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On Tribonacci functions and Gaussian Tribonacci functions

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This paper is dedicated to the occasion of Professor Gaston M. N'Guérékata's 70th birthday

Abstract. In this work, Gaussian Tribonacci functions are defined and investigated on the set of real numbers $\mathbb{R}$, *i.e.*, functions $f_G : \mathbb{R} \to \mathbb{C}$ such that for all $x \in \mathbb{R}$, $n \in \mathbb{Z}$, $f_G(x+n) = f(x+n) + if(x+n-1)$ where $f : \mathbb{R} \to \mathbb{R}$ is a Tribonacci function which is given as $f(x+3) = f(x+2) + f(x+1) + f(x)$ for all $x \in \mathbb{R}$. Then the concept of Gaussian Tribonacci functions by using the concept of f-even and f-odd functions is developed. Also, we present linear sum formulas of Gaussian Tribonacci functions. Moreover, it is showed that if f_G is a Gaussian Tribonacci function with Tribonacci function f, then $\lim\limits_{x\to\infty} \frac{f_G(x+1)}{f_G(x)} = \alpha$ and $\lim\limits_{x\to\infty} \frac{f_G(x)}{f(x)} = \alpha + i$, where α is the positive real root of equation $x^3 - x^2 - x - 1 = 0$ for which $\alpha > 1$. Finally, matrix formulations of Tribonacci functions and Gaussian Tribonacci functions are given. In the literature, there are several studies on the functions of linear recurrent sequences such as Fibonacci functions and Tribonacci functions. However, there are no study on Gaussian functions of linear recurrent sequences such as Gaussian Tribonacci and Gaussian Tetranacci functions and they are waiting for the investigating.
We also present linear sum formulas and matrix formulations of Tribonacci functions which have not been studied in the literature.

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Keywords: Tribonacci numbers, Tribonacci functions, Gaussian Tribonacci functions, f-even functions.

Contents

1. Introduction

A function f defined on the real numbers $\mathbb{R}$ is said to be a Fibonacci function if it satisfies the following relation

$$f(x+2) = f(x+1) + f(x)$$

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for all $x \in \mathbb{R}$. First and foremost, Elmore [2], Parker [8] and Spickerman [12] discovered useful properties of the Fibonacci functions. Later, many renowned researchers such as Fergy and Rabago [3], Han, et al. [5], Sroysang [14], and Gandhi [4], have devoted their study to the analysis of many properties of the Fibonacci function.

A function f defined on the real numbers $\mathbb{R}$ is said to be a Tribonacci function if it satisfies the following relation

$$f(x+3) = f(x+2) + f(x+1) + f(x)$$

for all $x \in \mathbb{R}$ (a short review on Tribonacci functions will be given in this section below). Some references on Tribonacci functions are Arolkar [1], Magnani [6], Parizi [7] and Sharma [10].

A function f defined on the real numbers $\mathbb{R}$ is said to be a Tetranacci function if it satisfies the following relation

$$f(x+4) = f(x+3) + f(x+2) + f(x+1) + f(x)$$

for all $x \in \mathbb{R}$. See Sharma [11] for more information on Tetranacci functions.

More generally, a function f defined on the real numbers $\mathbb{R}$ is said to be a k-step Fibonacci function if it satisfies the following relation

$$f(x+k) = f(x+k-1) + f(x+k-2) + f(x+k-3) + ... + f(x)$$

for all $x \in \mathbb{R}$. See Sriponpaew and Sassanapitax [13], and Wolfram [18] for more information on k-step Fibonacci functions.

Before giving a short review on Tribonacci functions, we recall the definition of a Tribonacci sequence. A Tribonacci sequence $\{T_n\}_{n\geq 0} = \{V_n(T_0, T_1, T_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$T_n = T_{n-1} + T_{n-2} + T_{n-3},$$

with the initial values $T_0 = 0, T_1 = 1, T_2 = 1$.

Next, we present the first few values of the Tribonacci numbers with positive and negative subscripts:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
T_n	0	1	1	2	4	7	13	24	44	81	149	274	504	927
T_{-n}	....	0	1	-1	0	2	-3	1	4	-8	5	7	-20	18

If we let $u_0 = 0, u_1 = 1, u_2 = 1$, then we consider the full (bilateral) Tribonacci sequence $\{u_n\}_{n=-\infty}^{\infty}$: . . . , $-3, 2, 0, -1, 1, 0, 0, 1, 1, 2, 4, 7, 13, ...$, i.e. $T_{-n} = T_n^2 + T_{2n} + T_{n+2}T_n - 4T_{n+1}T_n$ [see 5, Corollary 7] for $n > 0$ and $u_n = T_n$, the nth Tribonacci number.

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be Tribonacci function if it satisfies the formula

$$f(x+3) = f(x+2) + f(x+1) + f(x)$$

for all $x \in \mathbb{R}$ or equivalently

$$f(x) = f(x-1) + f(x-2) + f(x-3)$$

for all $x \in \mathbb{R}$.

Note that

$$f(x+n) = T_{n-1}f(x+2) + (T_{n-2} + T_{n-3})f(x+1) + T_{n-2}f(x) \tag{1.1}$$

for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$.

We next present the Binet's formula of f.

Lemma 1.1. *[5, p.141] The Binet's formula of f is*

$$f(x) = K_1\alpha^x + K_2\alpha^{-x/2}\cos(\theta x) + \alpha^{-x/2}\left(\frac{K_2\left(\frac{1-\alpha}{2}\right) + K_3}{\sqrt{\frac{1}{\alpha} - \left(\frac{\alpha-1}{2}\right)^2}}\right)\sin(\theta x),$$

where

$$
\begin{aligned}
K_1 &= \frac{f(0)}{\alpha(\alpha-1)(3\alpha+1)} + \frac{f(1)}{3\alpha+1} + \frac{f(2)}{(\alpha-1)(3\alpha+1)}, \\
K_2 &= f(0) - K_1, \\
K_3 &= \frac{K_1 - \alpha(f(2)-f(1)-f(0))}{\alpha^2}, \\
\theta &= \arccos\left(\frac{1-\alpha}{2}\sqrt{\alpha}\right).
\end{aligned}
$$

Note that this formula does not use complex roots.

Next, we list some examples of Tribonacci functions.

Example 1.2.

(a) *[5, Example 2.1]*

$$f : \mathbb{R} \to \mathbb{R},\ f(x) = \alpha^x$$

is a Tribonacci function, where α is a positive root of equation $x^3 - x^2 - x - 1 = 0$ and α is greater than one and given as

$$\alpha = \frac{1 + \sqrt[3]{19+3\sqrt{33}} + \sqrt[3]{19-3\sqrt{33}}}{3}.$$

The other two roots are

$$\beta = \frac{1 + \omega\sqrt[3]{19+3\sqrt{33}} + \omega^2\sqrt[3]{19-3\sqrt{33}}}{3},$$

and

$$\gamma = \frac{1 + \omega^2\sqrt[3]{19+3\sqrt{33}} + \omega\sqrt[3]{19-3\sqrt{33}}}{3}.$$

(b) *[5, Example 2.2] Let $\{u_n\}_{n=-\infty}^{\infty}, \{v_n\}_{n=-\infty}^{\infty}$ and $\{w_n\}_{n=-\infty}^{\infty}$ be full Tribonacci sequences and define a function $f(x)$ by $f(x) = u_{\lfloor x \rfloor} + v_{\lfloor x \rfloor} t + w_{\lfloor x \rfloor} t^2$, where $t = x - \lfloor x \rfloor \in (0,1)$ and $x \in \mathbb{R}$, $\lfloor x \rfloor$ is the greatest integer function (floor function). Then*

$$f(x+3) = f(x+2) + f(x+1) + f(x)$$

so that f is a Tribonacci function.

