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Weighted pseudo S-asymptotically Bloch type periodic solutions for a class of mean field stochastic fractional evolution equations

MAMADOU MOUSTAPHA MBAYE[1*], **AMADOU DIOP**[2] AND **MOUSTAPHA DIEYE**[3]

[1] *Département de Mathématiques et Informatique, Faculté des Sciences et Technique, Université Cheikh Anta Diop, Dakar, BP5005, Sénégal.*
[2] *Laboratoire d'Analyse Numérique et Informatique, Université Gaston Berger, Saint-Louis, BP234, Sénégal.*
[3] *Ecole Polytechnique de Thiès, Thiès, BPA10, Sénégal.*

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Abstract. This paper concerns a class of mean-field stochastic fractional evolution equations. Initially, we establish some auxiliary results for weighted pseudo S-asymptotically Bloch type periodic stochastic processes. Without a compactness assumption on the resolvent operator and some additional conditions on forced terms, the existence and uniqueness of weighted pseudo S-asymptotically Bloch type periodic mild solutions on the real line of the referred equation are obtained. In addition, we show the existence of weighted pseudo S-asymptotically Bloch type periodic mild solutions with sublinear growth assumptions on the drift term and compactness conditions. Finally, an example is provided to verify the main outcomes.

AMS Subject Classifications: 30D45, 34C25, 60H15, 60G22.

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Contents

*Corresponding author. Email address: **mamadoumoustapha3.mbaye@ucad.edu.sn** (M.M Mbaye), **amadou.diop@ugb.edu.sn** (A. Diop), **mdieye@ept.sn** (M. Dieye)

1. Introduction

Periodicity is a key topic in the qualitative property of differential equations, because of its importance in both pure mathematics and applications. However, not all the phenomena in the real world can satisfy the periodicity criteria. Some phenomena have a behaviour that is not periodic, but rather almost periodic, asymptotically almost periodic, ω-periodicity, asymptotically ω-periodicity, Bloch periodic, and so on. Bloch periodic phenomena are related to the conductivity of crystalline solids, as F. Bloch showed in [10]. N'Guérékata and Hasler introduced the concept of Bloch-type periodic functions in [19], which generalizes the classical notions of ω-periodicity and ω-anti-periodicity. Some publications have also explored the effects of perturbations on Bloch periodic functions, by defining some quasi-Bloch periodicity concepts. For instance, [18, 20, 32] studied asymptotically Bloch periodic functions and their applications, while [31–34] investigated (pseudo) S-asymptotically Bloch periodic functions and their applications. These quasi-Bloch periodic functions are extensions of the corresponding asymptotically ω-periodic and (pseudo) S-asymptotically ω-periodic functions in the deterministic case. In [27, 28], the authors investigated the notion of S-asymptotically ω-periodicity for stochastic processes and established some results on their existence, uniqueness, and asymptotic stability. Recently, in [5], the concepts of square-mean (weighted) pseudo S-asymptotically Bloch-type periodicity for stochastic process was introduced, which is a type of periodicity that can capture more stochastic phenomena. Moreover, the authors of [5] investigated the existence and uniqueness of the mild solution of some stochastic evolution equations.

In contrast, mean-field stochastic differential equations (SDEs), also known as McKean-Vlasov equations, represent weak interactions between particles within a large system. Kac [17] first investigated it in relation to the Boltzmann equation for particle density in diluted monoatomic gases. He also studied it in the stochastic toy model for the Vlasov kinetic equation for plasma. McKean [13] examined how chaos spreads in physical systems containing N particles interacting with one another. In his work, he emphasized the importance of the Boltzmann equation, which describes the statistical behavior of gases with low densities. In [3, 4], Sznitman examines chaos and the limit equation from a different perspective. As with the previously mentioned SDEs, he described the limit equation using an evolution equation. A study of the dynamics of the polymers was carried out by E and Shen in [30]. To approximate the description of the polymers, they used stochastic partial differential equations of the mean field type. In [2, 11] they addressed similar issues related to stochastic differential equations in infinite dimensional spaces.

A number of current research investigations have focused primarily on the existence and uniqueness of solutions for stochastic fractional order evolution equations of the McKean-Vlasov type, with little or no results from periodic or quasi-periodic solutions for the referred class of equations. Therefore, the above literature motivates us to explore the existence and uniqueness of weighted pseudo S-asymptotically Bloch type periodicity mild solutions of the following abstract mean field stochastic fractional evolution equations

$$\begin{cases} \partial_t^\alpha v(t) = Av(t) + \displaystyle\int_{-\infty}^{t} b(t-s)Av(s)ds + g(t, v(t), \mathbb{P}_{v(t)}) \\ \qquad\qquad\qquad + f(t, v(t), \mathbb{P}_{v(t)})\dfrac{d\mathbb{W}(t)}{dt}, \quad t \in \mathbb{R}, \\ \mathbb{P}_{v(t)} = \text{ Probability distribution of } v(t). \end{cases} \tag{1.1}$$

