R}, L^2(P,H))$ and $\phi \in \mathcal{E}(\mathbb{R}, L^2(P,H), \mu, \nu, \infty)$. If $SPAA(\mathbb{R}, L^2(P,H), \mu, \nu, \infty)$ is translation invariant, then

$$\overline{\{f(t), t \in \mathbb{R}\}} \supset \{g(t), t \in \mathbb{R}\}. \tag{3.1}$$

The proof of Theorem 3.8 is similar to the proof of Theorem 4.1 in [5]

Theorem 3.9. *Let $\mu,\nu \in \mathcal{M}$. Assume that $SPAA(\mathbb{R}, L^2(P,H), \mu, \nu, \infty)$ is a Banach space with the norm $||\cdot||_\infty$.*

The proof of the theorem above is similar to the proof of Theorem 4.9 in [5].
Next, we study the composition of square-mean (μ,ν) pseudo almost automorphic processes.

Definition 3.10. *Let $\mu,\nu \in \mathcal{M}$. A continuous function $f(t,x) : \mathbb{R} \times L^2(P,H) \to L^2(P,H)$ is said to be square mean (μ,ν)-pseudo almost automorphic in t for any $x \in L^2(P,H)$ if it can be decomposed as $f = g + \phi$, where $g \in SAA(\mathbb{R} \times L^2(P,H), L^2(P,H))$, $\phi \in \mathcal{E}(\mathbb{R} \times L^2(P,H), \mu, \nu, \infty)$. Denote the set of all such stochastically continuous processes by $SPAA(\mathbb{R} \times L^2(P,H), L^2(P,H), \mu, \nu, \infty)$*

Theorem 3.11. *[11] Let $f : \mathbb{R} \times L^2(P,H) \to L^2(P,H)$, $(t,x) \mapsto f(t,x)$ be square-mean almost automorphic in $t \in \mathbb{R}$ for each $x \in L^2(P,H)$, and assume that f satisfies the Lipschitz condition in the following sense:*

$$\mathbb{E}||f(t,x) - f(t,y)||^2 \leqslant L.\mathbb{E}||x-y||^2$$

for all x, $y \in L^2(P,H)$ and for each $t \in \mathbb{R}$, where $L > 0$ is independent of t. Then for any square-mean almost automorphic process $x : \mathbb{R} \to L^2(P,H)$, the stochastic process $F : \mathbb{R} \to L^2(P,H)$ given by $F(t) := f(t,x(t))$ is square-mean almost automorphic.

Theorem 3.12. *Let* $\mu, \nu \in \mathcal{M}$*,* $\phi = \phi_1 + \phi_2 \in SPAA(\mathbb{R} \times L^2(P,H); L^2(P,H), \mu, \nu, \infty)$ *with* $\phi_1 \in SAA(\mathbb{R} \times L^2(P,H); L^2(P,H))$*,* $\phi_2 \in \mathcal{E}(\mathbb{R} \times L^2(P,H); L^2(P,H), \mu, \nu, \infty)$ *and* $h \in SPAA(\mathbb{R}; L^2(P,H), \mu, \nu, \infty)$*. Assume:*

$i)$ $\phi_1(t,x)$ *is uniformly continuous on any bounded subset uniformly for* $t \in \mathbb{R}$*.*
$ii)$ *there exist a nonnegative function* $L_\phi \in L^p(\mathbb{R}),\ (1 \le p \le \infty)$ *such that*

$$\mathbb{E}||\phi(t,x_1) - \phi(t,x_2)||^2 \le L_\phi(t)\mathbb{E}||x_1 - x_2||^2, \quad \textit{for all}\ \ t \in \mathbb{R} \quad \textit{and for all}\ \ x_1, x_2 \in L^2(P,H). \tag{3.2}$$

If

$$\beta = \lim_{\tau \to +\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \Big(\sup_{\theta \in]-\infty,t]} L_\phi(\theta) \Big) d\mu(t) < \infty \tag{3.3}$$

then the function $t \to \phi(t,h(t))$ *belongs to* $SPAA(\mathbb{R}; L^2(P,H), \mu, \nu, \infty)$*.*

To prove the theorem, we need the following lemma.

Lemma 3.13. *Assume* (***H₃***) *holds and let* $f \in SBC(\mathbb{R}; L^2(P,H))$*. Then* $f \in \mathcal{E}(\mathbb{R}; L^2(P,H), \mu, \nu, \infty)$ *if and only if for any* $\varepsilon > 0$*,*

$$\lim_{\tau \to +\infty} \frac{\mu(M_{\tau,\varepsilon}(f))}{\nu([-\tau,\tau]} = 0$$

where

$$M_{\tau,\varepsilon}(f) = \{t \in [-\tau,\tau] : \sup_{\theta \in]-\infty,t]} \mathbb{E}||f(\theta)||^2 \ge \varepsilon\}.$$

Proof. Suppose that $f \in \mathcal{E}(\mathbb{R}; L^2(P,H), \mu, \nu, \infty)$. Then

$$\begin{aligned}
\frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} \sup_{\theta \in]-\infty,t]} \mathbb{E}||f(\theta)||^2 d\mu(t) &= \frac{1}{\nu([-\tau,\tau])} \int_{M_{\tau,\varepsilon}(f)} \sup_{\theta \in]-\infty,t]} \mathbb{E}||f(\theta)||^2 d\mu(t) \\
&\quad + \frac{1}{\nu([-\tau,\tau])} \int_{[-\tau,\tau] \setminus M_{\tau,\varepsilon}(f)} \sup_{\theta \in]-\infty,t]} \mathbb{E}||f(\theta)||^2 d\mu(t) \\
&\ge \frac{1}{\nu([-\tau,\tau])} \int_{M_{\tau,\varepsilon}(f)} \sup_{\theta \in]-\infty,t]} \mathbb{E}||f(\theta)||^2 d\mu(t) \\
&\ge \frac{\varepsilon \mu(M_{\tau,\varepsilon}(f))}{\nu([-\tau,\tau])}.
\end{aligned}$$

Consequently

$$\lim_{\tau \to +\infty} \frac{\mu(M_{\tau,\varepsilon}(f))}{\nu([-\tau,\tau]} = 0.$$

Suppose that $f \in SBC(\mathbb{R}; L^2(P,H))$ such that for any $\varepsilon > 0$,

$$\lim_{\tau \to +\infty} \frac{\mu(M_{\tau,\varepsilon}(f))}{\nu([-\tau,\tau]} = 0.$$

Assume $\mathbb{E}||f(t)||^2 \leq N$ for all $t \in \mathbb{R}$, then using $(\mathbf{H_3})$, it follows that

$$\begin{aligned}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2d\mu(t) &= \frac{1}{\nu([-\tau,\tau])}\int_{M_{\tau,\varepsilon}(f)}\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2d\mu(t)\\ &\quad+\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]\setminus M_{\tau,\varepsilon}(f)}\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2d\mu(t)\\ &\leq \frac{N}{\nu([-\tau,\tau])}\int_{M_{\tau,\varepsilon}(f)}d\mu(t)\\ &\quad+\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]\setminus M_{\tau,\varepsilon}(f)}\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2d\mu(t)\\ &\leq \frac{N}{\nu([-\tau,\tau])}\int_{M_{\tau,\varepsilon}(f)}d\mu(t)+\frac{\varepsilon}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}d\mu(t)\\ &\leq \frac{N\mu(M_{\tau,\varepsilon}(f))}{\nu([-\tau,\tau])}+\frac{\varepsilon\mu([-\tau,\tau])}{\nu([-\tau,\tau])}.\end{aligned}$$

Which implies that

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in]-\infty,t]}\mathbb{E}||f(\theta)||^2d\mu(t)\leq\alpha\varepsilon \text{ for any } \varepsilon>0.$$

Therefore $f \in \mathcal{E}(\mathbb{R}; L^2(P,H),\mu,\nu,\infty)$.■
The following proof is for the Theorem(3.12).
Proof. Assume that $\phi = \phi_1 + \phi_2,\ h = h_1 + h_2$ where $\phi_1 \in AA(\mathbb{R}\times L^2(P,H); L^2(P,H)),\ \phi_2 \in \mathcal{E}(\mathbb{R}\times L^2(P,H); L^2(P,H),\mu,\nu,\infty)$ and $h_1 \in AA(\mathbb{R}; L^2(P,H)),\ h_2 \in \mathcal{E}(\mathbb{R}; L^2(P,H),\mu,\nu,\infty)$. Consider the following decomposition

$$\phi(t,h(t)) = \phi_1(t,h_1(t)) + [\phi(t,h(t)) - \phi(t,h_1(t))] + \phi_2(t,h_1(t)).$$

From [11], $\phi_1(.,h_1(.)) \in SAA(\mathbb{R}; L^2(P,H))$. It remains to prove that both $\phi(.,h(.)) - \phi(.,h_1(.))$ and $\phi_2(.,h_1(.))$ belong to $\mathcal{E}(\mathbb{R}; L^2(P,H),\mu,\nu,\infty)$. Clearly, $\phi(t,h(t)) - \phi(t,h_1(t))$ is bounded and continuous. Assume $\mathbb{E}||\phi(t,h(t)) - \phi(t,h_1(t))||^2 \leq N,\ \forall t\in\mathbb{R}$. Since $h(t),\ h_1(t)$ are bounded, choose a bounded subset $B\subset\mathbb{R}$ such that $h(\mathbb{R}), h_1(\mathbb{R})\subset B$. Under assumption (ii), for a given $\varepsilon>0$, $\mathbb{E}||x_1-x_2||^2\leq\varepsilon$, implies that $\mathbb{E}||\phi(t,x_1)-\phi(t,x_2)||^2\leq\varepsilon L_\phi(t)$, for all $t\in\mathbb{R}$. Since for $\delta\in\mathcal{E}(\mathbb{R}; L^2(P,H),\mu,\nu,\infty)$, Lemma 3.13 yields that

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau]}\mu(M_{\tau,\varepsilon}(\delta)) = 0.$$

Consequently

$$\begin{aligned}
&\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||\phi(\theta,h(\theta))-\phi(\theta,h_1(\theta))||^2\Big)d\mu(t)\\
&=\frac{1}{\nu([-\tau,\tau])}\int_{M_{\tau,\varepsilon}(\delta)}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||\phi(\theta,h(\theta))-\phi(\theta,h_1(\theta))||^2\Big)d\mu(t)\\
&+\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]\setminus M_{\tau,\varepsilon}(\delta)}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||\phi(\theta,h(\theta))-\phi(\theta,h_1(\theta))||^2\Big)d\mu(t)\\
&\leq\frac{N}{\nu([-\tau,\tau])}\int_{M_{\tau,\varepsilon}(\delta)}d\mu(t)+\frac{\varepsilon}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]\setminus M_{\tau,\varepsilon}(\delta)}\Big(\sup_{\theta\in]-\infty,t]}|L_\phi(\theta)|\Big)d\mu(t)\\
&\leq\frac{N}{\nu([-\tau,\tau])}\int_{M_{\tau,\varepsilon}(\delta)}d\mu(t)+\frac{\varepsilon}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\Big(\sup_{\theta\in]-\infty,t]}|L_\phi(\theta)|\Big)d\mu(t)\\
&\leq\frac{N\mu(M_{\tau,\varepsilon}(\delta))}{\nu([-\tau,\tau])}+\frac{\varepsilon}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\Big(\sup_{\theta\in]-\infty,t]}|L_\phi(\theta)|\Big)d\mu(t).
\end{aligned}$$