(c) *[5, Proposition 2.3 and Example 2.4] Let f be a Tribonacci function and define $g(x) = f(x+t+t^2)$ for any $x \in \mathbb{R}$, where $t \in \mathbb{R}$. Then g is also a Tribonacci function. If $f(x) = \alpha^x$ which is a Tribonacci function, then $g(x) = \alpha^{x+t+t^2} = \alpha^{t+t^2} f(x)$ is a Tribonacci function.*

We now present the concepts of f-even and f-odd functions which were defined by Han, et al. [5] in 2012.

Definition 1.3. *Suppose that $a(x)$ is a real-valued function of real variable such that if $a(x)h(x) = 0$ and $h(x)$ is continuous. Then $h(x) \equiv 0$. The map $a(x)$ is called an f-even function if $a(x+1) = a(x)$ and f-odd function if $a(x+1) = -a(x)$) for all $x \in \mathbb{R}$.*

We present an f-even and an f-odd function.

Example 1.4.

(a) *If $a(x) = x - \lfloor x \rfloor$ then $a(x)$ is an f-even function.*

(b) *If $a(x) = \sin(\pi x)$ then $a(x)$ is an f-odd function.*

Solution.

(a) If $a(x) = x - \lfloor x \rfloor$ then $a(x)h(x) = 0$ implies $h(x) \equiv 0$ if $x \notin \mathbb{Z}$. By continuity of $h(x)$, it follows that $h(n) = \lim_{x \to n} h(x) = 0$ for any integer $n \in \mathbb{Z}$ and therefore $h(x) \equiv 0$. Since

$$a(x+1) = (x+1) - \lfloor x+1 \rfloor = (x+1) - (\lfloor x \rfloor + 1) = x - \lfloor x \rfloor = a(x),$$

we see that $a(x)$ is an f-even function.

(b) If $a(x) = \sin(\pi x)$, then $a(x)h(x) \equiv 0$ implies that $h(x) = 0$ if $x \neq n\pi$ for any integer $n \in \mathbb{Z}$. Since $h(x)$ is continuous, it follows that $h(n\pi) = \lim_{x \to n\pi} h(x) = 0$ for $n \in \mathbb{Z}$, and therefore, $h(x) \equiv 0$. Since

$$a(x+1) = \sin(\pi x + \pi) = \sin(\pi x)\cos(\pi) = -\sin(\pi x) = -a(x),$$

we see that $a(x)$ is an f-odd function. □

The following theorem is given in ([7], Theorem 3.3).

Theorem 1.5. *Assume that $f(x) = a(x)g(x)$ is a function, where $a(x)$ is an f-even function and $g(x)$ is a continuous function. Then $f(x)$ is a Tribonacci function if and only if $g(x)$ is a Tribonacci function.*

The following theorem shows that the limit of quotient of a Tribonacci function exists.

Theorem 1.6. *If $f(x)$ is a Tribonacci function, then the limit of quotient $\frac{f(x+1)}{f(x)}$ exists (5, Theorem 4.1]) and $\lim_{x \to \infty} \frac{f(x+1)}{f(x)} = \alpha$ (take $p = 1$ in Magnani [5, Theorem 14].*

We also have a result on the limit of quotient $\frac{f(x+2)}{f(x)}$ which we need for calculation of the limit of the quotient of a Gaussian Tribonacci function.

Theorem 1.7. *If $f(x)$ is a Tribonacci function, then, for $0 \leq k, m \in \mathbb{N}$ we have*

$$\lim_{x \to \infty} \frac{f(x+k)}{f(x+m)} = \alpha^{k-m}. \tag{1.2}$$

Proof. If $0 \leq k, m \leq 1$ then (1.2) is true (Theorem 1.6). We give the proof in three stages:
Stage I:

$$\lim_{x \to \infty} \frac{f(x+2)}{f(x)} = \alpha^2.$$

Stage II: for $3 \leq k \in \mathbb{N}$,

$$\lim_{x \to \infty} \frac{f(x+k)}{f(x)} = \alpha^k.$$

Stage III: for $3 \leq k, m \in \mathbb{N}$,

$$\lim_{x \to \infty} \frac{f(x+k)}{f(x+m)} = \alpha^{k-m}.$$

Proof of Stage I:
Given $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ and $n \in \mathbb{N}$ such that $x = y + n$. Then, using the formula

$$f(x+n) = T_{n-1}f(x+2) + (T_{n-2} + T_{n-3})f(x+1) + T_{n-2}f(x),$$

we get

$$\begin{aligned}\frac{f(x+2)}{f(x)} &= \frac{f(y+n+2)}{f(y+n)} \\ &= \frac{T_{n+1}f(y+2)+(T_n+T_{n-1})f(y+1)+T_nf(y)}{T_{n-1}f(y+2)+(T_{n-2}+T_{n-3})f(y+1)+T_{n-2}f(y)} \\ &= \frac{T_n}{T_{n-2}}\frac{\frac{T_{n+1}}{T_n}\frac{f(y+2)}{f(y)}+(1+\frac{T_{n-1}}{T_n})\frac{f(y+1)}{f(y)}+1}{\frac{T_{n-1}}{T_{n-2}}\frac{f(y+2)}{f(y)}+(1+\frac{T_{n-3}}{T_{n-2}})\frac{f(y+1)}{f(y)}+1}\end{aligned}$$

Since

$$\lim_{x\to\infty}\frac{f(x+1)}{f(x)} = \lim_{y\to\infty}\frac{f(y+1)}{f(y)} = \alpha$$

and

$$\lim_{n\to\infty}\frac{T_{n+p}}{T_{n+q}} = \alpha^{p-q}, \;\; p,q\in\mathbb{Z}$$

and

$$\lim_{x\to\infty}\frac{f(x+2)}{f(x)} = \lim_{y\to\infty}\frac{f(y+2)}{f(y)} = u \quad \text{(say)}$$

we obtain

$$u = \alpha^2\frac{\alpha u+(1+\alpha^{-1})\alpha+1}{\alpha u+(1+\alpha^{-1})\alpha+1}$$

and so

$$u = \lim_{x\to\infty}\frac{f(x+2)}{f(x)} = \alpha^2.$$

This completes the proof of Stage I.

Proof of Stage II:

Given $x\in\mathbb{R}$, there exists $y\in\mathbb{R}$ and $n\in\mathbb{N}$ such that $x=y+n$. Then, by using Stage I, we get

$$\begin{aligned}\frac{f(x+k)}{f(x)} &= \frac{f(y+n+k)}{f(y+n)} \\ &= \frac{T_{n+k-1}f(y+2)+(T_{n+k-2}+T_{n+k-3})f(y+1)+T_{n+k-2}f(y)}{T_{n-1}f(y+2)+(T_{n-2}+T_{n-3})f(y+1)+T_{n-2}f(y)} \\ &= \frac{T_{n+k-2}}{T_{n-2}}\frac{\frac{T_{n+k-1}}{T_{n+k-2}}\frac{f(y+2)}{f(y)}+(1+\frac{T_{n+k-3}}{T_{n+k-2}})\frac{f(y+1)}{f(y)}+1}{\frac{T_{n-1}}{T_{n-2}}\frac{f(y+2)}{f(y)}+(1+\frac{T_{n-3}}{T_{n-2}})\frac{f(y+1)}{f(y)}+1}\end{aligned}$$

and so

$$\begin{aligned}\lim_{x\to\infty}\frac{f(x+k)}{f(x)} &= \lim_{y\to\infty}\lim_{n\to\infty}\frac{T_{n+k-2}}{T_{n-2}}\frac{\frac{T_{n+k-1}}{T_{n+k-2}}\frac{f(y+2)}{f(y)}+(1+\frac{T_{n+k-3}}{T_{n+k-2}})\frac{f(y+1)}{f(y)}+1}{\frac{T_{n-1}}{T_{n-2}}\frac{f(y+2)}{f(y)}+(1+\frac{T_{n-3}}{T_{n-2}})\frac{f(y+1)}{f(y)}+1} \\ &= \alpha^k\frac{\alpha\alpha^2+(1+\alpha^{-1})\alpha+1}{\alpha\alpha^2+(1+\alpha^{-1})\alpha+1} \\ &= \alpha^k\end{aligned}$$

which completes the proof of Stage II.