Here ∂_t^α denotes the Weyl fractional derivative of order $\alpha > 0$, $A : D(A) \subseteq \mathbb{L}^2(\Omega, \mathbb{H}) \to \mathbb{L}^2(\Omega, \mathbb{H})$ is a closed linear operator on a complex Hilbert space $\mathbb{L}^2(\Omega, \mathbb{H})$ (where $\mathbb{L}^2(\Omega, \mathbb{H})$ is an appropriate function space specified in Section 2) and generate an α-resolvent family $\{\mathcal{R}_\alpha(t)\}_{t\geq 0}$ on $\mathbb{H}$, $g,\ f$ are $\mathbb{H}$-valued stochastic processes. Here $(\mathbb{W}(t))_{t\in\mathbb{R}}$ represents a two-sided and standard one-dimensional Brownian motion on $\mathbb{H}$ and $\mathbb{P}_{v(t)} = \mathbb{P} \circ [v(t)]^{-1}$ is the probability distribution of $v(t)$ under $\mathbb{P}$ (i.e $\mathbb{P}_{v(t)}(K) = \mathbb{P}(\{x \in \Omega : v(t, x) \in K\})$ for each $K \in \mathcal{B}(\mathbb{H})$, where $\mathcal{B}(\mathbb{H})$ represents the Borel class on $\mathbb{H}$.

Returning to the literature, when $\alpha = 1$ and $b(t) = 0$, the problem (1.1) degrades to a classical stochastic differential equations of Mckean–Vlasov type for which have been investigated by many researchers through different methods [7, 14–16, 21, 22]. For example, let consider an q-particle system $v^{q,1}, \cdots, v^{q,q}$ given by the following weakly interacting stochastic partial differential equations

$$dv^{q,j}(t) = Av^{q,j}(t)dt + g(t, v^{q,j}(t), \mu^q(t))dt + f(t, v^{q,j}(t), \mu^q(t))dW^j(t), \tag{1.2}$$

$j = 1, 2, \cdots, q$, where $\mu^q(t) = \dfrac{1}{q}\sum_{j=1}^{q} \delta_{v^{q,j}(t)}$, $t \in \mathbb{R}_+$ represents the empirical distributions, $(W^j(t))_{j=1,\cdots,q}$ are independent standard cylindrical Brownian motions and A generates a C_0-semigroup. It is proved that under suitable conditions on the Equ.(1.2), it's possible to describe the limit by the following McKean-Vlasov equation

$$dv(t) = Av(t)dt + g(t, v(t), \mathbb{P}_{v(t)}) + f(t, v(t), \mathbb{P}_{v(t)})dW(t) \tag{1.3}$$

Moreover, if $f \equiv 0$ and $g(t, v(t), \mathbb{P}_{v(t)}) \equiv g(t, v(t))$, the existence and uniqueness of almost automorphic, asymptotically periodic, almost periodic, asymptotically ω-periodic solutions, S-asymptotically ω-periodic solutions, asymptotically almost periodic and asymptotically almost automorphic, (ω, c)-periodic and pseudo S-asymptotically (ω, k)-Bloch periodic mild solutions of problem (1.1) have been investigated in deterministic cases by various authors [6, 8, 9, 24, 35, 36, 38]. In this work, the problem (1.1) captures fading memory behaviors, and randomness of the dynamical processes. We examine a more general class of above mentioned problems under the situation that diffusion and drift terms f, g are weighted speudo-S-asymptotically Bloch periodic, and depend on the probability distribution of the process at times. The obtained outcomes are mainly relied upon on the Wassertein topology, resolvent operator theory, Banach and Krasnoselki's fixed point theorem and stochastic analysis. We firstly provide some convolutions and composition results under some suitable conditions and continuity assumptions. Next, we establish existence and uniqueness result (see Theorem 3.10) which needs no compactness condition on the resolvent operator under global Lipschitz conditions on f, g and additional suitable conditions. Finally, we relax the Lipschitz condition of g to some sublinear growth conditions (see Theorem 3.14). Consequently, our research study can be viewed as an extension and continuation of investigation in [5, 6, 8, 9, 24, 35, 36, 38]. Additionally, this work generalize various papers on S-asymptotically ω-antiperiodic (or ω-periodic) mild solutions of to square-mean weighted pseudo S-asymptotically (ω, k)-periodic mild solutions for some stochastic fractional evolution equations.

The remainder of the paper is arranged as follows: Section 2 discusses some basic results regarding weighted pseudo square-mean S-asymptotically Bloch type periodicity processes. Section 3.2 is devoted to the existence and uniqueness of weighted pseudo S-asymptotically Bloch type periodicity mild solutions of Eq.(1.1). To summarize this work, we provide an example that illustrates our results, in Section 4.

2. Background

We suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ represents a probability space, $\mathbb{H}$ is complex separable Hilbert space, and $\mathbb{K}$ indicates a real separable Hilbert space. For convenience, the same notations $\|\cdot\|$ and $(\cdot,\cdot)$ are applied to denote the norms and the inner products in $\mathbb{H}$ and $\mathbb{K}$. We denote by $\mathcal{L}(\mathbb{K}, \mathbb{H})$ the Banach space of all bounded linear operators from $\mathbb{K}$ to $\mathbb{H}$ endowed with the topology defined by the operator norm, and $\mathbb{L}^2(\Omega, \mathbb{H})$ stands for the collection of all strongly-measurable, square-integrable $\mathbb{H}$-valued random variables, which is a complex Hilbert space endowed with the norm

$$\|v\|_{\mathbb{L}^2} = (\mathsf{E}\|v\|^2)^{1/2},\ v \in \mathbb{L}^2(\Omega, \mathbb{H})$$

where $\mathsf{E}(\cdot)$ is the expectation defined by $\mathsf{E}\|v\|^2 = \displaystyle\int_\Omega \|v\|^2 d\mathbb{P}$.