Which implies that

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||\phi(\theta,h(\theta))-\phi(\theta,h_1(\theta))||^2\Big)d\mu(t)\leq\varepsilon\beta\ \text{ for any } \varepsilon>0,$$

which shows that $t\mapsto\phi(t,h(t))-\phi(t,h_1(t))$ is (μ,ν)-ergodic of infinite class.
Now to complete the proof, it is enough to prove that $t\mapsto\phi_2(t,h(t))$ is (μ,ν)-ergodic of infinite class. Since ϕ_2 is uniformly continuous on the compact set $\Omega=\overline{\{h_1(t):\ t\in\mathbb{R}\}}$ with respect to the second variable x, then for given $\varepsilon>0$, there exists $\delta>0$ such that, for all $t\in\mathbb{R}$, ξ_1 and $\xi_2\in\Omega$, one has

$$\mathbb{E}||\xi_1-\xi_2||^2\leq\delta\Rightarrow\mathbb{E}||\phi_2(t,\xi_1(t))-\phi_2(t,\xi_2(t))||^2\leq\varepsilon.$$

Therefore, there exist $n(\varepsilon)$ and $\{z_i\}_{i=1}^{n(\varepsilon)}\subset\Omega$, such that

$$\Omega\subset\bigcup_{i=1}^{n(\varepsilon)}B_\delta(z_i,\delta)$$

and then

$$\mathbb{E}||\phi_2(t,h_1(t))||^2\leq\varepsilon+\sum_{n=1}^{n(\varepsilon)}\mathbb{E}||\phi_2(t,z_i)||^2$$

Since

$$\forall i\in\{1,...,n(\varepsilon)\},\quad\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||\phi_2(\theta,z_i)||^2\Big)d\mu(t)=0,$$

then

$$\forall\varepsilon>0,\quad\limsup_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||\phi_2(\theta,h_1(t))||^2\Big)d\mu(t)\leq\varepsilon,$$

that implies

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\sup_{\theta\in]-\infty,t]}\mathbb{E}||\phi_2(\theta,h_1(t))||^2\Big)d\mu(t)=0.$$

Consequently $t\mapsto\phi_2(t,h(t))$ is (μ,ν)-ergodic of infinite class.■

4. Square-mean pseudo almost automorphic solutions of infinite class

($\mathbf{H_5}$): g is a stochastically bounded process.

Theorem 4.1. *Assume that* ($\boldsymbol{H_0}$), ($\boldsymbol{H_1}$), ($\boldsymbol{H_4}$) *and* ($\boldsymbol{H_5}$) *hold and the semigroup* $(\mathcal{U}(t))_{t\geqslant 0}$ *is hyperbolic. If* f *is bounded and continuous on* $\mathbb{R}$*, then there exists a unique bounded solution* u *of equation (1.1) on* $\mathbb{R}$ *given by*

$$\begin{aligned} u_t = & \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds \\ & + \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s) + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda (X_0 g(s))dW(s) \end{aligned}$$

$\forall \quad t \geq 0$*, where* $\widetilde{B}_\lambda = \lambda(\lambda I - \widetilde{\mathcal{A}}_{\mathcal{U}})^{-1}$, Π^s *and* Π^u *are the projections of* $\mathcal{B}_A$ *onto the stable and unstable subspaces.*

Proof. Let $u_t = v(t) + \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)$
$+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))dW(s)\forall \quad t\geq 0$, where
$v(t) = \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds$

Let us first prove that u_t exists. The existence of $v(t)$ have proved by [1]. Now, we show that the limit $\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)$ exist.
For $t\in\mathbb{R}$ and using the Ito's isometry property of the stochastic integral we have,

$$\begin{aligned} \mathbb{E}\left|\left|\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s\ (\widetilde{B}_\lambda X_0 g(s))dW(s)\right|\right|^2 & \leqslant \mathbb{E}\int_{-\infty}^{t}\overline{M}^2 e^{-2w(t-s)}|\Pi^s|^2||\ (\widetilde{B}_\lambda X_0 g(s))||^2 ds \\ & \leqslant \overline{M}^2\mathbb{E}\int_{-\infty}^{t} e^{-2w(t-s)}|\Pi^s|^2||\ (\widetilde{B}_\lambda X_0 g(s))||^2 ds \\ & \leqslant \overline{M}^2\widetilde{M}^2|\Pi^s|^2\mathbb{E}\int_{-\infty}^{t} e^{-2w(t-s)}||g(s)||^2 ds \\ & \leqslant \overline{M}^2\widetilde{M}^2|\Pi^s|^2\sum_{n=1}^{\infty}\mathbb{E}\left(\int_{t-n}^{t-n+1} e^{-2w(t-s)}||g(s)||^2 ds\right). \end{aligned}$$

then, using the Holders inequality, we obtain

$$\begin{aligned} & \mathbb{E}\left|\left|\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s\ (\widetilde{B}_\lambda X_0 g(s))dW(s)\right|\right|^2 \\ & \leqslant \overline{M}^2\widetilde{M}^2|\Pi^s|^2\sum_{n=1}^{+\infty}\left(\int_{t-n}^{t-n+1} e^{-4w(t-s)}ds\right)^{\frac{1}{2}}\mathbb{E}\left(\int_{t-n}^{t-n+1}||g(s)||^2 ds\right)^{\frac{1}{2}} \\ & \leqslant \overline{M}^2\widetilde{M}^2|\Pi^s|^2\frac{1}{2\sqrt{w}}\sum_{n=1}^{\infty}\left(e^{-4w(n-1)} - e^{-4wn}\right)^{\frac{1}{2}}\mathbb{E}\left(\int_{t-n}^{t-n+1}||g(s)||^2 ds\right)^{\frac{1}{2}} \\ & \leqslant \overline{M}^2\widetilde{M}^2|\Pi^s|^2\frac{1}{2\sqrt{w}}(e^{4wn}-1)^{\frac{1}{2}}\sum_{n=1}^{\infty}e^{-2wn}\times\mathbb{E}\left(\int_{t-n}^{t-n+1}||g(s)||^2 ds\right)^{\frac{1}{2}} \end{aligned}$$

Since the serie $\sum_{n=1}^{\infty} e^{-2wn}$ is convergent, then it exists a constant $c > 0$ such that $\sum_{n=1}^{\infty} e^{-2wn} \leqslant c$, moreover it follows that

$$\begin{aligned}\mathbb{E}\Big|\Big|\int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s\,(\widetilde{B}_\lambda X_0 g(s))dW(s)\Big|\Big|^2 &\leqslant \overline{M}\widetilde{M}^2|\Pi^s|^2\frac{1}{2\sqrt{w}}(e^{4w}-1)^{\frac{1}{2}}\mathbb{E}||g(s)||\sum_{n=1}^{\infty} e^{-2wn}\\ &\leqslant \gamma\sum_{n=1}^{\infty} e^{-2wn}\\ &\leqslant \gamma c,\end{aligned}$$

where, $\gamma = \overline{M}^2\widetilde{M}^2|\Pi^s|^2\frac{1}{2\sqrt{w}}(e^{4w}-1)^{\frac{1}{2}}\mathbb{E}||g(s)||$.

Let $F(n,s,t) = \mathcal{U}^s(t-s)\Pi^s(\,\widetilde{B}_\lambda X_0 g(s))$ for $n \in \mathbb{N}$ for $s \leqslant t$.

For n is sufficiently large and $\sigma \leqslant t$ and using the Ito's isometry property of the stochastic integral we get the following result

$$\begin{aligned}&\mathbb{E}\Big|\Big|\int_{-\infty}^{\sigma} F(n,s,t)dW(s)\Big|\Big|^2\\ &\leqslant \overline{M}^2\widetilde{M}^2|\Pi^s|^2\sum_{n=1}^{+\infty}\Big(\int_{\sigma-n}^{\sigma-n+1} e^{-4w(t-s)}ds\Big)^{\frac{1}{2}}\times\mathbb{E}\left(\int_{\sigma-n}^{\sigma-n+1}||g(s)||^2ds\right)^{\frac{1}{2}}\\ &\leqslant \overline{M}^2\widetilde{M}^2|\Pi^s|^2\frac{1}{2\sqrt{\omega}}\Big(\sum_{n=1}^{\infty}\left(e^{-4\omega(t-\sigma+n-1)}-e^{-4\omega(t-\sigma+n)}\right)^{\frac{1}{2}}\\ &\times\mathbb{E}\Big(\int_{\sigma-n}^{\sigma-n+1}||g(s)||^2ds\Big)^{\frac{1}{2}}\Big)\\ &\leqslant \overline{M}^2\widetilde{M}^2|\Pi^s|^2\frac{1}{2\sqrt{\omega}}e^{-2\omega(t-\sigma)}(e^{4\omega}-1)^{\frac{1}{2}}\sum_{n=1}^{\infty} e^{-2\omega n}\times\mathbb{E}\left(\int_{\sigma-n}^{\sigma-n+1}||g(s)||^2ds\right)^{\frac{1}{2}}\\ &\leqslant \gamma c e^{-2w(t-\sigma)}\end{aligned}$$

It follow that for n and m sufficiently large and $\sigma \leqslant t$, we have

$$\begin{aligned}\mathbb{E}\Big|\Big|\int_{-\infty}^{t} F(n,s,t)dW(s)-\int_{\infty}^{t} F(m,s,t)dW(s)\Big|\Big|^2 &\leqslant \mathbb{E}\Big|\Big|\int_{-\infty}^{\sigma} F(n,s,t)dW(s)+\int_{\sigma}^{t} F(n,s,t)dW(s)\\ &-\int_{-\infty}^{\sigma} F(m,s,t)dW(s)-\int_{\sigma}^{t} F(m,s,t)dW(s)\Big|\Big|\\ &\leqslant 3\mathbb{E}\Big|\Big|\int_{-\infty}^{\sigma} F(n,s,t)dW(s)\Big|\Big|^2+3\mathbb{E}\Big|\Big|\int_{-\infty}^{\sigma} F(m,s,t)dW(s)\Big|\Big|^2\\ &+3\mathbb{E}\Big|\Big|\int_{\sigma}^{t} F(n,s,t)dW(s)-\int_{\sigma}^{t} F(m,s,t)dW(s)\Big|\Big|^2\\ &\leqslant 6\gamma c e^{-2\omega(t-\sigma)}+3\mathbb{E}\Big|\Big|\int_{\sigma}^{t} F(n,s,t)dW(s)-\int_{\sigma}^{t} F(m,s,t)dW(s)\Big|\Big|^2\end{aligned}$$