Proof of Stage III:

By using Stage II, we obtain

$$\lim_{x\to\infty}\frac{f(x)}{f(x+m)} = \frac{1}{\lim_{x\to\infty}\frac{f(x+m)}{f(x)}} = \frac{1}{\alpha^m} = \alpha^{-m}.$$

Now, it follows that

$$\lim_{x\to\infty}\frac{f(x+k)}{f(x+m)}=\lim_{x\to\infty}\frac{f(x+k)}{f(x)}\lim_{x\to\infty}\frac{f(x)}{f(x+m)}=\alpha^k\alpha^{-m}=\alpha^{k-m}$$

which completes the proof of Stage III. □

2. Gaussian Tribonacci Function

Gaussian Tribonacci numbers $\{GT_n\}_{n\geq 0}=\{GT_n(GT_0,GT_1,GT_2)\}_{n\geq 0}$ are defined by

$$GT_n=GT_{n-1}+GT_{n-2}+GT_{n-3},$$

with the initial conditions $GT_0=0$, $GT_1=1$ and $GT_2=1+i$. Note that

$$GT_n=T_n+iT_{n-1}.$$

The first few values of Gaussian Tribonacci numbers with positive and negative subscript are given in the following table.

n	0	1	2	3	4	5	6	7	8	9
GT_n	0	1	$1+i$	$2+i$	$4+2i$	$7+4i$	$13+7i$	$24+13i$	$44+24i$	$81+44i$
GT_{-n}	0	i	$1-i$	-1	$2i$	$2-3i$	$-3+i$	$1+4i$	$4-8i$	$-8+5i$

The full Gaussian Tribonacci sequence, where $Gu_n=GT_n$ the n^{th} Gaussian Tribonacci numbers, are: . . . , $-3+i, 2-3i, 2i, -1, 1-i, i, 0, 1, 1+i, 2+i, 4+2i, 7+4i, 13+7i, ...$

Definition 2.1. *A Gaussian function f_G on the real numbers $\mathbb{R}$ is said to be a Gaussian Tribonacci function if it satisfies the formula*

$$f_G(x+n)=GT_{n-1}f(x+2)+(GT_{n-2}+GT_{n-3})f(x+1)+GT_{n-2}f(x) \tag{2.1}$$

for all $x\in\mathbb{R}$ and $n\in\mathbb{Z}$ where f is a Tribonacci function.

To emphasize which Tribonacci function used we can say that f_G is a Gaussian Tribonacci function with Tribonacci function f.

The following theorem gives an equivalent characterization of a Gaussian Tribonacci function.

Theorem 2.2. *A Gaussian function f_G on the real numbers $\mathbb{R}$ is a Gaussian Tribonacci function if and only if*

$$f_G(x+n)=f(x+n)+if(x+n-1) \tag{2.2}$$

for $x\in\mathbb{R}$, $n\in\mathbb{Z}$ where f is a Tribonacci function.

Proof.

(:⇒) Assume that f_G is a Gaussian Tribonacci function, i.e., f_G satisfies (2.1). Then

$$\begin{aligned} f_G(x+n) &= (T_{n-1}+iT_{n-2})f(x+2)+((T_{n-2}+iT_{n-3}) \\ &\quad +(T_{n-3}+iT_{n-4}))f(x+1)+(T_{n-2}+iT_{n-3})f(x) \\ &= T_{n-1}f(x+2)+(T_{n-2}+T_{n-3})f(x+1)+T_{n-2}f(x)+ \\ &\quad i(T_{n-2}f(x+2)+(T_{n-3}+T_{n-4})f(x+1)+T_{n-3}f(x)) \\ &= f(x+n)+if(x+n-1) \end{aligned}$$

since

$$GT_n=T_n+iT_{n-1}$$

and

$$f(x+n)=T_{n-1}f(x+2)+(T_{n-2}+T_{n-3})f(x+1)+T_{n-2}f(x)$$

($\Leftarrow$:) If we suppose that (2.2) holds then by (1.1), we obtain

$$\begin{aligned}
& f_G(x+n) \\
&= f(x+n)+if(x+n-1) \\
&= T_{n-1}f(x+2)+(T_{n-2}+T_{n-3})f(x+1)+T_{n-2}f(x)+ \\
&\quad +(iT_{n-2}f(x+2)+(T_{n-3}+T_{n-4})f(x+1)+T_{n-3}f(x)) \\
&= (T_{n-1}+iT_{n-2})f(x+2)+((T_{n-2}+iT_{n-3})+(T_{n-3}+iT_{n-4}))f(x+1)+(T_{n-2}+iT_{n-3})f(x) \\
&= GT_{n-1}f(x+2)+(GT_{n-2}+GT_{n-3})f(x+1)+GT_{n-2}f(x)
\end{aligned}$$

□

Remark 2.3. *Using the Binet's formula of Tribonacci function f (see Lemma 1.1) and (2.1) or equivalenly (2.2), the Binet's formula of Gaussian Tribonacci function can be found.*

Now, we present an example of a Tribonacci function.

Example 2.4. *The function $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\alpha^x$, considered in Example 1.2, is a Tribonacci function. Then*

$$f_G(x+n)=f(x+n)+if(x+n-1)=\alpha^{x+n}+i\alpha^{x+n-1}=(1+i\alpha^{-1})\alpha^{x+n}$$

is a Gaussian Tribonacci function.

The following example shows that using floor function, a Tribonacci function and a Gaussian Tribonacci function can be obtained.

Example 2.5. *Let $\{Gu_n\}_{n=-\infty}^{\infty}$, $\{Gv_n\}_{n=-\infty}^{\infty}$, and $\{Gw_n\}_{n=-\infty}^{\infty}$, be full (bilateral) Gaussian Tribonacci sequences. We define a function f_G by $f_G(x+n)=Gu_{\lfloor x\rfloor+n}+Gv_{\lfloor x\rfloor+n}t+Gw_{\lfloor x\rfloor+n}t^2=Gu_{\lfloor x+n\rfloor}+Gv_{\lfloor x+n\rfloor}t+Gw_{\lfloor x+n\rfloor}t^2$ and $f(x)=u_{\lfloor x\rfloor}+v_{\lfloor x\rfloor}t+w_{\lfloor x\rfloor}t^2$, where $t=x-\lfloor x\rfloor\in(0,1)$ and $x\in\mathbb{R}$. Then, f is a Tribonacci function and f_G is a Gaussian Tribonacci function.*

Solution. Since

$$\begin{aligned}
f(x) &= u_{\lfloor x\rfloor}+v_{\lfloor x\rfloor}t+w_{\lfloor x\rfloor}t^2 \\
f(x+1) &= u_{\lfloor x+1\rfloor}+v_{\lfloor x+1\rfloor}t+w_{\lfloor x+1\rfloor}t^2=u_{\lfloor x\rfloor+1}+v_{\lfloor x\rfloor+1}t+w_{\lfloor x\rfloor+1}t^2 \\
f(x+2) &= u_{\lfloor x+2\rfloor}+v_{\lfloor x+2\rfloor}t+w_{\lfloor x+2\rfloor}t^2=u_{\lfloor x\rfloor+2}+v_{\lfloor x\rfloor+2}t+w_{\lfloor x\rfloor+2}t^2
\end{aligned}$$