Definition 2.1. *A stochastic process $v : \mathbb{R} \to \mathbb{L}^2(\Omega, \mathbb{H})$ is said to be*

(i) *stochastically bounded if there exists a constant $M > 0$ such that*

$$\mathsf{E}\|v(t)\|^2 = \int_\Omega \|v(t)\|^2 d\mathbb{P} < M \quad \textit{for all} \quad t \in \mathbb{R};$$

(ii) *stochastically continuous if* $\lim_{t\to s} \mathsf{E}\|v(t) - v(s)\|^2 = 0 \quad \textit{for all} \quad s \in \mathbb{R}.$

We denote by $\mathcal{BC}\left(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H})\right)$ the complex Banach space of all bounded and continuous stochastic processes v from $\mathbb{R}$ into $\mathbb{L}^2(\Omega, \mathbb{H})$ with the norm $\|v\|_\infty = \left(\sup_{t\in\mathbb{R}} \mathsf{E}\|v(t)\|^2\right)^{1/2}$. We denote by $\mathbb{P}_{v(t)} = \mathbb{P} \circ [v(t)]^{-1} = \mu(v(t))$ the distribution of all random variable $v(t) : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{H}$.

2.1. Wasserstein distances

Let $(\mathbb{H}, d)$ be a separable complete metric space and $\mathcal{P}(\mathbb{H})$ be the space of Borel probability measures on $\mathbb{H}$. For $\mu_1, \mu_2 \in \mathcal{P}(\mathbb{H})$, we define

$$d_{BL}(\mu_1, \mu_2) = \sup_{\|\psi\|_{BL}\le 1} \left| \int_{\mathbb{H}} \psi \, d(\mu_1 - \mu_2) \right|, \tag{2.1}$$

where ψ are Lipschitz continuous functions on $\mathbb{H}$ with the norm

$$\|\psi\|_L = \sup \left\{ \frac{|\psi(z_1) - \psi(z_2)|}{\|z_1 - z_2\|} \; ; \, z_1, \, z_2 \in \mathbb{H}, \, z_1 \neq z_2 \right\}$$

$$\|\psi\|_{BL} = \max\{\|\psi\|_\infty, \|\psi\|_L\} \; , \; \|\psi\|_\infty := \sup_{k\in\mathbb{Y}} |\psi(k)| < \infty.$$

It is known that d_{BL} is a complete metric on $\mathcal{P}(\mathbb{Y})$ which generates the weak topology [29]. For any $p \geq 1$,we denote by $\mathcal{P}_p(\mathbb{H})$ the subspace of $\mathcal{P}(\mathbb{H})$ consisting of the probability measures of order p. For any $p \geq 1$ and $u, \tilde{u} \in \mathcal{P}_p(\mathbb{H})$, the p-Wasserstein distance $\mathsf{W}_p(u, \tilde{u})$ is defined by :

$$\mathsf{W}_p(u, \tilde{u}) = \inf \left\{ \Big[\int_{\mathbb{H}\times\mathbb{H}} |x - y|^p \mu(dx, dy) \Big]^{1/p} : \mu \in \mathcal{P}_p(\mathbb{H} \times \mathbb{H}) \text{ with marginals } u \text{ and } \tilde{u} \right\}$$

The following lemma is of great importance in our analysis.

Lemma 2.2 (Carmona and Delarue [25], Corollary 5.4)**.** *If $(\mathcal{P}_1(\mathbb{H}), d)$ is a complete separable metric space, and $\mu, \tilde{\mu} \in \mathcal{P}_1(\mathbb{H})$, then*

$$\mathsf{W}_1(\mu, \tilde{\mu}) = \sup_{\psi:|\psi(x)-\psi(y)|\le d(x,y)} \int_{\mathbb{H}} \psi(z)(\mu - \tilde{\mu})(dz)$$

where the supremum is taken over all the $1-$lipschitz functions..

Notice that if v and $\widetilde{v}$ are random variables of order p, then

$$\mathsf{W}_p(\mathbb{P}_v, \mathbb{P}_{\widetilde{v}}) \le (\mathbb{E}\|v - \widetilde{v}\|^p)^{1/p}$$

and the Hölder inequality implies that

$$\mathsf{W}_p(\mu, \widetilde{\mu}) \le \mathsf{W}_q(\mu, \widetilde{\mu}), \; \mu, \widetilde{\mu} \in \mathcal{P}_p(\mathbb{H}), \; 1 \le p \le q.$$

2.2. Weighted square-mean S-asymptotically Bloch type periodic process

In this segment, we recall some definitions and properties of weighted square-mean S-asymptotically Bloch type periodic processes. We refer to [5] for a more detailed analysis. Let Λ denote the set of all functions $\rho : \mathbb{R} \to (0, \infty)$, which are locally integrable over $\mathbb{R}$ such that $\rho > 0$ almost everywhere. For a given $r > 0$ and each $\rho \in \Lambda$, we set

$$m(r, \rho) = \int_{-r}^{r} \rho(s)ds.$$

Throughout the work, we suppose that following condition hold:

$$(\mathbf{H}^{\rho}) \ : \ \text{For all } \ \zeta \in \mathbb{R}, \ \limsup_{|t|\to\infty} \frac{\rho(t+\zeta)}{\rho(t)} < +\infty.$$

Define the $\Lambda_{\infty} = \{\rho \in \Lambda : \lim_{r\to+\infty} m(r, \rho) = \infty\}$.