Since $\lim_{n\to+\infty}\mathbb{E}\Big|\Big|\int_{\sigma}^{t} F(n,s,t)dW(s)\Big|\Big|^2$ exists, then

$$\limsup_{n,m\to+\infty}\mathbb{E}\Big|\Big|\int_{-\infty}^{t} F(n,s,t)dW(s)-\int_{-\infty}^{t} F(m,s,t)dW(s)\Big|\Big|^2 \leqslant 6\gamma c e^{-2\omega(t-\sigma)}$$

If $\sigma \to -\infty$, then

$$\limsup_{n,m\to+\infty} \mathbb{E}\Big|\Big|\int_{-\infty}^{t} F(n,s,t)dW(s) - \int_{-\infty}^{t} F(m,s,t)dW(s)\Big|\Big|^2 = 0.$$

We deduce that the limit

$$\lim_{n\to+\infty} \mathbb{E}\Big|\Big|\int_{-\infty}^{t} F(n,s,t)dW(s)\Big|\Big|^2 = \lim_{n\to+\infty} \mathbb{E}\Big|\Big|\int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_n X_0 g(s))dW(s)\Big|\Big|^2$$

exists. Therefore, $\lim_{n\to+\infty}\int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_n X_0 g(s))dW(s)$ exists. In addition, one can show that the function

$$t \to \lim_{n\to+\infty} \mathbb{E}\Big|\Big|\int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_n X_0 g(s))ds\Big|\Big|^2$$

is bounded on $\mathbb{R}$. Similary, we can show that the function

$$t \to \lim_{n\to+\infty}\int_{t}^{+\infty} \mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_n X_0 g(s))dW(s)$$

is well defined and bounded on $\mathbb{R}$.■

Theorem 4.2. *Assume that (**H₃**) holds. Let $\mu,\nu \in \mathcal{M}$ and $\phi \in SPAA_c(\mathbb{R}, L^2(P,H),\mu,\nu,\infty)$ then the function $t \to \phi_t$, belongs to $SPAA_c(C(]-\infty,0], L^2(P,H)),\mu,\nu,\infty)$.*

Proof. Assume that $\phi = v + h$, where $v \in SAA_c(\mathbb{R}, L^2(P,H))$ and $h \in \mathcal{E}(\mathbb{R}, L^2(P,H),\mu,\nu,\infty)$. We have $\phi_t = v_t + h_t$. Firstly, we show that $v_t \in SAA_c(\mathbb{R}, L^2(P,H))$.
Let $(s_m)_{m\in\mathbb{N}}$ of real numbers, fix a subsequence $(s_n)_{n\in\mathbb{N}}$ and $w \in SBC(\mathbb{R}, L^2(P,H)))$ such that $v(s+s_n) \longrightarrow w(s)$ uniformly on compact subsets of $\mathbb{R}$. Let $K \subset [-L;L]$. For $\varepsilon > 0$ fix $N_{\varepsilon,L} \in \mathbb{N}$ such that $\mathbb{E}||v(s+s_n) - w(s)||^2 \leqslant \varepsilon$ for $s \in [-L;L]$. Whenerver $n \geqslant N_{\varepsilon,L}$. For $t \in K$ and $n \geqslant N_{\varepsilon,L}$ we have

$$\begin{aligned}\mathbb{E}||v_{t+s_n} - w_t||^2 &\leqslant \sup_{\theta\in[-L;L]} \mathbb{E}||v(\theta+s_n) - w(\theta)||^2 \\ &\leqslant \varepsilon\end{aligned}$$

then, v_{t-s_n} converges to w_t uniformly in K. Simlarly, we can show prove that w_{t-s_n} converges to v_t uniformly in K.
Finaly, we show that $h_t \in \mathcal{E}(\mathbb{R}, L^2(P,H),\mu,\nu,\infty)$

$$M_\alpha = \frac{1}{\nu_\alpha([-\tau,\tau])}\int_{-\tau}^{\tau}(\sup_{\theta\in]-\infty,t]} \mathbb{E}||h(\theta)||^2)d\mu_\alpha(t).$$

Where μ_α and ν_α are the positive measures defined by equation (4.3). By using Lemma (2.20), it follows that μ_α and μ are equivalent and ν_α and ν are also equivalent. Then by using Theorem (3.8) we have $\mathcal{E}(\mathbb{R}, L^2(P,H),\mu_\alpha,\nu_\alpha,\infty) = \mathcal{E}(\mathbb{R}, L^2(P,H),\mu,\nu,\infty)$ therefore $h \in \mathcal{E}(\mathbb{R}, L^2(P,H),\mu_\alpha,\nu_\alpha,\infty)$ that is $\lim_{\tau\to+\infty} M_\alpha(\tau) = 0$ for all $\alpha \in \mathbb{R}$. On the other hand, for $r > 0$ we have

$$\begin{aligned}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\left(\sup_{\eta\in]-\infty,0]}\left(\mathbb{E}||h(\theta+\eta)||^2\right)\right)d\mu(t) &\leqslant \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\left(\mathbb{E}||h(\theta)||^2\right)d\mu(t) \\
\leqslant \quad & \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big[\sup_{\theta\in]-\infty,t-r]}\left(\mathbb{E}||h(\theta)||^2\right) + \sup_{\theta\in]-\infty,t]}\left(\mathbb{E}||h(\theta)||^2\right)\Big]d\mu(t) \\
\leqslant \quad & \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,\tau+r]}\left(\mathbb{E}||h(\theta)||^2\right)d\mu(t+r) + \int_{-\tau}^{\tau}\sup_{\theta\in]\infty,t]}\left(\mathbb{E}||h(\theta)||^2\right)d\mu(t) \\
\leqslant \quad & \frac{\nu([-\tau-r,\tau+r])}{\nu([-\tau,\tau])} \times \frac{1}{\nu([-\tau-r,\tau+r])}\int_{-\tau-r}^{\tau+r}\sup_{\theta\in]\infty,t]}\mathbb{E}||h(\theta)||^2 d\mu(t+r) \\
+ \quad & \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\mathbb{E}||h(\theta)||^2 d\mu(t)\end{aligned}$$

Consequently,

$$\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\left(\sup_{\eta\in]-\infty,0]}\left(\mathbb{E}||h(\theta+\eta)||^2\right)\right)d\mu(t) \leqslant \frac{\nu([-\tau-r,\tau+r])}{\nu([-\tau,\tau])}\times M_r(\tau+r) + \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\mathbb{E}||h(\theta)||^2 d\mu(t)$$

which shows using Lemma(2.21) and Lemma (2.20) that ϕ_t belongs to $SPAA_c(C(]-\infty,0],L^2(P,H)),\mu,\nu,\infty)$. Thus, we obtain the desired result ■

Theorem 4.3. *Let f, $g\in SAA_c(\mathbb{R},X)$ and Γ be the mapping defined for $t\in\mathbb{R}$ by*

$$\Gamma(f,g)(t) = \Big[\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds + \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s) + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)\Big](0)$$

Then $\Gamma(f,g)\in SAA_c(\mathbb{R},L^2(P,H))$.

Proof. Let $(s_m)_{m\in\mathbb{N}}$ of real numbers, fix a subsequence $(s_n)_{n\in\mathbb{N}}$ and $v,h\in SBC(\mathbb{R},L^2(P,H))$ such that $f(t+s_n)$ converges to $v(t)$ and $g(t+s_n)$ converges to $h(t)$ uniformly on compact subsets of $\mathbb{R}$. using Lemma 2.4 and Theorem 2.10, we get the following estimates

$$\lim_{\lambda\to+\infty}||\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))||^2 \leqslant \overline{M}^2\widetilde{M}^2|\Pi^s|^2 e^{-2\omega(t-s)}||f(s)||^2 \tag{4.1}$$

$$\lim_{\lambda\to+\infty}||\mathcal{U}^u(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))||^2 \leqslant \overline{M}^2\widetilde{M}^2|\Pi^u|^2 e^{2\omega(t-s)}||f(s)||^2 \tag{4.2}$$

$$\lim_{\lambda\to+\infty}||\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))||^2 \leqslant \overline{M}^2\widetilde{M}^2|\Pi^s|^2 e^{-2\omega(t-s)}||g(s)||^2 \tag{4.3}$$

and

$$\lim_{\lambda\to+\infty}||\mathcal{U}^u(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))||^2 \leqslant \overline{M}^2\widetilde{M}^2|\Pi^u|^2 e^{2\omega(t-s)}||g(s)||^2 \tag{4.4}$$

Therefore, if

$$w(t+s_n) = \Big[\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s+s_n))ds + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s+s_n))ds + \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s+s_n))dW(s) + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s+s_n))dW(s)\Big]$$

then by Equations.(4.1), (4.2), (4.3) and (4.4) and the Lebesgue Dominated convergence Theorem, we have $w(t+s_n)$ that converges to $v(t)$.

$$v(t) = \Big[\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds + \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s) + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)\Big]$$

Now, It remains to prove that the convergence is uniform on all compact subset of $\mathbb{R}$. Let $K\subset\mathbb{R}$ be an arbitrary compact and let $\varepsilon>0$. We fix $L>0$ and $N_\varepsilon\in\mathbb{N}$ such that $K\subset\left[\frac{-L}{2};\frac{L}{2}\right]$ with,

$$\int_{\frac{L}{2}}^{+\infty}e^{-2\omega s}ds<\varepsilon.$$

$$\mathbb{E}||f(s+s_n)-v(s)||^2 \leqslant \varepsilon \text{ for } n \geqslant N_\varepsilon \text{ and } s \in [-L,L]. \tag{4.5}$$

and

$$\mathbb{E}||g(s+s_n)-h(s)||^2 \leqslant \varepsilon \text{ for } n \geqslant N_\varepsilon \text{ and } s \in [-L,L]. \tag{4.6}$$