and

$$\begin{aligned}
f(x+2)+f(x+1)+f(x) &= (u_{\lfloor x\rfloor+2}+v_{\lfloor x\rfloor+2}t+w_{\lfloor x\rfloor+2}t^2)+(u_{\lfloor x\rfloor+1}+v_{\lfloor x\rfloor+1}t+w_{\lfloor x\rfloor+1}t^2) \\
&\quad +u_{\lfloor x\rfloor}+v_{\lfloor x\rfloor}t+w_{\lfloor x\rfloor}t^2 \\
&= (u_{\lfloor x\rfloor+2}+u_{\lfloor x\rfloor+1}+u_{\lfloor x\rfloor})+(v_{\lfloor x\rfloor+2}+v_{\lfloor x\rfloor+1}+v_{\lfloor x\rfloor})t \\
&\quad +(w_{\lfloor x\rfloor+2}+w_{\lfloor x\rfloor+1}+w_{\lfloor x\rfloor})t^2 \\
&= u_{\lfloor x\rfloor+3}+v_{\lfloor x\rfloor+3}t+w_{\lfloor x\rfloor+3}t^2=u_{\lfloor x+3\rfloor}+v_{\lfloor x+3\rfloor}t+w_{\lfloor x+3\rfloor}t^2 \\
&= f(x+3)
\end{aligned}$$

f is a Tribonacci function and since

$$\begin{aligned} Gu_{\lfloor x \rfloor+n} &= u_{\lfloor x \rfloor+n} + iu_{\lfloor x \rfloor+n-1}, \\ Gv_{\lfloor x \rfloor+n} &= v_{\lfloor x \rfloor+n} + iv_{\lfloor x \rfloor+n-1}, \\ Gw_{\lfloor x \rfloor+n} &= w_{\lfloor x \rfloor+n} + iw_{\lfloor x \rfloor+n-1}, \end{aligned}$$

we get

$$\begin{aligned} f_G(x+n) &= Gu_{\lfloor x \rfloor+n} + Gv_{\lfloor x \rfloor+n}t + Gwu_{\lfloor x \rfloor+n}t^2 \\ &= (u_{\lfloor x \rfloor+n} + iu_{\lfloor x \rfloor+n-1}) + (v_{\lfloor x \rfloor+n} + iv_{\lfloor x \rfloor+n-1})t + (w_{\lfloor x \rfloor+n} + iw_{\lfloor x \rfloor+n-1})t^2 \\ &= (u_{\lfloor x \rfloor+n} + v_{\lfloor x \rfloor+n}t + w_{\lfloor x \rfloor+n}t^2) + (u_{\lfloor x \rfloor+n-1} + v_{\lfloor x \rfloor+n-1}t + w_{\lfloor x \rfloor+n-1}t^2)i \\ &= f(x+n) + if(x+n-1). \end{aligned}$$

Therefore, f_G is a Gaussian Tribonacci function. □

Lemma 2.6. *Let f_G be a Gaussian Tribonacci function, i.e., $f_G(x+n) = f(x+n) + if(x+n-1)$ for $x \in \mathbb{R}$, $n \in \mathbb{Z}$ where f is a Tribonacci function. We define $g_G(x+n) = f_G(x+t+n)$ and $g(x) = f(x+t)$ for any $x \in \mathbb{R}$ where $t \in \mathbb{R}$. Then g is a Tribonacci function and g_G is a Gaussian Tribonacci function.*

Proof. Let $x \in \mathbb{R}$. Since f_G is a Gaussian Tribonacci function and f is a Tribonacci function, it follows that

$$\begin{aligned} g(x+3) &= f(x+3+t) = f(x+t+3) \\ &= f(x+t+2) + f(x+t+1) + f(x+t) \\ &= g(x+2) + g(x+1) + g(x) \end{aligned}$$

which shows that g is a Tribonacci function and

$$\begin{aligned} g_G(x+n) &= f_G(x+t+n) \\ &= f(x+t+n) + if(x+t+n-1) \\ &= g(x+n) + ig(x+n-1) \end{aligned}$$

which shows that g_G is a Gaussian Tribonacci function. □

Lemma 2.7. *Let $\{u_n\}$ and $\{Gu_n\}$ be the full Tribonacci and Gaussian Tribonacci sequences, respectively. Then*

$$\begin{aligned} Gu_{\lfloor x \rfloor+n} &= GT_{n-1}u_{\lfloor x \rfloor+2} + (GT_{n-2} + GT_{n-3})u_{\lfloor x \rfloor+1} + GT_{n-2}u_{\lfloor x \rfloor}, \\ Gu_{\lfloor x \rfloor+n-1} &= GT_{n-1}u_{\lfloor x \rfloor+1} + (GT_{n-2} + GT_{n-3})u_{\lfloor x \rfloor} + GT_{n-2}u_{\lfloor x \rfloor-1}, \\ Gu_{\lfloor x \rfloor+n-2} &= GT_{n-1}u_{\lfloor x \rfloor} + (GT_{n-2} + GT_{n-3})u_{\lfloor x \rfloor-1} + GT_{n-2}u_{\lfloor x \rfloor-2}. \end{aligned}$$

Proof. The functions $f_G(x+n) = Gu_{\lfloor x \rfloor+n} + Gv_{\lfloor x \rfloor+n}t + Gw_{\lfloor x \rfloor+n}t^2$ and $f(x) = u_{\lfloor x \rfloor} + v_{\lfloor x \rfloor}t + w_{\lfloor x \rfloor}t^2$ where $t = x - \lfloor x \rfloor \in (0,1)$ and $x \in \mathbb{R}$, considered in Example 2.5, are Gaussian Tribonacci and Tribonacci functions, respectively. So, if we let $v_{\lfloor x \rfloor} = u_{\lfloor x \rfloor-1}$, $Gv_{\lfloor x \rfloor+n} = Gu_{\lfloor x \rfloor+n-1}$ and $w_{\lfloor x \rfloor} = u_{\lfloor x \rfloor-2}$, $Gw_{\lfloor x \rfloor+n} = Gu_{\lfloor x \rfloor+n-2}$ then $f(x)$ and $f_G(x)$ are Tribonacci function and Gaussian Tribonacci function, respectively. Note that

$$\begin{aligned} f(x) &= u_{\lfloor x \rfloor} + v_{\lfloor x \rfloor}t + w_{\lfloor x \rfloor}t^2 = u_{\lfloor x \rfloor} + u_{\lfloor x \rfloor-1}t + u_{\lfloor x \rfloor-2}t^2 \\ f(x+1) &= u_{\lfloor x+1 \rfloor} + v_{\lfloor x+1 \rfloor}t + w_{\lfloor x+1 \rfloor}t^2 = u_{\lfloor x \rfloor+1} + v_{\lfloor x \rfloor+1}t + w_{\lfloor x \rfloor+1}t^2 = u_{\lfloor x \rfloor+1} + u_{\lfloor x \rfloor}t + u_{\lfloor x \rfloor-1}t^2 \\ f(x+2) &= u_{\lfloor x+2 \rfloor} + v_{\lfloor x+2 \rfloor}t + w_{\lfloor x+2 \rfloor}t^2 = u_{\lfloor x \rfloor+2} + v_{\lfloor x \rfloor+2}t + w_{\lfloor x \rfloor+2}t^2 = u_{\lfloor x \rfloor+2} + u_{\lfloor x \rfloor+1}t + u_{\lfloor x \rfloor}t^2 \end{aligned}$$