Definition 2.3 ([5]). *A stochastic process $v \in \mathcal{BC}\left(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H})\right)$ is said to be square-mean weighted pseudo-S-asymptotically (ω, k)-Bloch periodic if for given $\omega \in \mathbb{R}, k \in \mathbb{R}$,*

$$\lim_{r\to+\infty} \frac{1}{m(r, \rho)} \int_{-r}^{r} \mathbb{E}\|v(t+\omega) - e^{ik\omega}v(t)\|^2 \rho(t)dt = 0,$$

for each $t \in \mathbb{R}$. We denote the space of all such processes by $\mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho)$ and and

$$\mathcal{WSABP}_{\omega,k}(\mathbb{R} \times \mathbb{L}^2(\Omega, \mathbb{H}) \times \mathcal{P}_2(\mathbb{H}), \mathbb{L}^2(\Omega, \mathbb{H})) = \Big\{ h(\cdot, v, \mu) \in \mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho) \ \textit{for any} \quad v \in \mathbb{L}^2(\Omega, \mathbb{H}), \ \mu \in \mathcal{P}_2(\mathbb{H}) \Big\}.$$

From definition 2.3, we can formulate, the following concepts. By taking

1. $k\omega = \pi$, we obtain the notion of square-mean weighted pseudo-S-asymptotically ω-antiperiodic stochastic processes, i.e
$$\lim_{r\to+\infty} \frac{1}{m(r, \rho)} \int_{-r}^{r} \mathbb{E}\|v(t+\omega) + v(t)\|^2 \rho(t)dt = 0, \ \text{for each } t \in \mathbb{R};$$
2. $k\omega = 2\pi$, we get the concept of square-mean weighted pseudo-S-asymptotically ω-periodic stochastic processes, i.e
$$\lim_{r\to+\infty} \frac{1}{m(r, \rho)} \int_{-r}^{r} \mathbb{E}\|v(t+\omega) - v(t)\|^2 \rho(t)dt = 0, \ \text{for each } t \in \mathbb{R}.$$

Lemma 2.4 ([5]). *Let $\rho \in \Lambda_{\infty}$, and $X_1, X_2, X \in \mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho)$, then the following results hold:*

(a) $X_1 + X_2 \in \mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho)$*, and* $aX \in \mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho)$ *for each* $a \in \mathbb{C}$.

(b) $X_a \in \mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho)$ *for each* $a \in \mathbb{R}$*, where* $X_a(t) := X(t+a)$ *for each* $t \in \mathbb{R}$.

(c) $\mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho)$ *is a Banach space endowed with the norm* $\|\cdot\|_{\infty}$.

Throughout the paper, we define the set

$$\mathcal{BC}(\mathbb{R} \times \mathbb{L}^2(\Omega, \mathbb{H}) \times \mathcal{P}_2(\mathbb{H}), \mathbb{L}^2(\Omega, \mathbb{H})) = \Big\{ h(\cdot, v, \mu) \in \mathcal{BC}\left(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H})\right) \ \text{for any} \quad v \in \mathbb{L}^2(\Omega, \mathbb{H}), \ \mu \in \mathcal{P}_2(\mathbb{H}) \Big\}.$$

3. Discussion on solutions existence

This section of the paper is mainly concerned with demonstrating that for each weighted pseudo S-asymptotically (ω, k)-periodic input, the output is still a bounded and continuous mild solutions to the fractional stochastic evolution equation (1.1), which is also weighted pseudo S-asymptotically (ω, k)-periodic. To achieve that, we provide some auxiliary outcomes where we establish some useful superposition results.

3.1. Some auxiliary results

Let $h \in \mathcal{BC}(\mathbb{R} \times \mathbb{L}^2(\Omega, \mathbb{H}) \times \mathcal{P}_2(\mathbb{H}), \mathbb{L}^2(\Omega, \mathbb{H}))$ and consider the following assumptions:

(H0) For all $u \in \mathbb{L}^2(\Omega, \mathbb{H})$,

$$\lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} \mathsf{E}\|h(t+\omega, u, \mathbb{P}_u) - e^{ik\omega} h(t, e^{-ik\omega}u, \mathbb{P}_{e^{-ik\omega}u})\|^2 \rho(t)dt = 0$$

uniformly on any bounded set of $\mathbb{L}^2(\Omega, \mathbb{H})$.

(H1) There exists a number $L > 0$ such that for any $u, v \in \mathbb{L}^2(\Omega, \mathbb{H})$ and $\mu_1, \mu_2 \in \mathcal{P}_2(\mathbb{H})$,

$$\mathsf{E}\|h(t, u, \mu_1) - h(t, v, \mu_2)\|^2 \leq L. \bigg(\mathsf{E}\|u - v\|^2 + \mathsf{W}_2^2(\nu_1, \nu_2)\bigg),$$

uniformly for all $t \in \mathbb{R}$.