Then, for each $t \in K$, ones has

$$\begin{aligned}
&\mathbb{E}||w(t+s_n)-z(t)||^2 \\
&= \mathbb{E}\Big|\Big| \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s+s_n))ds + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s+s_n))ds \\
&+ \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s+s_n))dW(s) + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s+s_n))dW(s) \\
&- \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 v(s))ds - \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^s(\widetilde{B}_\lambda X_0 v(s))ds \\
&- \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 h(s))dW(s) - \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^s(\widetilde{B}_\lambda X_0 h(s))dW(s)\Big|\Big|^2 \\
&\leqslant 4\Big(\mathbb{E}\Big|\Big|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0(f(s+s_n)-v(s)))ds\Big|\Big|^2 \\
&+ \mathbb{E}\Big|\Big|\lim_{\lambda\to+\infty}\int_{t}^{+\infty}\mathcal{U}^u(t-s)\Pi^s(\widetilde{B}_\lambda X_0(f(s+s_n)-v(s)))ds\Big|\Big|^2 \\
&+ \mathbb{E}\Big|\Big|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda (X_0 g(s+s_n)-h(s)))dW(s)\Big|\Big|^2 \\
&+ \mathbb{E}\Big|\Big|\lim_{\lambda\to+\infty}\int_{t}^{+\infty}\mathcal{U}^u(t-s)\Pi^s(\widetilde{B}_\lambda X_0(g(s+s_n)-h(s)))dW(s)\Big|\Big|^2\Big)
\end{aligned}$$

progressively, we increase each terms of previous inegalitie.

$$\begin{aligned}
&\mathbb{E}\Big|\Big|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0(f(s+s_n)-v(s)))ds\Big|\Big|^2 \\
&\leqslant \mathbb{E}\Big(\lim_{\lambda\to+\infty}\Big|\Big|\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0(f(s+s_n)-v(s)))ds\Big|\Big|^2\Big) \\
&\leqslant \mathbb{E}\Big(\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\Big|\Big|\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0(f(s+s_n)-v(s)))ds\Big|\Big|^2\Big) \\
&\leqslant \mathbb{E}\Big(\int_{-\infty}^{t}\overline{M}^2\widetilde{M}^2 e^{-2\omega(t-s)}|\Pi^s|^2\Big|\Big|f(s+s_n)-v(s)\Big|\Big|^2 ds\Big) \\
&\leqslant \int_{-\infty}^{t}\overline{M}^2\widetilde{M}^2 e^{-2\omega(t-s)}|\Pi^s|^2\mathbb{E}\Big|\Big|f(s+s_n)-v(s)\Big|\Big|^2 ds \\
&\leqslant \widetilde{M}^2|\Pi^s|^2\int_{-\infty}^{-L}e^{-2\omega(t-s)}\mathbb{E}\Big|\Big|f(s+s_n)-v(s)\Big|\Big|^2 ds \\
&+ \overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-L}^{t}e^{-2\omega(t-s)}\mathbb{E}\Big|\Big|f(s+s_n)-v(s)\Big|\Big|^2 ds
\end{aligned}$$

$$\begin{aligned}
&\mathbb{E}\Big|\Big|\lim_{\lambda\to+\infty}\int_{t}^{+\infty}\mathcal{U}^u(t-s)\Pi^s(\widetilde{B}_\lambda X_0(f(s+s_n)-v(s)))ds\Big|\Big|^2 \\
&\leqslant \mathbb{E}\Big(\lim_{\lambda\to+\infty}\int_{t}^{+\infty}\Big|\Big|\mathcal{U}^u(t-s)\Pi^s(\widetilde{B}_\lambda X_0(f(s+s_n)-v(s)))ds\Big|\Big|^2\Big) \\
&\leqslant \mathbb{E}\Big(\int_{t}^{+\infty}\overline{M}^2\widetilde{M}^2 e^{-2\omega(t-s)}|\Pi^u|^2\Big|\Big|f(s+s_n)-v(s)\Big|\Big|^2 ds\Big) \\
&\leqslant \overline{M}^2\widetilde{M}^2|\Pi^u|^2\int_{t}^{+\infty}e^{-2\omega(t-s)}\mathbb{E}\Big|\Big|f(s+s_n)-v(s)\Big|\Big|^2 ds
\end{aligned}$$

Using Ito's isometry property of stochastic integral, we obtain that

$$\begin{aligned}
&\mathbb{E}\Big|\Big|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda(X_0g(s+s_n)-h(s)))dW(s)\Big|\Big|^2\\
&\leqslant \mathbb{E}\Big(\lim_{\lambda\to+\infty}\Big|\Big|\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda(X_0g(s+s_n)-h(s)))dW(s)\Big|\Big|^2\Big)\\
&\leqslant \mathbb{E}\Big(\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\Big|\Big|\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda(X_0g(s+s_n)-h(s)))ds\Big|\Big|^2\Big)\\
&\leqslant \overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^{t}e^{-2\omega(t-s)}\mathbb{E}\Big|\Big|g(s+s_n)-h(s)\Big|\Big|^2ds\\
&\leqslant \overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^{-L}e^{-2\omega(t-s)}\mathbb{E}\Big|\Big|g(s+s_n)-h(s)\Big|\Big|^2ds\\
&+\ \overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-L}^{t}e^{-2\omega(t-s)}\mathbb{E}\Big|\Big|g(s+s_n)-h(s)\Big|\Big|^2ds
\end{aligned}$$

and,

$$\begin{aligned}
&\mathbb{E}\Big|\Big|\lim_{\lambda\to+\infty}\int_{t}^{+\infty}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda(X_0g(s+s_n)-h(s)))dW(s)\Big|\Big|^2\\
&\leqslant \mathbb{E}\Big(\lim_{\lambda\to+\infty}\Big|\Big|\int_{-\infty}^{t}\mathcal{U}^u(t-s)\Pi^s(\widetilde{B}_\lambda(X_0g(s+s_n)-h(s)))dW(s)\Big|\Big|^2\Big)\\
&\leqslant \mathbb{E}\Big(\lim_{\lambda\to+\infty}\int_{t}^{+\infty}\Big|\Big|\mathcal{U}^s(t-s)\Pi^u(\widetilde{B}_\lambda(X_0g(s+s_n)-h(s)))ds\Big|\Big|^2\Big)\\
&\leqslant \overline{M}^2\widetilde{M}^2|\Pi^u|^2\int_{t}^{+\infty}e^{-2\omega(t-s)}\mathbb{E}\Big|\Big|g(s+s_n)-h(s)\Big|\Big|^2ds
\end{aligned}$$

Consequently,

$$\begin{aligned}
\mathbb{E}||w(t+s_n)-z(t)||^2 \ \leqslant\ & 4\Big(\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^{-L}e^{-2\omega(t-s)}\mathbb{E}\Big|\Big|f(s+s_n)-v(s)\Big|\Big|^2ds\\
&+\ \overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-L}^{t}e^{-2\omega(t-s)}\mathbb{E}\Big|\Big|f(s+s_n)-v(s)\Big|\Big|^2ds\\
&+\ \overline{M}^2\widetilde{M}^2|\Pi^u|^2\int_{t}^{+\infty}e^{-2\omega(t-s)}\mathbb{E}\Big|\Big|f(s+s_n)-v(s)\Big|\Big|^2ds\\
&+\ \overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^{-L}e^{-2\omega(t-s)}\mathbb{E}\Big|\Big|g(s+s_n)-h(s)\Big|\Big|^2ds\\
&+\ \overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-L}^{t}e^{-2\omega(t-s)}\mathbb{E}\Big|\Big|g(s+s_n)-h(s)\Big|\Big|^2ds\\
&+\ \overline{M}^2\widetilde{M}^2|\Pi^u|^2\int_{t}^{+\infty}e^{-2\omega(t-s)}\mathbb{E}\Big|\Big|g(s+s_n)-h(s)\Big|\Big|^2ds\Big)\\
\mathbb{E}||w(t+s_n)-z(t)||^2 \ \leqslant\ & 4\Big(2\varepsilon\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^{-L}e^{-2\omega(t-s)}ds\\
&+\ \overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-L}^{t}e^{-2\omega(t-s)}\mathbb{E}\Big|\Big|f(s+s_n)-v(s)\Big|\Big|^2ds\\
&+\ \overline{M}^2\widetilde{M}^2|\Pi^u|^2\int_{t}^{+\infty}e^{-2\omega(t-s)}\mathbb{E}\Big|\Big|f(s+s_n)-v(s)\Big|\Big|^2ds\\
&+\ \overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-L}^{t}e^{-2\omega(t-s)}\mathbb{E}\Big|\Big|g(s+s_n)-h(s)\Big|\Big|^2ds\\
&+\ \overline{M}^2\widetilde{M}^2|\Pi^u|^2\int_{t}^{+\infty}e^{-2\omega(t-s)}\mathbb{E}\Big|\Big|g(s+s_n)-h(s)\Big|\Big|^2ds\Big).
\end{aligned}$$

Therefore,

$$
\begin{aligned}
\mathbb{E}||w(t+s_n)-z(t)||^2 &\leqslant 4\Big(2\varepsilon\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{t+L}^{+\infty}e^{-2\omega s}ds \\
&+ \overline{M}^2\widetilde{M}^2(|\Pi^s|^2+|\Pi^u|^2)\int_{-L}^{+\infty}e^{-2\omega(t-s)}\mathbb{E}\Big|\Big|f(s+s_n)-v(s)\Big|\Big|^2ds \\
&+ \overline{M}^2\widetilde{M}^2(|\Pi^s|^2+|\Pi^u|^2)\int_{-L}^{+\infty}e^{-2\omega(t-s)}\mathbb{E}\Big|\Big|g(s+s_n)-h(s)\Big|\Big|^2ds\Big) \\
&\leqslant 4\Big(2\varepsilon\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{\frac{L}{2}}^{+\infty}e^{-2\omega s}ds+2\varepsilon\overline{M}^2\widetilde{M}^2(|\Pi^s|^2+|\Pi^u|^2)\int_0^{+\infty}e^{-2\omega s}ds\Big) \\
&\leqslant \Big(8\varepsilon\overline{M}^2\widetilde{M}^2|\Pi^s|^2+8\overline{M}^2\widetilde{M}^2(|\Pi^s|^2+|\Pi^u|^2)\int_0^{+\infty}e^{-2\omega s}ds\Big)\varepsilon \\
&\leqslant \Big(8\varepsilon\overline{M}^2\widetilde{M}^2|\Pi^s|^2+\frac{4\overline{M}^2\widetilde{M}^2(|\Pi^s|^2+|\Pi^u|^2)}{\omega}\Big)\varepsilon
\end{aligned}
$$

which proves that the convergence is uniform on K, by the fact that the last estimate is independent of $t\in K$. Proceeding as previously, one can similarly prove that $z(t-s_n)$ converges to w uniformly on compact subsets in $\mathbb{R}$. This completes the proof.■

Theorem 4.4. *Assume that* (***H₃***) *and* (***H₅***) *holds. Let* $f,g\in\mathcal{E}(\mathbb{R},L^2(P,H),\mu,\nu,\infty)$ *then* $\Gamma(f,g)\in\mathcal{E}(\mathbb{R},L^2(P,H),\mu,\nu,\infty)$.