Then we obtain

$$\begin{aligned}
& Gu_{\lfloor x\rfloor+n}+Gu_{\lfloor x\rfloor+n-1}t+Gu_{\lfloor x\rfloor+n-2}t^2 \\
&= Gu_{\lfloor x\rfloor+n}+Gv_{\lfloor x\rfloor+n}t+Gw_{\lfloor x\rfloor+n}t^2=f_G(x+n) \\
&= GT_{n-1}f(x+2)+(GT_{n-2}+GT_{n-3})f(x+1)+GT_{n-2}f(x) \\
&= GT_{n-1}(u_{\lfloor x\rfloor+2}+u_{\lfloor x\rfloor+1}t+u_{\lfloor x\rfloor}t^2)+(GT_{n-2}+GT_{n-3})(u_{\lfloor x\rfloor+1}+u_{\lfloor x\rfloor}t+u_{\lfloor x\rfloor-1}t^2) \\
& GT_{n-2}(u_{\lfloor x\rfloor}+u_{\lfloor x\rfloor-1}t+u_{\lfloor x\rfloor-2}t^2) \\
&= (GT_{n-1}u_{\lfloor x\rfloor+2}+GT_{n-1}u_{\lfloor x\rfloor+1}t+GT_{n-1}u_{\lfloor x\rfloor}t^2) \\
& +((GT_{n-2}+GT_{n-3})u_{\lfloor x\rfloor+1}+(GT_{n-2}+GT_{n-3})u_{\lfloor x\rfloor}t+(GT_{n-2}+GT_{n-3})u_{\lfloor x\rfloor-1}t^2) \\
& (GT_{n-2}u_{\lfloor x\rfloor}+GT_{n-2}u_{\lfloor x\rfloor-1}t+GT_{n-2}u_{\lfloor x\rfloor-2}t^2) \\
&= (GT_{n-1}u_{\lfloor x\rfloor+2}+(GT_{n-2}+GT_{n-3})u_{\lfloor x\rfloor+1}+GT_{n-2}u_{\lfloor x\rfloor}) \\
& +(GT_{n-1}u_{\lfloor x\rfloor+1}+(GT_{n-2}+GT_{n-3})u_{\lfloor x\rfloor}+GT_{n-2}u_{\lfloor x\rfloor-1})t \\
& +(GT_{n-1}u_{\lfloor x\rfloor}+(GT_{n-2}+GT_{n-3})u_{\lfloor x\rfloor-1}+GT_{n-2}u_{\lfloor x\rfloor-2})t^2
\end{aligned}$$

This completes the proof. □

By taking $\{u_n\}=\{T_n\}$ in the last theorem, we have the following corollary.

Corollary 2.8. *For $x\in\mathbb{R}$, we have the following formulas:*

$$\begin{aligned}
GT_{\lfloor x\rfloor+n} &= GT_{n-1}T_{\lfloor x\rfloor+2}+(GT_{n-2}+GT_{n-3})T_{\lfloor x\rfloor+1}+GT_{n-2}T_{\lfloor x\rfloor}, \\
GT_{\lfloor x\rfloor+n-1} &= GT_{n-1}T_{\lfloor x\rfloor+1}+(GT_{n-2}+GT_{n-3})T_{\lfloor x\rfloor}+GT_{n-2}T_{\lfloor x\rfloor-1}, \\
GT_{\lfloor x\rfloor+n-2} &= GT_{n-1}T_{\lfloor x\rfloor}+(GT_{n-2}+GT_{n-3})T_{\lfloor x\rfloor-1}+GT_{n-2}T_{\lfloor x\rfloor-2}.
\end{aligned}$$

By taking $\lfloor x\rfloor=m\in\mathbb{Z}$ in the last corollary, we see that for all integers m,n we have

$$\begin{aligned}
GT_{m+n} &= GT_{n-1}T_{m+2}+(GT_{n-2}+GT_{n-3})T_{m+1}+GT_{n-2}T_m, \\
GT_{m+n-1} &= GT_{n-1}T_{m+1}+(GT_{n-2}+GT_{n-3})T_m+GT_{n-2}T_{m-1}, \\
GT_{m+n-2} &= GT_{n-1}T_m+(GT_{n-2}+GT_{n-3})T_{m-1}+GT_{n-2}T_{m-2}.
\end{aligned}$$

Theorem 2.9. *Let $f_G(x)=a(x)g_G(x)$ be a function, $g(x)$ and $f(x)=a(x)g(x)$ be Tribonacci functions, where $a(x)$ is an f-even function, and suppose that $g_G(x)$ and $g(x)$ are continuous functions. Then $f_G(x)$ is a Gaussian Tribonacci function with Tribonacci function $f(x)$ if and only if $g_G(x)$ is a Gaussian Tribonacci function with Tribonacci function $g(x)$.*

Proof. By definition of the function f_G and since $a(x)$ is an f-even function, we have

$$f_G(x+n)=a(x+n)g_G(x+n)=a(x)g_G(x+n). \tag{2.3}$$

Suppose that f_G is a Gaussian Tribonacci function. Then, since $a(x)$ is an f-even function, we obtain

$$\begin{aligned}
f_G(x+n) &= f(x+n)+if(x+n-1) \\
&= a(x+n)g(x+n)+ia(x+n-1)g(x+n-1) \\
&= a(x)g(x+n)+ia(x)g(x+n-1) \\
&= a(x)(g(x+n)+ig(x+n-1)).
\end{aligned} \tag{2.4}$$

From the equations (2.3) and (2.4), we get

$$a(x)(g_G(x+n)-g(x+n)-ig(x+n-1))\equiv 0$$

and so

$$g_G(x+n) - g(x+n) - ig(x+n-1) \equiv 0$$

i.e.,

$$g_G(x+n) = g(x+n) + ig(x+n-1).$$

Therefore, g_G is a Gaussian Tribonacci function.
On the other hand, if g_G is a Gaussian Tribonacci function, then

$$g_G(x+n) = g(x+n) + ig(x+n-1) \tag{2.5}$$

Since $f(x) = a(x)g(x)$ and $a(x)$ is an f-even function, we obtain

$$\begin{aligned} f(x+n) &= a(x+n)g(x+n) = a(x)g(x+n), \\ f(x+n-1) &= a(x+n-1)g(x+n-1) = a(x)g(x+n-1). \end{aligned}$$

Then, since $f_G(x) = a(x)g_G(x)$ and $a(x)$ is an f-even function, the equation (2.5) implies that

$$\begin{aligned} f_G(x+n) &= a(x+n)g_G(x+n) = a(x)g_G(x+n) \\ &= a(x)(g(x+n) + ig(x+n-1)) \\ &= a(x)g(x+n) + ia(x)g(x+n-1) \\ &= f(x+n) + if(x+n-1). \end{aligned}$$

Hence, f_G is a Gaussian Tribonacci function. □

3. Sums of Tribonacci and Gaussian Tribonacci Functions

In this section, we discuss the sums of the terms of a Tribonacci function and a Gaussian Tribonacci function. The following corollary gives linear sum formulas of Tribonacci numbers.

Corollary 3.1. *For $n \geq 0$, Tribonacci numbers have the following property.*

$$\sum_{k=0}^{n} T_k = \frac{1}{2}(T_{n+3} - T_{n+1} - 1).$$

Proof. For a proof see [17]. □

The following theorem gives linear sum formulas of Tribonacci functions.

Theorem 3.2. *Suppose that f is a Tribonacci function. Then for all $x \in \mathbb{R}$ and $n \geq 0$, the following sum formula holds:*

$$\sum_{k=0}^{n} f(x+k) = \frac{1}{2}(f(x+n+4) - f(x+n+2) - 2f(x+n+1) - f(x+2) + f(x)).$$

Proof. We use corollary 3.1. Since

$$\begin{aligned} \sum_{k=0}^{n} T_{k-1} &= \frac{1}{2}(T_{n+3} - T_{n+1} - 2T_n - 1), \\ \sum_{k=0}^{n} T_{k-2} &= \frac{1}{2}(T_{n+3} - T_{n+1} - 2T_{n-1} - 2T_n + 1), \\ \sum_{k=0}^{n} T_{k-3} &= \frac{1}{2}(T_{n+3} - T_{n+1} - 2T_n - 2T_{n-1} - 2T_{n-2} - 1), \end{aligned}$$

and

$$f(x+n) = T_{n-1}f(x+2) + (T_{n-2}+T_{n-3})f(x+1) + T_{n-2}f(x),$$

we obtain

$$\begin{aligned}\sum_{k=0}^{n} f(x+k) &= f(x+2)\sum_{k=0}^{n} T_{k-1} + f(x+1)\sum_{k=0}^{n}(T_{k-2}+T_{k-3}) + f(x)\sum_{k=0}^{n} T_{k-2} \\ &= f(x+2)\times\frac{1}{2}(T_{n+3}-T_{n+1}-2T_n-1) \\ &\quad +f(x+1)\times\frac{1}{2}(T_{n+2}+T_{n+1}-(T_n+T_{n-1})-2(T_{n-1}+T_{n-2})) \\ &\quad +f(x)\times\frac{1}{2}(T_{n+2}-T_n-2T_{n-1}+1) \\ &= \frac{1}{2}(f(x+n+4)-f(x+n+2)-2f(x+n+1)-f(x+2)+f(x))\end{aligned}$$

□

Note that if we consider the Tribonacci function

$$f(x) = \alpha^x$$

then, using Theorem 3.2, we have the sum formula

$$\sum_{k=0}^{n} \alpha^{x+k} = \frac{1}{2}(\alpha^{x+n+4} - \alpha^{x+n+2} - 2\alpha^{x+n+1} - \alpha^{x+2} + \alpha^x)$$

for all $x \in \mathbb{R}$ and $n \geq 0$.