(H*1) For any $\epsilon > 0$ and any bounded subset $D \subset \mathbb{L}^2(\Omega, \mathbb{H})$, there exist constants $T_{\epsilon,D} > 0$ and $\delta_{\epsilon,D} > 0$ such that

$$\mathsf{E}\|h(t, v_1, \mathbb{P}_{v_1}) - h(t, v_2, \mathbb{P}_{v_2})\|^2 \leq \epsilon$$

for all $v_1, v_2 \in D$ with $\mathsf{E}\|v_1 - v_2\|^2 \leq \delta_{\epsilon,D}$ and $t \geq T_{\epsilon,D}$.

Remark 3.1. *The condition* **(H*1)** *mean that* $h : \mathbb{R} \times \mathbb{L}^2(\Omega, \mathbb{H}) \times \mathcal{P}_2(\mathbb{H})$ *is asymptotically uniformly continuous on bounded sets of* $\mathbb{L}^2(\Omega, \mathbb{H})$.

Remark 3.2. *Particularly, by choosing*

1. $k\omega = \pi$*, condition* **(H0)** *degrades to assumption* **(H*0)** *given by: for all* $v \in \mathbb{L}^2(\Omega, \mathbb{H})$*,*

$$\lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} \mathsf{E}\|h(t+\omega, v, \mathbb{P}_v) + h(t, -v, \mathbb{P}_{-v})\|^2 \rho(t)dt = 0$$

uniformly on any bounded set of $\mathbb{L}^2(\Omega, \mathbb{H})$.

2. $k\omega = 2\pi$*, condition* **(H0)** *degrades to assumption* **(H**0)** *given by: for all* $v \in \mathbb{L}^2(\Omega, \mathbb{H})$*,*

$$\lim_{r\to\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} \mathsf{E}\|h(t+\omega, v, \mathbb{P}_v) - h(t, v, \mathbb{P}_v)\|^2 \rho(t)dt = 0$$

uniformly on any bounded set of $\mathbb{L}^2(\Omega, \mathbb{H})$.

We have the following composition theorem.

Theorem 3.3. *Let* $\rho \in \Lambda_\infty$*. If* $h \in \mathcal{BC}(\mathbb{R} \times \mathbb{L}^2(\Omega, \mathbb{H}) \times \mathcal{P}_2(\mathbb{H}), \mathbb{L}^2(\Omega, \mathbb{H}))$ *satisfies* $(\mathbf{H0}) - (\mathbf{H1})$*, then we have* $h(\cdot, v(\cdot), \mathbb{P}_{v(\cdot)}) \in \mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho)$ *for every* $v \in \mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho)$.

Proof. Since $v \in \mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho)$ then $\sup_{t\in\mathbb{R}} \mathsf{E}\|v(t)\|^2 < \infty$. Therefore

$$\begin{aligned}\sup_{t\in\mathbb{R}} \mathsf{E}\|h(t, v(t), \mathbb{P}_{v(t)})\|^2 &\le \sup_{t\in\mathbb{R}} \mathsf{E}\|h(t, v(t), \mathbb{P}_{v(t)}) - h(t, 0, 0\|^2 + \sup_{t\in\mathbb{R}} \mathsf{E}\|h(t, 0, 0)\|^2 \\ &\le L \sup_{t\in\mathbb{R}} \Big(\mathsf{E}\|v(t)\|^2 + \mathsf{W}_2^2(\mathbb{P}_{v(t)}, 0)\Big) + \sup_{t\in\mathbb{R}} \mathsf{E}\|h(t, 0, 0)\|^2 \\ &\le 2L \sup_{t\in\mathbb{R}} \mathsf{E}\|v(t)\|^2 + \sup_{t\in\mathbb{R}} \mathsf{E}\|h(t, 0, 0)\|^2 < \infty.\end{aligned}$$

Let $t, t_0 \in \mathbb{R}$ and $v \in \mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho)$. Then

$$\begin{aligned}\mathsf{E}\left\|h(t, v(t), \mathbb{P}_{v(t)}) - h(t_0, v(t_0), \mathbb{P}_{v(t_0)})\right\|^2 &\le 3\mathsf{E}\left\|h(t, v(t), \mathbb{P}_{v(t)}) - h(t_0, v(t), \mathbb{P}_{v(t)})\right\|^2 \\ &\quad + 3\mathsf{E}\left\|h(t_0, v(t), \mathbb{P}_{v(t)}) - h(t_0, v(t_0), \mathbb{P}_{v(t)})\right\|^2 \\ &\quad + 3\mathsf{E}\left\|h(t_0, v(t_0), \mathbb{P}_{v(t)}) - h(t_0, v(t_0), \mathbb{P}_{v(t_0)})\right\|^2 \\ &\le 3\mathsf{E}\left\|h(t, v(t), \mathbb{P}_{v(t)}) - h(t_0, v(t), \mathbb{P}_{v(t)})\right\|^2 \\ &\quad + 3L\mathsf{E}\left\|v(t) - v(t_0)\right\|^2 + 3L\mathsf{W}_2^2\big(v(t), v(t_0)\big) \\ &\le 3\mathsf{E}\left\|h(t, v(t), \mathbb{P}_{v(t)}) - h(t_0, v(t), \mathbb{P}_{v(t)})\right\|^2 \\ &\quad + 6L\mathsf{E}\left\|v(t) - v(t_0)\right\|^2.\end{aligned}$$