Proof.

$$
\begin{aligned}
\Gamma(f,g)(t) &= \lim_{\lambda\to+\infty}\int_{-\infty}^t\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0f(s))ds+\lim_{\lambda\to+\infty}\int_{+\infty}^t\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0f(s))ds \\
&+\lim_{\lambda\to+\infty}\int_{-\infty}^t\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0g(s))dW(s)+\lim_{\lambda\to+\infty}\int_{+\infty}^t\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0g(s))dW(s)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\Big|\Big|\Gamma(f,g)(\theta)\Big|\Big|^2 &= \mathbb{E}\Big|\Big|\lim_{\lambda\to+\infty}\int_{-\infty}^t\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0f(s))ds+\lim_{\lambda\to+\infty}\int_{+\infty}^t\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0f(s))ds \\
&+\lim_{\lambda\to+\infty}\int_{-\infty}^t\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0g(s))dW(s)+\lim_{\lambda\to+\infty}\int_{+\infty}^t\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0g(s))dW(s)\Big|\Big|^2.
\end{aligned}
$$

$$
\begin{aligned}
\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\mathbb{E}||\Gamma(f,g)(\theta)||^2d\mu(t) &\leqslant \int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\Big[4\mathbb{E}\Big(\widetilde{M}^2\overline{M}^2\int_{-\infty}^{\theta}e^{-2\omega(t-s)}|\Pi^s|^2||f(s)||^2ds \\
&+\widetilde{M}^2\overline{M}^2\int_{\theta}^{+\infty}e^{2\omega(t-s)}|\Pi^u|^2||f(s)||^2ds \\
&+\widetilde{M}^2\overline{M}^2\int_{-\infty}^{\theta}e^{-2\omega(t-s)}|\Pi^s|^2||g(s)||^2ds \\
&+\widetilde{M}^2\overline{M}^2\int_{\theta}^{+\infty}e^{2\omega(t-s)}|\Pi^u|^2||g(s)||^2ds\Big)\Big]d\mu(t) \\
&\leqslant 4\widetilde{M}^2\overline{M}^2\Big[\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\Big(\int_{-\infty}^{\theta}e^{-2\omega(t-s)}|\Pi^s|^2\mathbb{E}||f(s)||^2ds\Big)d\mu(t) \\
&+\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\Big(\int_{\theta}^{+\infty}e^{2\omega(t-s)}|\Pi^u|^2\mathbb{E}||f(s)||^2ds\Big)d\mu(t)
\end{aligned}
$$

$$+\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\Big(\int_{-\infty}^{\theta}e^{-2\omega(t-s)}|\Pi^s|^2\mathbb{E}||g(s)||^2ds+\int_{\theta}^{+\infty}e^{2\omega(t-s)}|\Pi^u|^2\mathbb{E}||g(s)||^2ds\Big)d\mu(t)\Big]$$
$$\leqslant 4\widetilde{M}^2\overline{M}^2\Big[|\Pi^s|^2\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\Big(\int_{-\infty}^{\theta}e^{-2\omega(t-s)}(\mathbb{E}||f(s)||^2+\mathbb{E}||g(s)||^2)ds\Big)d\mu(t)$$
$$+|\Pi^u|^2\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\Big(\int_{\theta}^{+\infty}e^{2\omega(t-s)}(\mathbb{E}||f(s)||^2+\mathbb{E}||g(s)||^2)ds\Big)d\mu(t)\Big]$$

one the one hand using Fubini's theorem, we have

$$|\Pi^s|^2\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\Big(\int_{-\infty}^{\theta}e^{-2\omega(t-s)}(\mathbb{E}||f(s)||^2+\mathbb{E}||g(s)||^2)ds\Big)d\mu(t)$$
$$|\Pi^s|^2\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\int_{-\infty}^{\theta}e^{-2\omega(t-s)}(\mathbb{E}||f(s)||^2+\mathbb{E}||g(s)||^2)ds$$
$$\leqslant\quad e^{2\omega r}|\Pi^s|^2\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\Big(\int_{-\infty}^{\theta}e^{-2\omega(t-s)}(\mathbb{E}||f(s)||^2+\mathbb{E}||g(s)||^2)ds\Big)d\mu(t)$$
$$\leqslant\quad e^{2\omega r}|\Pi^s|^2\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\Big(\int_{-\infty}^{t}e^{-2\omega(t-s)}(\mathbb{E}||f(s)||^2+\mathbb{E}||g(s)||^2)ds\Big)d\mu(t)$$
$$\leqslant\quad e^{2\omega r}|\Pi^s|^2\int_{-\tau}^{\tau}\Big(\int_{-\infty}^{t}e^{-2\omega(t-s)}(\mathbb{E}||f(s)||^2+\mathbb{E}||g(s)||^2)ds\Big)d\mu(t)$$
$$\leqslant\quad e^{2\omega r}|\Pi^s|^2\int_{-\tau}^{\tau}\Big(\int_{0}^{+\infty}e^{-2\omega s}(\mathbb{E}||f(t-s)||^2+\mathbb{E}||g(t-s)||^2)ds\Big)d\mu(t)$$
$$\leqslant\quad e^{2\omega r}|\Pi^s|^2\int_{0}^{+\infty}e^{-2\omega s}\int_{-\tau}^{\tau}\Big(\mathbb{E}||f(t-s)||^2+\mathbb{E}||g(t-s)||^2\Big)d\mu(t)ds$$

By using Theorem(2.22) we deduce that
$\lim_{\tau\to+\infty}\frac{e^{-2\omega s}}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\mathbb{E}||f(t-s)||^2+\mathbb{E}||g(t-s)||^2\Big)d\mu(t)\to 0$ for all $s\in\mathbb{R}^+$ and
$\frac{e^{-2\omega s}}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\mathbb{E}||f(t-s)||^2+\mathbb{E}||g(t-s)||^2\Big)d\mu(t)\leqslant\frac{e^{-\omega s}}{\nu([-\tau,\tau])}\Big(||f||_\infty^2+||g||_\infty^2\Big)$
Since f and g are bounded functions, then the function $s\mapsto\frac{e^{-\omega s}}{\nu([-\tau,\tau])}\Big(||f||_\infty^2+||g||_\infty^2\Big)$ belongs to $L^1([0,+\infty[)$ in view of the Lebesgue dominated convergence Theorem, it follows that $e^{\omega r}\lim_{\tau\to+\infty}\int_0^{+\infty}\frac{e^{-2\omega s}}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\Big(\mathbb{E}||f(t-s)||^2+\mathbb{E}||g(t-s)||^2\Big)d\mu(t)ds\to 0.$
On the other hand by Fubinis theorem, we also have

$$|\Pi^u|^2\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\Big(\int_{\theta}^{+\infty}e^{2\omega(t-s)}(\mathbb{E}||f(s)||^2+\mathbb{E}||g(s)||^2)ds\Big)d\mu(t)$$
$$\leqslant\quad |\Pi^u|^2\int_{-\tau}^{\tau}\sup_{\theta\in]-\infty,t]}\Big(\int_{t-r}^{+\infty}e^{2\omega(t-s)}(\mathbb{E}||f(s)||^2+\mathbb{E}||g(s)||^2)ds\Big)d\mu(t)$$
$$\leqslant\quad |\Pi^u|^2\int_{-\tau}^{\tau}\Big(\int_{t-r}^{+\infty}e^{2\omega(t-s)}(\mathbb{E}||f(s)||^2+\mathbb{E}||g(s)||^2)ds\Big)d\mu(t)$$

$$\leqslant |\Pi^u|^2 \int_{-\tau}^{\tau} \Big(\int_{-\infty}^{r} e^{2\omega s} (\mathbb{E}||f(s)||^2 + \mathbb{E}||g(s)||^2) ds \Big) d\mu(t)$$
$$\leqslant |\Pi^u|^2 \int_{-\infty}^{r} \Big(\int_{-\tau}^{\tau} e^{2\omega s} (\mathbb{E}||f(s)||^2 + \mathbb{E}||g(s)||^2) d\mu(t) \Big) ds$$

Since the function $s \mapsto \dfrac{e^{2\omega s}}{\nu([-\tau,\tau])} \Big(||f||_\infty^2 + ||g||_\infty^2 \Big)$ belongs to $L^1(]-\infty, r])$ resoning like above, it follows that
$\lim\limits_{\tau \to +\infty} \displaystyle\int_{-\infty}^{r} e^{\omega s} \times \frac{1}{\nu([-\tau,\tau])} \Big(\int_{-\tau}^{\tau} e^{2\omega s} (\mathbb{E}||f(s)||^2 + \mathbb{E}||g(s)||^2) d\mu(t) \Big) ds = 0$
Consequently

$$\lim_{\tau \to +\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \sup_{\theta \in]-\infty, t]} \mathbb{E}||\Gamma(f,g)(\theta)||^2 d\mu(t) = 0$$

Thus, we obtain the desired result.■
For proof of existence of square-mean compact pseudo almost automorphic solution of infinite class , we need the following assertion.
(**H$_6$**) $f, g : \mathbb{R} \longrightarrow L^2(P,H)$ are square-mean compact pseudo almost automorpic of infinite class

Theorem 4.5. *Assume (**H$_0$**), (**H$_1$**) and (**H$_6$**) hold. Then Eq (4.1) has a unique pseudo almost automorpic solution of infinite class*

Proof. Since f and g are pseudo almost periodic functions, f has a decomposition $f = f_1 + f_2$ and $g = g_1 + g_2$ where $f_1, g_1 \in SAA_c(\mathbb{R}; L^2(P,H))$ and $f_2, g_2 \in \mathcal{E}(\mathbb{R}; L^2(P,H), \mu, \nu, \infty)$. Using Theorem 4.1, Theorem 4.3 and Theorem 4.4, we get the desired result. ■
Our next objective is to show the existence of square mean (μ,ν)-pseudo almost automorphic solutions of infinite class for the following problem

$$du(t) = [Au(t) + L(u_t) + f(t,u_t)]dt + g(t,u_t)dW(t) \text{ for } t \in \mathbb{R} \tag{4.7}$$

where $f : \mathbb{R} \times \mathcal{B} \to L^2(P,H)$ and $g : \mathbb{R} \times \mathcal{B} \to L^2(P,H)$ are two stochastic continuous processes. To prove our result, we formulate the following assumptions

(**H$_7$**) Let μ, $\nu \in \mathcal{M}$ and $f : \mathbb{R} \times C(]-\infty,0], L^2(P,H)) \to L^2(P,H)$ square mean $cl(\mu,\nu)$-pseudo automorphic periodic of infinite class such that there exists a function L_f such that $\mathbb{E}\Big|\Big| f(t,\phi_1) - f(t,\phi_2) \Big|\Big|^2 \leqslant L_f(t)\mathbb{E}||\phi_1 - \phi_2||^2$ for all $t \in \mathbb{R}$ and $\phi_1, \phi_2 \in C(]-\infty,0], L^2(P,H))$.