The following corollary gives linear sum formulas of Gaussian Tribonacci numbers.

Corollary 3.3. *For $n \geq 1$ we have the following formulas:*

$$\sum_{k=1}^{n} GT_k = \frac{1}{2}(GT_{n+3} - GT_{n+1} - (1+i)).$$

Proof. It is given in [16].

The following theorem gives linear sum formulas of Gaussian Tribonacci functions.

Theorem 3.4. *Suppose that f_G is a Gaussian Tribonacci function with Tribonacci function f. Then for all $x \in \mathbb{R}$ and $n \geq 1$ the following sum formula holds:*

$$\sum_{k=1}^{n} f_G(x+k) = \frac{1}{2}(f_G(x+n+3) - f_G(x+n+1) - (1+i)f(x+2) + (-1+i)f(x))$$

Proof. We use corollary 3.3. Since

$$\begin{aligned}\sum_{k=1}^{n} GT_k &= \frac{1}{2}(GT_{n+3} - GT_{n+1} - (1+i)), \\ \sum_{k=1}^{n} GT_{k-1} &= \frac{1}{2}(GT_{n+2} - GT_n - (1+i)), \\ \sum_{k=1}^{n} GT_{k-2} &= \frac{1}{2}(GT_{n+1} - GT_{n-1} - 1 + i), \\ \sum_{k=1}^{n} GT_{k-3} &= \frac{1}{2}(GT_n - GT_{n-2} + 1 - i),\end{aligned}$$

and

$$f_G(x+n) = GT_{n-1}f(x+2) + (GT_{n-2} + GT_{n-3})f(x+1) + GT_{n-2}f(x),$$

we get

$$\begin{aligned}\sum_{k=1}^{n} f_G(x+k) &= f(x+2)\sum_{k=1}^{n} GT_{k-1} + f(x+1)(\sum_{k=1}^{n} GT_{k-2} + \sum_{k=1}^{n} GT_{k-3}) + f(x)\sum_{k=1}^{n} GT_{k-2} \\ &= \frac{1}{2}(f_G(x+n+3) - f_G(x+n+1) - (1+i)f(x+2) + (-1+i)f(x)).\end{aligned}$$

□

Note that if we consider the Tribonacci function

$$f(x) = \alpha^x$$

and the Gaussian Tribonacci function

$$f_G(x+n) = (1+i\alpha^{-1})\alpha^{x+n}$$

then, using Theorem 3.4, we have the sum formula

$$\sum_{k=1}^{n}(1+i\alpha^{-1})\alpha^{x+k} = \frac{1}{2}((1+i\alpha^{-1})\alpha^{x+n+3} - (1+i\alpha^{-1})\alpha^{x+n+1} - (1+i)\alpha^{x+2} + (-1+i)\alpha^{x})$$

for all $x \in \mathbb{R}$ and $n \geq 1$.

4. Ratio of Gaussian Tribonacci Functions

In this section, we discuss the limit of the quotient of a Gaussian Tribonacci function. Note that since

$$\lim_{n\to\infty}\frac{T_{n+p}}{T_{n+q}} = \alpha^{p-q}, \quad p, q \in \mathbb{Z}$$

we have

$$\begin{aligned}\lim_{n\to\infty}\frac{GT_{n+p}}{GT_{n+q}} &= \lim_{n\to\infty}\frac{T_{n+p} + iT_{n+p-1}}{T_{n+q} + iT_{n+q-1}} \\ &= \lim_{n\to\infty}\frac{\frac{T_{n+p}}{T_{n+q}} + i\frac{T_{n+p-1}}{T_{n+q}}}{\frac{T_{n+q}}{T_{n+q}} + i\frac{T_{n+q-1}}{T_{n+q}}} \\ &= \frac{\alpha^{p-q} + i\alpha^{p-1-q}}{1+i\alpha^{-1}}.\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty}\frac{GT_{n+p}}{T_{n+q}} &= \lim_{n\to\infty}\frac{T_{n+p} + iT_{n+p-1}}{T_{n+q}} \\ &= \lim_{n\to\infty}\frac{T_{n+p}}{T_{n+q}} + i\lim_{n\to\infty}\frac{T_{n+p-1}}{T_{n+q}} \\ &= \alpha^{p-q} + i\alpha^{p-1-q}.\end{aligned}$$

Theorem 4.1. *If f_G is a Gaussian Tribonacci function, then the limit of quotient*

$$\frac{f_G(x+k)}{f_G(x+m)}$$

exists and given by

$$\lim_{x\to\infty}\frac{f_G(x+k)}{f_G(x+m)}=\alpha^{k-m}$$

for all $k,m\in\mathbb{Z}$.

Proof. Suppose that f_G is a Gaussian Tribonacci function with Tribonacci function f. Note that from Theorem 1.6 and Theorem 1.7, the limit of quotients $\frac{f(x+1)}{f(x)}$ and $\frac{f(x+2)}{f(x)}$ exists and $\lim\limits_{x\to\infty}\frac{f(x+1)}{f(x)}=\alpha$ and $\lim\limits_{x\to\infty}\frac{f(x+2)}{f(x)}=\alpha^2$. We use the formula, by definition,

$$f_G(x+n)=GT_{n-1}f(x+2)+(GT_{n-2}+GT_{n-3})f(x+1)+GT_{n-2}f(x).$$

Given $x\in\mathbb{R}$, there exists $y\in\mathbb{R}$ and $n\in\mathbb{N}$ such that $x=y+n$.Then

$$\begin{aligned}\lim_{x\to\infty}\frac{f_G(x+k)}{f_G(x+m)}&=\lim_{y\to\infty}\lim_{n\to\infty}\frac{f_G(y+n+k)}{f_G(y+n+m)}=\lim_{x\to\infty}\lim_{n\to\infty}\frac{f_G(x+n+k)}{f_G(x+n+m)}\\&=\lim_{x\to\infty}\lim_{n\to\infty}\frac{\frac{GT_{n+k-1}}{GT_{n+m-1}}\frac{f(x+2)}{f(x)}+(\frac{GT_{n+k-2}}{GT_{n+m-1}}+\frac{GT_{n+k-3}}{GT_{n+m-1}})\frac{f(x+1)}{f(x)}+\frac{GT_{n+k-2}}{GT_{n+m-1}}}{\frac{f(x+2)}{f(x)}+(\frac{GT_{n+m-2}}{GT_{n+m-1}}+\frac{GT_{n+m-3}}{GT_{n+m-1}})\frac{f(x+1)}{f(x)}+\frac{GT_{n+m-2}}{GT_{n+m-1}}}.\end{aligned}$$