Since $v \in \mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho)$ and $h \in \mathcal{BC}(\mathbb{R} \times \mathbb{L}^2(\Omega, \mathbb{H}) \times \mathcal{P}_2(\mathbb{H}), \mathbb{L}^2(\Omega, \mathbb{H}))$ then

$$\lim_{t\to t_0} \mathsf{E}\left\|h(t, v(t), \mathbb{P}_{v(t)}) - h(t_0, v(t), \mathbb{P}_{v(t)})\right\|^2 = \lim_{t\to t_0} \mathsf{E}\left\|v(t) - v(t_0)\right\|^2 = 0.$$

It follows that $h(\cdot, v(\cdot), \mathbb{P}_{v(\cdot)}) \in \mathcal{BC}\left(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H})\right)$. Next, we prove that

$$\lim_{r\to+\infty} \frac{1}{m(r,\rho)} \int_{-r}^{r} \mathsf{E}\|h(t+\omega, v(t+\omega), \mathbb{P}_{v(t+\omega)})) - e^{ik\omega} h(t, v(t), \mathbb{P}_{v(t)}))\|^2 \rho(t) dt = 0.$$

We have

$$\begin{aligned}&\frac{1}{m(r,\rho)} \int_{-r}^{r} \mathsf{E}\|h(t+\omega, v(t+\omega), \mathbb{P}_{v(t+\omega)})) - e^{ik\omega} h(t, v(t), \mathbb{P}_{v(t)}))\|^2 \rho(t) dt \\ \le& \frac{1}{m(r,\rho)} \int_{-r}^{r} \mathsf{E}\|h(t+\omega, v(t+\omega), \mathbb{P}_{v(t+\omega)}) - e^{ik\omega} h(t, e^{-ik\omega} v(t+\omega), \mathbb{P}_{e^{-ik\omega} v(t+\omega)})\|^2 \rho(t) dt \\ &+ \frac{2}{m(r,\rho)} \int_{-r}^{r} \mathsf{E}\|e^{ik\omega} h(t, e^{-ik\omega} v(t+\omega), \mathbb{P}_{e^{-ik\omega} v(t+\omega)}) - e^{ik\omega} h(t, v(t), \mathbb{P}_{v(t)}))\|^2 \rho(t) dt \\ \le& \frac{2}{m(r,\rho)} \int_{-r}^{r} \mathsf{E}\|h(t+\omega, v(t+\omega), \mathbb{P}_{v(t+\omega)}) - e^{ik\omega} h(t, e^{-ik\omega} v(t+\omega), \mathbb{P}_{e^{-ik\omega} v(t+\omega)})\|^2 \rho(t) dt \\ &+ \frac{2L}{m(r,\rho)} \int_{-r}^{r} \Big(\mathsf{E}\|e^{-ik\omega} v(t+\omega)) - v(t)\|^2 + \mathsf{W}_2^2\big(\mathbb{P}_{e^{-ik\omega} v(t+\omega)}, \mathbb{P}_{v(t)}\big)\Big) \rho(t) dt \\ \le& \frac{2}{m(r,\rho)} \int_{-r}^{r} \mathsf{E}\|h(t+\omega, v(t+\omega), \mathbb{P}_{v(t+\omega)}) - e^{ik\omega} h(t, e^{-ik\omega} v(t+\omega), \mathbb{P}_{e^{ik\omega} v(t+\omega)})\|^2 \rho(t) dt \\ &+ \frac{4L}{m(r,\rho)} \int_{-r}^{r} \mathsf{E}\|e^{-ik\omega} v(t+\omega) - v(t)\|^2 \rho(t) dt \\ &\to 0 \text{ as } r \to +\infty \text{ by } (\mathbf{H0}) \text{ and the fact that } v \in \mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho).\end{aligned}$$

Then it follows that

$$\lim_{r\to+\infty}\frac{1}{m(r,\rho)}\int_{-r}^{r}\mathsf{E}\|h(t+\omega,v(t+\omega),\mathbb{P}_{v(t+\omega)}))-e^{ik\omega}h(t,v(t),\mathbb{P}_{v(t)}))\|^2\rho(t)dt=0.$$

Which means that $h(\cdot,v(\cdot),\mathbb{P}_{v(\cdot)})\in\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)$. ■

From Theorem 3.3, we derive the following corollaries.

Corollary 3.4. *Let $\rho\in\Lambda_\infty$ and v be a square-mean weighted pseudo-S-asymptotically ω-antiperiodic stochastic process. If $h\in\mathcal{BC}(\mathbb{R}\times\mathbb{L}^2(\Omega,\mathbb{H})\times\mathcal{P}_2(\mathbb{H}),\mathbb{L}^2(\Omega,\mathbb{H}))$ satisfies* **(H*0)-(H1)***, then $t\longmapsto h(t,v(t),\mathbb{P}_{v(t)})$ is square-mean weighted pseudo-S-asymptotically ω-antiperiodic.*

Corollary 3.5. *Let $\rho\in\Lambda_\infty$ and v be a square-mean weighted pseudo-S-asymptotically ω-periodic stochastic process. If $h\in\mathcal{BC}(\mathbb{R}\times\mathbb{L}^2(\Omega,\mathbb{H})\times\mathcal{P}_2(\mathbb{H}),\mathbb{L}^2(\Omega,\mathbb{H}))$ satisfies* **(H**0)-(H1)***, then $t\longmapsto h(t,v(t),\mathbb{P}_{v(t)})$ is square-mean weighted pseudo-S-asymptotically ω-periodic.*