(**H$_8$**) Let μ, $\nu \in \mathcal{M}$ and $g : \mathbb{R} \times C(]-\infty,0], L^2(P,H)) \to L^2(P,H)$ square mean $cl(\mu,\nu)$-pseudo almost periodic of infinite class such that there exists a function L_g such that $\mathbb{E}\Big|\Big| g(t,\phi_1) - g(t,\phi_2) \Big|\Big|^2 \leqslant L_g(t)\mathbb{E}||\phi_1 - \phi_2||^2$ for all $t \in \mathbb{R}$ and ϕ_1, $\phi_2 \in C(]-\infty,0], L^2(P,H))$. Where L_f and $L_g \in L^p(\mathbb{R})$, $(1 \leqslant p < \infty)$

(**H$_9$**) Let $k = \max(L_f, L_g)$.

(**H$_{10}$**) The instable space $U \equiv \{0\}$

Theorem 4.6. *Assume that $\mathcal{B}$ satisfies (**A$_1$**), (**A$_2$**), (**B**), (**C$_1$**), (**C$_2$**) and (**H$_0$**), (**H$_1$**), (**H$_3$**) , (**H$_4$**), (**H$_4$**), (**H$_6$**), (**H$_7$**), (**H$_8$**), (**H$_9$**) and (**H$_{10}$**) hold. Then Eq.(4.7) has a unique $cl(\mu,\nu)$- square mean pseudo compact almost automorphic mild solution of infinite class.*

Proof. Let x be a function in $SPAA_c(\mathbb{R}; L^2(P,H),\mu,\nu,\infty)$ from Theorem 4.2 the function $t \to x_t$ belongs to $SPAA_c(C(]-\infty,0]; L^2(P,H)),\mu,\nu,\infty)$. Hence Theorem implies that the function $g(.) := f(.,x_.)$ is in $SPAA_c(\mathbb{R}; L^2(P,H),\mu,\nu,\infty)$. Since the instable space $U \equiv \{0\}$, then $\Pi^u \equiv 0$. Consider now the following mapping

$$\mathcal{H} : SPAA_c(\mathbb{R}; L^2(P,H),\mu,\nu,\infty) \to SPAA_c(\mathbb{R}; L^2(P,H),\mu,\nu,\infty)$$

defined for $t \in \mathbb{R}$ by

$$(\mathcal{H}x)(t) = \Big[\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s,x_s))ds + \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s,x_s))dW(s)\Big](0)$$

From Theorem 4.3, Theorem 4.4, Theorem 4.4 and Theorem 4.1 we obtain that $\mathcal{H}$ maps $SPAA_c(\mathbb{R}; L^2(P,H),\mu,\nu,\infty)$ into $SPAA_c(\mathbb{R}; L^2(P,H),\mu,\nu,\infty)$.
It remains now to show that the operator $\mathcal{H}$ has a unique fixed point in $SPAA_c(\mathbb{R}; L^2(P,H),\mu,\nu,\infty)$.
Since $\mathcal{B}$ is a uniform fading memory space, by the Lemma (2.7), choose the function K constant and the function M such that $M(t) \to 0$ as $t \to +\infty$. Let $\eta = \max_{t\in\mathbb{R}}\Big\{\sup_{t\in\mathbb{R}}|K(t)|^2, \sup_{t\in\mathbb{R}}|M(t)|^2\Big\}$ **Case 1:** $L_f, L_g \in L^1(\mathbb{R},\mathbb{R}^+)$
Let $x_1, x_2 \in SPAA_c(\mathbb{R}; L^2(P,H),\mu,\nu,\infty)$. Then we have

$$\begin{aligned}\mathbb{E}\Big|\Big|\mathcal{H}x_1(t) - \mathcal{H}x_2(t)\Big|\Big|^2 &\leqslant 2\mathbb{E}\Big|\Big|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0[f(s,x_{1s}) - f(s,x_{2s}))ds\Big|\Big|^2 \\ &+ 2\mathbb{E}\Big|\Big|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0\Big[g(s,x_{1s}) - g(s,x_{2s})\Big])dW(s)\Big|\Big|^2\end{aligned}$$

Using Ito's isometry we have

$$\begin{aligned}&\mathbb{E}\Big|\Big|\mathcal{H}x_1(t) - \mathcal{H}x_2(t)\Big|\Big|^2 \leqslant 2\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^{t}e^{-2\omega(t-s)}L_f(s)\mathbb{E}||x_{1s} - x_{2s}||^2_{\mathcal{B}}ds \\ &+ 2\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^{t}e^{-2\omega(t-s)}L_g(s)\mathbb{E}||x_{1s} - x_{2s}||^2_{\mathcal{B}}ds \\ &\leqslant 2\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^{t}e^{-2\omega(t-s)}L_f(s)\mathbb{E}\Big(K(s)\sup_{0\leqslant\xi\leqslant s}||x_1(\xi) - x_2(\xi)|| + M(s)||x_{1_0} - x_{2_0}||\Big)^2ds \\ &+ 2\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^{t}e^{-2\omega(t-s)}L_g(s)\mathbb{E}\Big(K(s)\sup_{0\leqslant\xi\leqslant s}||x_1(\xi) - x_2(\xi)|| + M(s)||x_{1_0} - x_{2_0}||\Big)^2ds \\ &\leqslant 4\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^{t}e^{-2\omega(t-s)}k(s)\mathbb{E}\Big(K(s)\sup_{0\leqslant\xi\leqslant s}||x_1(\xi) - x_2(\xi)|| + M(s)||x_{1_0} - x_{2_0}||\Big)^2ds \\ &\leqslant 8\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^{t}e^{-2\omega(t-s)}k(s)\mathbb{E}\Big(K^2(s)\sup_{0\leqslant\xi\leqslant s}||x_1(\xi) - x_2(\xi)||^2 + M^2(s)||x_{1_0} - x_{2_0}||^2\Big)ds \\ &\leqslant 8\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^{t}e^{-2\omega(t-s)}k(s)\Big(K^2(s)\sup_{0\leqslant\xi\leqslant s}\mathbb{E}||x_1(\xi) - x_2(\xi)||^2 + M^2(s)\mathbb{E}||x_{1_0} - x_{2_0}||^2\Big)ds \\ &\leqslant 16\overline{M}^2\widetilde{M}^2|\Pi^s|^2\eta\Big(\int_{-\infty}^{t}k(s)ds\Big)||x_1 - x_2||^2_\infty\end{aligned}$$

It follows that

$$\mathbb{E}\Big|\Big|\mathcal{H}^2x_1(t)-\mathcal{H}^2x_2(t)\Big|\Big|^2 \leqslant 2\mathbb{E}\Big|\Big|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s\Big(\widetilde{B}_\lambda X_0\Big[f(s,\mathcal{H}x_{1s})-f(s,\mathcal{H}x_{2s})\Big]\Big)ds\Big|\Big|^2$$
$$+\ 2\mathbb{E}\Big|\Big|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s\Big(\widetilde{B}_\lambda X_0\Big[g(s,\mathcal{H}x_{1s})-g(s,\mathcal{H}x_{2s})\Big]\Big)dW(s)\Big|\Big|^2$$
$$\leqslant \Big(16\overline{M}^2\widetilde{M}^2|\Pi^s|^2\eta\Big)^2\Big(\int_{-\infty}^{t}k(s)ds\Big)^2||x_1-x_2||_\infty^2$$

By induction on n we obtain the following inequalitie

$$\mathbb{E}\Big|\Big|\mathcal{H}^nx_1(t)-\mathcal{H}^nx_2(t)\Big|\Big|^2 \leqslant (16\overline{M}^2\widetilde{M}^2|\Pi^s|^2\eta)^n\Big(\int_{-\infty}^{t}k(s)ds\Big)^n||x_1-x_2||_\infty^2$$

Therefore

$$\Big|\Big|\mathcal{H}^nx_1(t)-\mathcal{H}^nx_2(t)\Big|\Big|_\infty \leqslant (4\overline{M}\widetilde{M}|\Pi^s|\sqrt{\eta})^n|k|^n_{L^1(\mathbb{R})}||x_1-x_2||_\infty$$

Let n_0 be such that $(4\overline{M}\widetilde{M}|\Pi^s|\sqrt{\eta})^{n_0}|k|^{n_0}_{L^1(\mathbb{R})}<1$. By Banach fix point Theorem, $\mathcal{H}$ has a unique point fixed and this fixed point satisfies the integral equation

$$u_t=\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s\widetilde{B}_\lambda(X_0f(s))ds+\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s\widetilde{B}_\lambda(X_0g(s))dW(s)$$

Case 2: $L_g, L_f\in L^p(\mathbb{R})$; $(1<p<\infty)$
First, put

$$\mu(t)=\int_{-\infty}^{t}(k(s))^pds.$$

Then we define an equivalent norm over $SPAA(\mathbb{R},L^2(P,H),\mu,\nu,r)$ as follows

$$||f||_c=\sup_{t\in\mathbb{R}}\Big(e^{-c\mu(t)}\mathbb{E}||f(t)||^2\Big)^{\frac{1}{2}}$$

where c is a fixed positive number to be precised later. Using the Holder inequality and Ito's isometry we have

$$\mathbb{E}\Big|\Big|\mathcal{H}x_1(t)-\mathcal{H}x_2(t)\Big|\Big|^2 \leqslant 2\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^{t}e^{-2\omega(t-s)}L_f(s)\mathbb{E}||x_{1s}-x_{2s}||^2_{\mathcal{B}}ds$$
$$+\ 2\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^{t}e^{-2\omega(t-s)}L_g(s)\mathbb{E}||x_{1s}-x_{2s}||^2_{\mathcal{B}}ds$$