Since

$$\begin{aligned}\lim_{n\to\infty}\frac{T_{n+p}}{T_{n+q}}&=\alpha^{p-q},\quad p,q\in\mathbb{Z},\\\lim_{n\to\infty}\frac{GT_{n+p}}{GT_{n+q}}&=\frac{\alpha^{p-q}+i\alpha^{p-1-q}}{1+i\alpha^{-1}},\quad p,q\in\mathbb{Z},\\\lim_{x\to\infty}\frac{f(x+1)}{f(x)}&=\alpha,\\\lim_{x\to\infty}\frac{f(x+2)}{f(x)}&=\alpha^2,\end{aligned}$$

it follows that

$$\lim_{x\to\infty}\frac{f_G(x+k)}{f_G(x+m)}=\alpha^{k-m}$$

□

Note that if we consider the Tribonacci function

$$f(x)=\alpha^x$$

and the Gaussian Tribonacci function

$$f_G(x+n)=(1+i\alpha^{-1})\alpha^{x+n}$$

then, we see that

$$\lim_{x\to\infty}\frac{f_G(x+k)}{f_G(x+m)}=\lim_{x\to\infty}\frac{(1+i\alpha^{-1})\alpha^{x+k}}{(1+i\alpha^{-1})\alpha^{x+m}}=\lim_{x\to\infty}\frac{1}{\alpha^{m+x}}\alpha^{k+x}=\alpha^{k-m}.$$

Also, it follows from Theorem 4.1, that

$$\lim_{x\to\infty}\frac{f_G(x+k)}{f_G(x+m)}=\alpha^{k-m}.$$

Corollary 4.2. *If f_G is a Gaussian Tribonacci function, then*

$$\lim_{x\to\infty}\frac{f_G(x+1)}{f_G(x)}=\alpha.$$

Proof. Take $k=1, m=0$ in Theorem 4.1. $\square$

Theorem 4.3. *If f_G is a Gaussian Tribonacci function with Tribonacci function f, then the limit of quotient*

$$\frac{f_G(x+k)}{f(x)}$$

exists and given by

$$\lim_{x\to\infty}\frac{f_G(x+k)}{f(x)}=GT_{k-1}\alpha^2+(GT_{k-2}+GT_{k-3})\alpha+GT_{k-2}$$

for all $k\in\mathbb{Z}$.

Proof. Suppose that f_G is a Gaussian Tribonacci function with Tribonacci function f. Note that from Theorem 1.6, the limit of quotients $\frac{f(x+1)}{f(x)}$ and $\frac{f(x+2)}{f(x)}$ exists. Using the formula, by definition,

$$f_G(x+n)=GT_{n-1}f(x+2)+(GT_{n-2}+GT_{n-3})f(x+1)+GT_{n-2}f(x).$$

we get

$$\begin{aligned}\lim_{x\to\infty}\frac{f_G(x+k)}{f(x)}&=\lim_{x\to\infty}\frac{GT_{k-1}f(x+2)+(GT_{k-2}+GT_{k-3})f(x+1)+GT_{k-2}f(x)}{f(x)}\\&=GT_{k-1}\lim_{x\to\infty}\frac{f(x+2)}{f(x)}+(GT_{k-2}+GT_{k-3})\lim_{x\to\infty}\frac{f(x+1)}{f(x)}+GT_{k-2}.\end{aligned}$$

Hence, since the limit of quotients $\frac{f(x+1)}{f(x)}$ and $\frac{f(x+2)}{f(x)}$ exists, $\lim_{x\to\infty}\frac{f_G(x+k)}{f(x)}$ exists and

$$\lim_{x\to\infty}\frac{f_G(x+k)}{f(x)}=GT_{k-1}\alpha^2+(GT_{k-2}+GT_{k-3})\alpha+GT_{k-2}.$$

$\square$

Note that if we consider the Tribonacci function

$$f(x)=\alpha^x$$

and the Gaussian Tribonacci function

$$f_G(x+n)=(1+i\alpha^{-1})\alpha^{x+n}$$

then, we see that

$$\lim_{x\to\infty}\frac{f_G(x+k)}{f(x)}=\lim_{x\to\infty}\frac{(1+i\alpha^{-1})\alpha^{x+k}}{\alpha^x}=\lim_{x\to\infty}(1+i\alpha^{-1})\alpha^k=(1+i\alpha^{-1})\alpha^k. \tag{4.1}$$

Also,from Theorem 4.3, we know that

$$\lim_{x\to\infty}\frac{f_G(x+k)}{f(x)}=GT_{k-1}\alpha^2+(GT_{k-2}+GT_{k-3})\alpha+GT_{k-2}. \tag{4.2}$$

Therefore, comparing (4.1) and (4.2), we obtain

$$GT_{k-1}\alpha^2+(GT_{k-2}+GT_{k-3})\alpha+GT_{k-2}=(1+i\alpha^{-1})\alpha^k$$

for $k\in\mathbb{Z}$.

Corollary 4.4. *If f_G is a Gaussian Tribonacci function with Tribonacci function f, then*

$$\lim_{x\to\infty}\frac{f_G(x)}{f(x)} = 1+i(\alpha^2-\alpha-1),$$
$$\lim_{x\to\infty}\frac{f_G(x+1)}{f(x)} = \alpha+i,$$
$$\lim_{x\to\infty}\frac{f_G(x+2)}{f(x)} = \alpha^2+i\alpha.$$

Proof. Take $k=0,1,2$ in Theorem 4.3, respectively. □

We can generalize Theorem 4.3 as follows.

Theorem 4.5. *If f_G is a Gaussian Tribonacci function with Tribonacci function f, then the limit of quotient*

$$\frac{f_G(x+k)}{f(x+m)}$$

exists and given by

$$\lim_{x\to\infty}\frac{f_G(x+k)}{f(x+m)} = (\alpha+i)\alpha^{k-m-1}$$

for all $k,m\in\mathbb{Z}$.

Proof. Suppose that f_G is a Gaussian Tribonacci function with Tribonacci function f. Note that from Theorem 1.7, the limit of quotients $\frac{f(x+1)}{f(x)}$ and $\frac{f(x+2)}{f(x)}$ exists and $\lim_{x\to\infty}\frac{f(x+1)}{f(x)}=\alpha$ and $\lim_{x\to\infty}\frac{f(x+2)}{f(x)}=\alpha^2$. Given $x\in\mathbb{R}$, there exists $y\in\mathbb{R}$ and $n\in\mathbb{N}$ such that $x=y+n$. By using the formulas

$$f_G(x+n)=GT_{n-1}f(x+2)+(GT_{n-2}+GT_{n-3})f(x+1)+GT_{n-2}f(x)$$

and

$$f(x+n)=T_{n-1}f(x+2)+(T_{n-2}+T_{n-3})f(x+1)+T_{n-2}f(x)$$

we get

$$\begin{aligned}\lim_{x\to\infty}\frac{f_G(x+k)}{f(x+m)} &= \lim_{y\to\infty}\lim_{n\to\infty}\frac{f_G(y+n+k)}{f(y+n+m)}\\ &= \lim_{x\to\infty}\lim_{n\to\infty}\frac{f_G(x+n+k)}{f(x+n+m)}\\ &= \lim_{x\to\infty}\lim_{n\to\infty}\frac{\frac{GT_{n+k-1}}{T_{n+m-1}}\frac{f(x+2)}{f(x)}+(\frac{GT_{n+k-2}}{T_{n+m-1}}+\frac{GT_{n+k-3}}{T_{n+m-1}})\frac{f(x+1)}{f(x)}+\frac{GT_{n+k-2}}{T_{n+m-1}}}{\frac{f(x+2)}{f(x)}+(\frac{T_{n+m-2}}{T_{n+m-1}}+\frac{T_{n+m-3}}{T_{n+m-1}})\frac{f(x+1)}{f(x)}+\frac{T_{n+m-2}}{T_{n+m-1}}}\end{aligned}$$

Since

$$\lim_{n\to\infty}\frac{GT_{n+p}}{T_{n+q}} = \alpha^{p-q}+i\alpha^{p-1-q},\quad p,q\in\mathbb{Z},$$
$$\lim_{x\to\infty}\frac{f(x+1)}{f(x)} = \alpha,$$
$$\lim_{x\to\infty}\frac{f(x+2)}{f(x)} = \alpha^2,$$

it follows that

$$\lim_{x\to\infty}\frac{f_G(x+k)}{f(x+m)} = (\alpha+i)\alpha^{k-m-1}.\quad\square$$