Theorem 3.6. *Let $\rho\in\Lambda_\infty$. If $h\in\mathcal{BC}(\mathbb{R}\times\mathbb{L}^2(\Omega,\mathbb{H})\times\mathcal{P}_2(\mathbb{H}),\mathbb{L}^2(\Omega,\mathbb{H}))$ satisfies* **(H0)-(H*1)***, then we have $t\longrightarrow h(t,v(t),\mathbb{P}_{v(t)})\in\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)$ for every $v\in\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)$.*

Proof. The demonstration can be done similarly to Theorem 3.3 and Theorem 2.6 in [5] with minor modifications. ■

Now, we present some convolutions results.

Lemma 3.7. *If $\{\mathcal{K}(t)\}_{t\geq0}\subset\mathcal{B}(\mathbb{H})$ is uniformly $1-$integrable and strongly continuous family of operators, and $X\in\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)$, then*

$$t\longmapsto\Phi(t)=\int_{-\infty}^{t}\mathcal{K}(t-s)X(s)ds\in\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho).$$

Proof. Let $X\in\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)$. Since the operator family $\{\mathcal{K}(t)\}_{t\geq0}$ is uniformly $1-$integrable then there exist $M>0$ such that

$$\int_0^\infty\|\mathcal{K}(t)\|dt\leq M.$$

Then it is easy to show that $\Phi\in\mathcal{BC}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)$ using the Lebesgue dominated convergence theorem and the fact that $X\in\mathcal{BC}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)$. For any $r>0$ we have

$$\begin{aligned}
&\frac{1}{m(r,\rho)}\int_{-r}^{r}\mathsf{E}\|\Phi(t+\omega)-e^{ik\omega}\Phi(t)\|^2\rho(t)dt\\
&=\frac{1}{m(r,\rho)}\int_{-r}^{r}\mathsf{E}\left\|\int_{-\infty}^{t+\omega}\mathcal{K}(t+\omega-s)X(s)ds-e^{ik\omega}\int_{-\infty}^{t}\mathcal{K}(t-s)X(s)ds\right\|^2\rho(t)dt\\
&=\frac{1}{m(r,\rho)}\int_{-r}^{r}\mathsf{E}\left\|\int_{-\infty}^{t}\mathcal{K}(t-s)X(s+\omega)ds-e^{ik\omega}\int_{-\infty}^{t}\mathcal{K}(t-s)X(s)ds\right\|^2d\rho(t)dt\\
&=\frac{1}{m(r,\rho)}\int_{-r}^{r}\mathsf{E}\left\|\int_{-\infty}^{t}\mathcal{K}(t-s)[X(s+\omega)-e^{ik\omega}X(s)]ds\right\|^2\rho(t)dt\\
&\leq\frac{1}{m(r,\rho)}\int_{-r}^{r}\left[\int_{-\infty}^{t}\|\mathcal{K}(t-s)\|\,ds\int_{-\infty}^{t}\|\mathcal{K}(t-s)\|\;\mathsf{E}\|X(s+\omega)-e^{ik\omega}X(s)\|^2ds\right]\rho(t)dt\\
&\leq M\frac{1}{m(r,\rho)}\int_{-r}^{r}\left[\int_{-\infty}^{t}\|\mathcal{K}(t-s)\|\;\mathsf{E}\|X(s+\omega)-e^{ik\omega}X(s)\|^2ds\right]\rho(t)dt\\
&\leq M\frac{1}{m(r,\rho)}\int_{-r}^{r}\left[\int_{0}^{\infty}\|\mathcal{K}(s)\|\;\mathsf{E}\|X(t-s+\omega)-e^{ik\omega}X(t-s)\|^2ds\right]\rho(t)dt.
\end{aligned}$$

From the Fubini theorem, it follows that

$$\begin{aligned}
&M\frac{1}{m(r,\rho)}\int_{-r}^{r}\left[\int_0^\infty \|\mathcal{K}(s)\|\ \mathsf{E}\|X(t-s+\omega)-e^{ik\omega}X(t-s)\|^2 ds\right]\rho(t)dt\\
&=M\int_0^\infty \|\mathcal{K}(s)\|\left[\frac{1}{m(r,\rho)}\int_{-r}^{r}\mathsf{E}\|X(t-s+\omega)-e^{ik\omega}X(t-s)\|^2\rho(t)dt\right]ds\\
&=M\int_0^\infty \|\mathcal{K}(s)\|\left[\frac{1}{m(r,\rho)}\int_{-r}^{r}\mathsf{E}\|X_{-s}(t+\omega)-e^{ik\omega}X_{-s}(t)\|^2\rho(t)dt\right]ds.
\end{aligned}$$

Since $X\in\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)$, thanks to Lemma 2.4, we know that for any $s\in\mathbb{R}$, we have

$$\lim_{r\to+\infty}\frac{1}{m(r,\rho)}\int_{-r}^{r}\mathsf{E}\|X_{-s}(t+\omega)-e^{ik\omega}X_{-s}(t)\|^2\rho(t)dt=0.$$