$$\leqslant 8\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^{t} e^{-2\omega(t-s)}e^{c\mu(s)}e^{-c\mu(s)}k(s)\Big(K^2(s)\sup_{0\leqslant\xi\leqslant s}\mathbb{E}||x_1(\xi)-x_2(\xi)||^2$$
$$+M^2(s)\mathbb{E}||x_{1_0}-x_{2_0}||^2\Big)ds$$
$$\leqslant 16\overline{M}^2\widetilde{M}^2|\Pi^s|^2\eta\int_{-\infty}^{t} e^{-2\omega(t-s)}e^{c\mu(s)}k(s)\Big(\sup_{s\in\mathbb{R}}e^{-c\mu(s)}\mathbb{E}||x_1(\xi)-x_2(\xi)||^2\Big)ds$$
$$\leqslant 16\overline{M}^2\widetilde{M}^2|\Pi^s|^2\eta\int_{-\infty}^{t} e^{-2\omega(t-s)}e^{c\mu(s)}k(s)\Big(\sup_{s\in\mathbb{R}}\Big(e^{-c\mu(s)}\mathbb{E}||x_1(\xi)-x_2(\xi)||^2\Big)^{\frac{1}{2}}\Big)^2ds$$
$$\leqslant 16\overline{M}^2\widetilde{M}^2|\Pi^s|^2\eta\Big(\int_{-\infty}^{t} e^{-2\omega(t-s)}e^{c\mu(s)}k(s)ds\Big)||x_1-x_2||_c^2$$
$$\leqslant 16\overline{M}^2\widetilde{M}^2|\Pi^s|^2\eta\Big(\int_{-\infty}^{t} e^{-2q\omega(t-s)}ds\Big)^{\frac{1}{q}}\Big(\int_{-\infty}^{t} e^{pc\mu(s)}k^p(s)ds\Big)^{\frac{1}{p}}||x_1-x_2||_c^2$$
$$\leqslant 16\overline{M}^2\widetilde{M}^2|\Pi^s|^2\eta\Big(\int_{-\infty}^{t} e^{-2q\omega(t-s)}ds\Big)^{\frac{1}{q}}\Big(\int_{-\infty}^{t} e^{pc\mu(s)}\mu'(s)ds\Big)^{\frac{1}{p}}||x_1-x_2||_c^2$$
$$\leqslant 16\overline{M}^2\widetilde{M}^2|\Pi^s|^2\eta\Big(\frac{1}{(2\omega q)^{\frac{1}{q}}}\times\frac{1}{(pc)^{\frac{1}{p}}}\Big)e^{c\mu(t)}||x_1-x_2||_c^2$$
$$e^{-c\mu(t)}\mathbb{E}\Big|\Big|\mathcal{H}x_1(t)-\mathcal{H}x_2(t)\Big|\Big|^2\leqslant 16\overline{M}^2\widetilde{M}^2|\Pi^s|^2\eta\Big(\frac{1}{(2\omega q)^{\frac{1}{q}}}\times\frac{1}{(pc)^{\frac{1}{p}}}\Big)||x_1-x_2||_c^2$$
$$\Big(e^{-c\mu(t)}\mathbb{E}\Big|\Big|\mathcal{H}x_1(t)-\mathcal{H}x_2(t)\Big|\Big|^2\Big)^{\frac{1}{2}}\leqslant 4\overline{M}\widetilde{M}|\Pi^s|\sqrt{\eta}\Big(\frac{1}{(2\omega q)^{\frac{1}{2q}}}\times\frac{1}{(pc)^{\frac{1}{2p}}}\Big)||x_1-x_2||_c^2$$

Consequently,

$$||\mathcal{H}x_1(t)-\mathcal{H}x_2(t)||_c\leqslant\frac{4\overline{M}\widetilde{M}|\Pi^s|\sqrt{\eta}}{(2\omega q)^{\frac{1}{2q}}\times(pc)^{\frac{1}{2p}}}||x_1-x_2||_c$$

Fix $c>0$ so large, then the function $c\mapsto\dfrac{1}{(pc)^{\frac{1}{2p}}}$ converges to 0 when c converges to $+\infty$. It follows that for $c>0$ so large we have $\dfrac{4\overline{M}\widetilde{M}|\Pi^s|\sqrt{\eta}}{(2\omega q)^{\frac{1}{2q}}\times(pc)^{\frac{1}{2p}}}<1$. Thus $\mathcal{H}$ is a contractive mapping. we conclude that there is a unique pseudo almost automorphic integral solution to Eq.(4.7).

Proposition 4.7. *Assume that $\mathcal{B}$ is a uniform fading space and (A_1), (A_2), (C_1), (C_2), (**H_0**), (**H_1**), (**H_2**), (**H_4**) and (**H_5**) hold f and g are lipschitz continuous with respect the second argument if* $\max(Lip(f),Lip(g))<\dfrac{\omega}{2\sqrt{2M}\widetilde{M}|\Pi^s|\eta}$*.Then Eq(4.7) has a unique square-mean $cl(\mu,\nu)$-pseudo almost automorphic solution of infinite class, where $Lip(f)$ and $Lip(g)$ are respectively Lipschitz constant of f and g.*

Proof. Let us pose $k=\max(Lip(f),Lip(g))$, we have

$$\mathbb{E}\Big|\Big|\mathcal{H}x_1(t)-\mathcal{H}x_2(t)\Big|\Big|^2\leqslant 8\overline{M}^2\widetilde{M}^2|\Pi^s|^2\int_{-\infty}^{t}e^{-2\omega(t-s)}k(s)\Big(K^2(s)\sup_{0\leqslant\xi\leqslant s}\mathbb{E}||x_1(\xi)-x_2(\xi)||^2+M^2(s)\mathbb{E}||x_{1_0}-x_{2_0}||^2\Big)ds$$
$$\leqslant 16k\overline{M}^2\widetilde{M}^2|\Pi^s|^2\eta\Big(\int_{-\infty}^{t}e^{-2\omega(t-s)}ds\Big)||x_1-x_2||_\infty^2$$
$$\leqslant\frac{8\eta\overline{M}^2\widetilde{M}^2|\Pi^s|^2k}{\omega}||x_1-x_2||_\infty^2$$
$$\Big|\Big|\mathcal{H}x_1(t)-\mathcal{H}x_2(t)\Big|\Big|\leqslant\frac{2\sqrt{2M}\widetilde{M}|\Pi^s|k\eta}{\omega}||x_1-x_2||_\infty$$

Consequently $\mathcal{H}$ is a strict contraction if $k < \dfrac{\omega}{2\sqrt{2M}\widetilde{M}|\Pi^s|\eta}$.■

5. Application

For illustration, we propose to study the existence of solutions for the following model

$$\begin{cases} dz(t,x) = \dfrac{\partial^2}{\partial x^2}z(t,x)dt + \Big[\displaystyle\int_{-\infty}^0 G(\theta)z(t+\theta,x)d\theta + \sin\Big(\dfrac{1}{2+\cos(t)+\cos(\sqrt{2}t)}\Big) + \arctan(t) \\ + \displaystyle\int_{-\infty}^0 e^{\omega\theta}h(\theta,z(t+\theta,x)d\theta\Big]dt + \Big[\cos\Big(\dfrac{1}{\sin(t)+\sin(\sqrt{2}t)}\Big) + \sin(t) + \displaystyle\int_{-\infty}^0 e^{\omega\theta}h(\theta,z(t+\theta,x))d\theta\Big]dW(t) \\ z(t,0) = z(t,\pi) = 0 \text{ for } t \in \mathbb{R} \end{cases} \tag{5.1}$$

Where $G :]-\infty,0] \to \mathbb{R}$ define by $G(\theta) = e^{(\gamma+1)\theta}$ is a continuous function and $h :]-\infty,0]\times\mathbb{R} \to \mathbb{R}$ is continuous, Lipschitzian with respect to the second argument and ω is a positive positive real number.
For example, take $h(\theta,x) = \theta^3 + \cos\left(\dfrac{x}{3}\right)$ for $(\theta,x) \in]-\infty,0]\times\mathbb{R}$, it follows that

$$\Big|h(\theta,x_1) - h(\theta,x_2)\Big| \leqslant \frac{1}{2}|x_1 - x_2|$$

which implies $h :]-\infty,0]\times\mathbb{R} \longrightarrow \mathbb{R}$ is continuous and lipschitzian with respect to the second argument. $W(t)$ is a two-sided and standard one-dimensional Brownian notion defined on the filtered probability space $(\Omega,\mathcal{F},P,\mathcal{F}_t)$ with $\mathcal{F}_t = \sigma\{W(u) - W(v) \mid u,v \leqslant t\}$.
The phase $\mathcal{B} = C_\gamma$, $\gamma > 0$ where

$$C_\gamma = \Big\{\phi \in C(]-\infty,0];L^2(P,H)) : \lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta) \text{ exist in } L^2(P,H)\Big\}$$

With the following norm

$$||\phi||_\gamma = \sup_{\theta\leqslant 0}\Big(\mathbb{E}||e^{\gamma\theta}\phi(\theta)||^2\Big)^{\frac{1}{2}}$$

To rewrite equation (5.1) in the abstract form , we introduce the space $H = L^2((0,\pi))$. Let $A : D(A) \to L^2((0,\pi))$ defined by

$$\begin{cases} D(A) = \mathbf{H}^1((0,\pi)) \cap \mathbf{H}_0^1((0,1)) \\ Ay(t) = y''(t) \ \text{ for } t \in (0,\pi) \text{ and } y \in D(A) \end{cases}$$

Then A generates a C_0-semigroup $(\mathcal{U}(t))_{t\geqslant 0}$ on $L^2((0,\pi))$ given by

$$(\mathcal{U}(t)x)(r) = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} < x, e_n >_{L^2} e_n(r)$$

Where $e_n(r) = \sqrt{2}\sin(n\pi r)$ for $n = 1,2,...$, and $||\mathcal{U}(t)|| \leqslant e^{-\pi^2 t}$ for all $t \geqslant 0$. Thus $\overline{M} = 1$ and $\omega = \pi^2$. Then A satisfied the Hille-Yosida condition in $L^2((0,\pi))$. Moreover the part A_0 of A in $\overline{D(A)}$. It follows that $(\mathbf{H_0})$ and $(\mathbf{H_1})$ are satisfied.
We define $f : \mathbb{R}\times\mathcal{B} \to L^2((0,\pi))$, $g : \mathbb{R}\times\mathcal{B} \to L^2((0,\pi))$ and $L : \mathcal{B} \to L^2((0,\pi))$ as follows

$$f(t,\phi)(x) = \sin\Big(\frac{1}{2+\cos(t)+\cos(\sqrt{2}t)}\Big) + \arctan(t) + \int_{-\infty}^0 e^{\omega\theta}h(\theta,\phi(\theta)(x))d\theta$$

$$g(t,\phi)(x) = \cos\Big(\frac{1}{\sin(t)+\sin(\sqrt{2}t)}\Big) + \sin(t) + \int_{-\infty}^{\theta} e^{\omega\theta} h(\theta,\phi(\theta)(x))d\theta$$

$$L(\phi)(x) = \int_{-\infty}^{\theta} G(\theta,\phi(\theta)(x))d\theta \text{ for } -\infty < \theta \leqslant 0 \text{ and } x \in (0,\pi)$$

let us pose $v(t) = z(t,x)$. Then equation(5.1) takes the following abstract form

$$dv(t) = [Av(t) + L(v_t) + f(t,v_t)]dt + g(t,v_t)dW(t) \text{ for } t \in \mathbb{R}$$

Consider the measures μ and ν where its Radon-Nikodyn derivative are respectively ρ_1, $\rho_2 : \mathbb{R} \to \mathbb{R}$ defined by

$$\rho_1(t) = \begin{cases} 1 \text{ for } t > 0 \\ e^t \text{ for } t \leqslant 0 \end{cases}$$

and

$$\rho_2(t) = |t| \; for \; t \in \mathbb{R}$$

i.e $d\mu(t) = \rho_1(t)dt$ and $d\nu(t) = \rho_2(t)dt$ where dt denotes the Lebesgue measure on $\mathbb{R}$ and

$$\mu(A) = \int_A \rho_1(t)dt \text{ for } \nu(A) = \int_A \rho_2(t)dt \text{ for } A \in \mathcal{B}.$$

From [6] $\mu,\nu \in \mathcal{M}$, μ, ν satisfy $(\mathbf{H_4})$, $\sin\Big(\frac{1}{2+\cos(t)+\cos(\sqrt{2}t)}\Big)$ and $\cos\Big(\frac{1}{\sin(t)+\sin(\sqrt{2}t)}\Big)$ are almost automorphic.