5. Matrix Formulation of $f(x)$ and $f_G(x+n)$

The matrix method is very useful method in order to obtain some identities for special sequences. We define the square matrix M of order 3 as:

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det M = 1$. Note that for all $n \in \mathbb{Z}$, we have

$$M^n = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} T_{n+1} & T_n + T_{n-1} & T_n \\ T_n & T_{n-1} + T_{n-2} & T_{n-1} \\ T_{n-1} & T_{n-2} + T_{n-3} & T_{n-2} \end{pmatrix}. \tag{5.1}$$

Matrix formulation of T_n can be given as

$$\begin{pmatrix} T_{n+2} \\ T_{n+1} \\ T_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} T_2 \\ T_1 \\ T_0 \end{pmatrix}. \tag{5.2}$$

Consider the matrices N_T, E_T defined by as follows:

$$N_T = \begin{pmatrix} 1+i & 1 & 0 \\ 1 & 0 & i \\ 0 & i & 1-i \end{pmatrix},$$

$$E_T = \begin{pmatrix} GT_{n+2} & GT_{n+1} & GT_n \\ GT_{n+1} & GT_n & GT_{n-1} \\ GT_n & GT_{n-1} & GT_{n-2} \end{pmatrix}.$$

The following theorem presents the relations between M^n, N_T and E_T.

Theorem 5.1. *For all $n \in \mathbb{Z}$, we have*

$$M^n N_T = E_T.$$

Proof. For a proof, see [16]. □

Define

$$A_f = \begin{pmatrix} f(x+2) & f(x+1) & f(x) \\ f(x+1) & f(x) & f(x-1) \\ f(x) & f(x-1) & f(x-2) \end{pmatrix},$$

$$B_f = \begin{pmatrix} f(x+n+2) & f(x+n+1) & f(x+n) \\ f(x+n+1) & f(x+n) & f(x+n-1) \\ f(x+n) & f(x+n-1) & f(x+n-2) \end{pmatrix}.$$

Theorem 5.2. *For all integers $n \in \mathbb{Z}$ and $x \in \mathbb{R}$, we have*

$$M^n A_f = B_f \tag{5.3}$$

Proof. By using

$$f(x+n) = T_{n-1} f(x+2) + (T_{n-2} + T_{n-3}) f(x+1) + T_{n-2} f(x)$$

and

$$f(x+3) = f(x+2) + f(x+1) + f(x),$$

the case $n \geq 0$ can be proved by mathematical induction. Then for the case $n \leq 0$, we take $m = -n$ in 5.3 and then the case $m \geq 0$ can be proved by mathematical induction, as well. □

Note that if we consider the Tribonacci function

$$f(x) = \alpha^x$$

then, we see that

$$A_f = \begin{pmatrix} \alpha^{x+2} & \alpha^{x+1} & \alpha^x \\ \alpha^{x+1} & \alpha^x & \alpha^{x-1} \\ \alpha^x & \alpha^{x-1} & \alpha^{x-2} \end{pmatrix}, B_f = \begin{pmatrix} \alpha^{x+n+2} & \alpha^{x+n+1} & \alpha^{x+n} \\ \alpha^{x+n+1} & \alpha^{x+n} & \alpha^{x+n-1} \\ \alpha^{x+n} & \alpha^{x+n-1} & \alpha^{x+n-2} \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} \alpha^{x+2} & \alpha^{x+1} & \alpha^x \\ \alpha^{x+1} & \alpha^x & \alpha^{x-1} \\ \alpha^x & \alpha^{x-1} & \alpha^{x-2} \end{pmatrix} = \begin{pmatrix} \alpha^{x+n+2} & \alpha^{x+n+1} & \alpha^{x+n} \\ \alpha^{x+n+1} & \alpha^{x+n} & \alpha^{x+n-1} \\ \alpha^{x+n} & \alpha^{x+n-1} & \alpha^{x+n-2} \end{pmatrix}$$

for all $n \in \mathbb{Z}$ and $x \in \mathbb{R}$.

Define

$$D_{GT} = \begin{pmatrix} GT_{n+1} & GT_n + GT_{n-1} & GT_n \\ GT_n & GT_{n-1} + GT_{n-2} & GT_{n-1} \\ GT_{n-1} & GT_{n-2} + GT_{n-3} & GT_{n-2} \end{pmatrix}$$

and

$$C_{f_G} = \begin{pmatrix} f_G(x+n+2) & f_G(x+n+1) & f_G(x+n) \\ f_G(x+n+1) & f_G(x+n) & f_G(x+n-1) \\ f_G(x+n) & f_G(x+n-1) & f_G(x+n-2) \end{pmatrix}.$$

Theorem 5.3. *For all integers $n \in \mathbb{Z}$ and $x \in \mathbb{R}$, we have*

$$D_{GT} A_f = C_{f_G} \tag{5.4}$$

Proof. By using

$$f_G(x+n) = GT_{n-1} f(x+2) + (GT_{n-2} + GT_{n-3}) f(x+1) + GT_{n-2} f(x)$$

and

$$f(x+3) = f(x+2) + f(x+1) + f(x),$$

the case $n \geq 0$ can be proved by mathematical induction. Then for the case $n \leq 0$, we take $m = -n$ in 5.4 and then the case $m \geq 0$ can be proved by mathematical induction, as well. □

Note that if we consider the Tribonacci function

$$f(x) = \alpha^x$$

and the Gaussian Tribonacci function

$$f_G(x+n) = (1 + i\alpha^{-1})\alpha^{x+n}$$

then, we see that

$$A_f = \begin{pmatrix} \alpha^{x+2} & \alpha^{x+1} & \alpha^x \\ \alpha^{x+1} & \alpha^x & \alpha^{x-1} \\ \alpha^x & \alpha^{x-1} & \alpha^{x-2} \end{pmatrix}, D_{GT} = \begin{pmatrix} GT_{n+1} & GT_n + GT_{n-1} & GT_n \\ GT_n & GT_{n-1} + GT_{n-2} & GT_{n-1} \\ GT_{n-1} & GT_{n-2} + GT_{n-3} & GT_{n-2} \end{pmatrix}$$

and

$$C_{f_G} = \begin{pmatrix} (1+i\alpha^{-1})\alpha^{x+n+2} & (1+i\alpha^{-1})\alpha^{x+n+1} & (1+i\alpha^{-1})\alpha^{x+n} \\ (1+i\alpha^{-1})\alpha^{x+n+1} & (1+i\alpha^{-1})\alpha^{x+n} & (1+i\alpha^{-1})\alpha^{x+n-1} \\ (1+i\alpha^{-1})\alpha^{x+n} & (1+i\alpha^{-1})\alpha^{x+n-1} & (1+i\alpha^{-1})\alpha^{x+n-2} \end{pmatrix},$$

and so

$$\begin{pmatrix} GT_{n+1} & GT_n + GT_{n-1} & GT_n \\ GT_n & GT_{n-1} + GT_{n-2} & GT_{n-1} \\ GT_{n-1} & GT_{n-2} + GT_{n-3} & GT_{n-2} \end{pmatrix} \begin{pmatrix} \alpha^{x+2} & \alpha^{x+1} & \alpha^{x} \\ \alpha^{x+1} & \alpha^{x} & \alpha^{x-1} \\ \alpha^{x} & \alpha^{x-1} & \alpha^{x-2} \end{pmatrix}$$
$$= \begin{pmatrix} (1+i\alpha^{-1})\alpha^{x+n+2} & (1+i\alpha^{-1})\alpha^{x+n+1} & (1+i\alpha^{-1})\alpha^{x+n} \\ (1+i\alpha^{-1})\alpha^{x+n+1} & (1+i\alpha^{-1})\alpha^{x+n} & (1+i\alpha^{-1})\alpha^{x+n-1} \\ (1+i\alpha^{-1})\alpha^{x+n} & (1+i\alpha^{-1})\alpha^{x+n-1} & (1+i\alpha^{-1})\alpha^{x+n-2} \end{pmatrix}.$$

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