We have

$$\limsup_{\tau\to+\infty} \frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])} = \limsup_{\tau\to+\infty} \frac{\int_{-\tau}^{0} e^t dt + \int_0^{\tau} dt}{2\int_{-\tau}^{0} t dt} = \limsup_{\tau\to+\infty} \frac{1-e^{-\tau}+\tau}{\tau^2} = 0 < \infty,$$

which implies that $(\mathbf{H_2})$ is satisfied.

For all $\theta \in \mathbb{R}$, $-1 \leqslant \sin(\theta) \leqslant 1$ then,

$$\begin{aligned} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \sup_{\theta\in]-\infty,t]} \mathbb{E}\Big|\sin(\theta)\Big|^2 dt &\leqslant \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} d\mu(t) \\ &\leqslant \frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])} \to 0 \text{ as } \tau \to +\infty \end{aligned}$$

Consequently,

$$\lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{+\tau} \sup_{\theta\in]-\infty,t]} \mathbb{E}\Big|\sin(\theta)\Big|^2 d\mu(t) = 0$$

It follows that $t \mapsto \sin(t)$ is square mean (μ,ν)-ergodic of infinite class , consequently, g is uniformly square mean (μ,ν)-pseudo almost automorphic of infinite class.

For all $\theta \in \mathbb{R}$, $\frac{-\pi}{2} < \arctan\theta < \frac{\pi}{2}$ then,

$$\begin{aligned} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \sup_{\theta\in]-\infty,t]} \mathbb{E}\Big|\arctan(\theta)\Big|^2 dt &\leqslant \frac{\pi}{2} \times \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} d\mu(t) \\ &\leqslant \frac{\pi}{2} \times \frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])} \to 0 \text{ as } \tau \to +\infty \end{aligned}$$

Consequently,

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\sup_{\theta\in]-\infty,t]}\mathbb{E}\Big|\arctan\theta\Big|^2 d\mu(t)=0$$

It follows that $t\mapsto\arctan t$ is square mean (μ,ν)-ergodic of infinite class , consequently, f is uniformly square mean (μ,ν)-pseudo almost automorphic of infinite class.
For $\phi\in C_\gamma$, $\gamma\in C(]-\infty,0];L^2(P,H))$ and $\lim\limits_{\theta\to-\infty}e^{\gamma\theta}\phi(\theta)=x_0$ exist in $L^2(P,H)$, then there exists $M\geqslant 0$ such that $\mathbb{E}||e^{\gamma\theta}\phi(\theta)||^2\leqslant M$ for $\theta\in]-\infty,0]$.

$$\begin{aligned}\mathbb{E}||L(\phi)(x)||^2&=\mathbb{E}\Big|\Big|\int_{-\infty}^0 G(\theta)\phi(\theta)(x)d\theta\Big|\Big|^2\\&\leqslant\int_{-\infty}^0\mathbb{E}||G(\theta)\phi(\theta)(x)||^2d\theta\\&\leqslant\int_{-\infty}^0 e^{2(\gamma+1)\theta}\mathbb{E}||e^{-\gamma\theta}e^{\gamma\theta}G(\theta)\phi(\theta)(x)||^2d\theta\\&\leqslant\int_{-\infty}^0 e^{2(\gamma+1)\theta}\times e^{-\gamma\theta}\mathbb{E}||e^{\gamma\theta}G(\theta)\phi(\theta)(x)||^2d\theta\\&\leqslant\int_{-\infty}^0 e^{2\theta}\mathbb{E}||e^{\gamma\theta}G(\theta)\phi(\theta)(x)||^2d\theta\\&\leqslant M\int_{-\infty}^0 e^{2\theta}d\theta<\infty\end{aligned}$$

Moreover,

$$\begin{aligned}\mathbb{E}||L(\phi)(x)||^2&\leqslant\Big(\int_{-\infty}^0 e^{2\theta}d\theta\Big)\sup_{\theta\leqslant 0}\mathbb{E}||e^{\gamma\theta}\phi(\theta)(x)||^2\\&\leqslant\Big(\int_{-\infty}^0 e^{2\theta}d\theta\Big)||\phi||^2_{\mathcal{B}}\end{aligned}$$

Then L is well defined and L is bounded linear operator from $\mathcal{B}$ to $L^2(P,L^2((0,\pi))$.

$$\begin{aligned}\mathbb{E}||f(t,\phi_1)(x)-f(t,\phi_2)(x)||^2&=\mathbb{E}\Big|\Big|\int_{-\infty}^0 e^{\omega\theta}\Big[h(\theta,\phi_1(\theta)(x))-h(\theta,\phi_2(\theta)(x))\Big]\Big|\Big|^2d\theta\\&\leqslant\int_{-\infty}^0 e^{2\omega\theta}\mathbb{E}\Big|\Big|h(\theta,\phi_1(\theta)(x))-h(\theta,\phi_2(\theta)(x))\Big|\Big|^2d\theta\\&\leqslant\frac{1}{9}\int_{-\infty}^0 e^{2\omega\theta}e^{-\frac{1}{2}\gamma\theta}e^{\frac{1}{2}\gamma\theta}\mathbb{E}\Big|\Big|\phi_1(\theta)(x)-\phi_2(\theta)(x)\Big|\Big|^2d\theta\\&\leqslant\frac{1}{9}\int_{-\infty}^0 e^{(2\omega-\frac{1}{2}\gamma)\theta}\mathbb{E}\Big|\Big|e^{2\gamma\theta}(\phi_1(\theta)(x)-\phi_2(\theta)(x))\Big|\Big|^2d\theta\\&\leqslant\frac{1}{9}\Big(\int_{-\infty}^0 e^{(2\omega-\frac{1}{2}\gamma)\theta}d\theta\Big)\Big|\Big|\phi_1-\phi_2\Big|\Big|^2_{\mathcal{B}}\end{aligned}$$

Consequently, we conclude that f and g are Lipschitz continuous and $cl(\mu,\nu)$-pseudo almost automorphic of infinite class .
Moreover, since h is Lipschitzian by consequently bounded i.e there exists a constant M_1 positive real number

such that $|h(\theta,x)| \leqslant M_1$, then we have

$$\mathbb{E}||g(t,\phi)(x)||^2 \leqslant 2+\int_{-\infty}^{0} e^{\omega\theta}\mathbb{E}\Big|h(\theta,\phi(\theta)(x))\Big|^2 d\theta$$
$$\leqslant 2+M_1^2\int_{-\infty}^{0} e^{\omega\theta}d\theta<\infty$$

Which implies that g verifies $(\mathbf{H_5})$

Lemma 5.1. *[9] If* $\int_{-\infty}^{0}|G(\theta)|d\theta<1$*, then the semigroup* $(\mathcal{U}(t))_{t\geqslant 0}$ *is hyperbolic and the instable space* $U\equiv\{0\}$.

Observe that $\int_{-\infty}^{0}|G(\theta)|d\theta=\lim_{t\to+\infty}\int_{-r}^{0}e^{(\gamma+1)\theta}d\theta = \lim_{r\to+\infty}\left[\frac{1}{\gamma+1}e^{(\gamma+1)\theta}\right]_{-r}^{0} = \frac{1}{\gamma+1} < 1$, then $(\mathbf{H_8})$ holds Then by Proposition(4.7) we deduce the following result.

Theorem 5.2. *Under the above assumptions, then equation (5.1) has a unique square mean* $cl(\mu,\nu)$*-pseudo almost automorphic solution of infinite class .*

References

[1] Mostafa Adimy, Abdelhai Elazzouzi and Khalil Ezzinbi, Reduction principle and dynamic behaviors for a class of partial functional differential equations, *Nonlinear Analysis: Theory, Methods & Applications*, **71**(2009), 1709–1727.

[2] Mostafa Adimy, Abdelhai Elazzouzi and Khalil Ezzimbi, Bohr-Neugebauer type theorem for some partial neutral functional differential equations, *Nonlinear Analysis, Theory, Methods and Applications*, **66(5)**(2007), 1145-1160.

[3] Mostafa Adimy, K. Ezzinbi, M. Laklach, Spectral decomposition for partial neutral functional differential equations, *Canadian Applied Mathematics Quarterly*, (1)(2001), 1–34.

[4] P. H. Bezandry and T. Diagana, Square-mean almost automorphic solutions nonautonomous stochastic differential equations, *Electronic Journal of Differential Equations*, 2007, 1-10

[5] J. Blot, P. Cieutat, K. Ezzinbi, Measure theory and pseudo almost automorphic functions: New developments and applications, *Nonlinear Analysis*, (2012), 2426-2447.

[6] J. Blot, P. Cieutat, K. Ezzinbi, New approach for weighted pseudo almost periodic functions under the light of measure theory, basic results and applications, *Applicable Analysis*, 2013, 493-526.

[7] M. Diop, K. Ezzinbi, M. Mbaye, Measure Theory and Square-Mean Pseudo Almost automorphic and Automorphic Process: Application to Stochastic Evolution Equations, *Bulletin of the Malaysian Mathematical Sciences Society*, 2018, 287–310.

[8] M. Diop, K. Ezzinbi, M. Mbaye, Measure theory and S^2- pseudo almost automorphic and automorphic process: application to stochastic evolution equation, *Afrika Matematika*, 2015, 779–812.

[9] K. Ezzinbi, S. Fatajou, G.M. N'Guérékata, C^n-almost automorphic solutions for partial neutral functional differential equations, *Applicable Analysis*, **86:9**(2007), 1127-1146.

[10] MOHAMADO KIEMA AND ISSA ZABSONRÉ, Square-mean pseudo almost periodic solutions of class r under the light of measure theory, *European Journal of Mathematics and Applications*, 2022, 1–23.

[11] MIAOMIAO FU AND ZHENXIN LIU, Square-mean almost automorphic solutions for some stochastic differential equations, *Proceedings of the American Mathematical Society*, **138(10)**(2010), 3689–3701.

[12] W RUDIN, *Real and Complex Analysis*, 3rd edn. Higher Mathematics Series, McGraw-Hill, New York, year, 1976.

[13] BERNT OKSENDAL, *Stochastic differential equations: An introduction with applications*The Mathematical Gazette, 89, 514, 2005, 178–179.