Then Lebesgue dominated convergence theorem yield that

$$\int_0^\infty \|\mathcal{K}(s)\|\left[\frac{1}{m(r,\rho)}\int_{-r}^{r}\mathsf{E}\|X_{-s}(t+\omega)-e^{ik\omega}X_{-s}(t)\|^2\rho(t)dt\right]ds\to 0 \text{ as } r\to+\infty.$$

Therefore

$$\lim_{r\to+\infty}\frac{1}{m(r,\rho)}\int_{-r}^{r}\mathsf{E}\|\Phi(t+\omega)-e^{ik\omega}\Phi(t)\|^2\rho(t)dt=0.$$

Which proves that $\Phi\in\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)$. ■

Lemma 3.8. *If $\{\mathcal{K}(t)\}_{t\geq 0}\subset\mathcal{B}(\mathbb{H})$ is uniformly $2-$integrable and strongly continuous family of operators, and $X\in\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)$, then*

$$t\longmapsto\widetilde{\Phi}(t)=\int_{-\infty}^{t}\mathcal{K}(t-s)X(s)d\mathbb{W}(s)\in\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho).$$

Proof. Since the operator family $\{\mathcal{K}(t)\}_{t\geq 0}$ is uniformly $2-$integrable then by the Ito's isometry property of stochastic integral, the Lebesgue dominated convergence theorem and the fact that $X\in\mathcal{BC}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)$, it is easy to show that $\widetilde{\Phi}\in\mathcal{BC}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)$ using the Lebesgue dominated convergence theorem and the fact that $X\in\mathcal{BC}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)$. We have that

$$\begin{aligned}
&\frac{1}{m(r,\rho)}\int_{-r}^{r}\mathsf{E}\|\widetilde{\Phi}(t+\omega)-e^{ik\omega}\widetilde{\Phi}(t)\|^2\rho(t)dt\\
&=\frac{1}{m(r,\rho)}\int_{-r}^{r}\mathsf{E}\left\|\int_{-\infty}^{t+\omega}\mathcal{K}(t+\omega-s)X(s)d\mathbb{W}(s)-e^{ik\omega}\int_{-\infty}^{t}\mathcal{K}(t-s)X(s)d\mathbb{W}(s)\right\|^2\rho(t)dt\\
&=\frac{1}{m(r,\rho)}\int_{-r}^{r}\mathsf{E}\left\|\int_{-\infty}^{t}\mathcal{K}(t-s)X(s+\omega)d\mathbb{W}(s+\omega)-e^{ik\omega}\int_{-\infty}^{t}\mathcal{K}(t-s)X(s)d\mathbb{W}(s)\right\|^2\rho(t)dt.
\end{aligned}$$

Let $\widetilde{W}(s)=\mathbb{W}(s+\omega)-\mathbb{W}(\omega)$. We know that $\widetilde{W}$ is a Brownian motion and has the same distribution as $\mathbb{W}$.

Using the Ito's isometry property of stochastic integral and the Fubini's theorem, we obtain that

$$\begin{aligned}
&\frac{1}{m(r,\rho)}\int_{-r}^{r}\mathsf{E}\|\widetilde{\Phi}(t+\omega)-e^{ik\omega}\widetilde{\Phi}(t)\|^2 d\mu(t)\\
&=\frac{1}{m(r,\rho)}\int_{-r}^{r}\mathsf{E}\left\|\int_{-\infty}^{t}\mathcal{K}(t-s)X(s+\omega)d\widetilde{W}(s)-e^{ik\omega}\int_{-\infty}^{t}\mathcal{K}(t-s)X(s)d\widetilde{W}(s)\right\|^2\rho(t)dt\\
&=\frac{1}{m(r,\rho)}\int_{-r}^{r}\mathsf{E}\left\|\int_{-\infty}^{t}\mathcal{K}(t-s)\left[X(s+\omega)-e^{ik\omega}X(s)\right]d\widetilde{W}(s)\right\|^2\rho(t)dt\\
&=\frac{1}{m(r,\rho)}\int_{-r}^{r}\int_{-\infty}^{t}\mathsf{E}\left\|\mathcal{K}(t-s)\left[X(s+\omega)-e^{ik\omega}X(s)\right]\right\|^2 ds\rho(t)dt\\
&\leq\frac{1}{m(r,\rho)}\int_{-r}^{r}\left[\int_{-\infty}^{t}\|\mathcal{K}(t-s)\|^2\,\mathsf{E}\|X(s+\omega)-e^{ik\omega}X(s)\|^2ds\right]\rho(t)dt\\
&\leq\frac{1}{m(r,\rho)}\int_{-r}^{r}\left[\int_{0}^{\infty}\|\mathcal{K}(s)\|^2\,\mathsf{E}\|X(t-s+\omega)-e^{ik\omega}X(t-s)\|^2ds\right]\rho(t)dt\\
&\leq\int_{0}^{\infty}\|\mathcal{K}(s)\|^2\left[\frac{1}{m(r,\rho)}\int_{-r}^{r}\mathsf{E}\|X(t-s+\omega)-e^{ik\omega}X(t-s)\|^2\rho(t)dt\right]ds.
\end{aligned}$$

Since $X\in\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)$ and $\{\mathcal{K}(t)\}_{t\geq 0}$ is uniformly $2-$integrable, therefore invoking Lemma 2.4 and Lebesgue dominated converge theorem, we get that

$$\lim_{r\to+\infty}\frac{1}{m(r,\rho)}\int_{-r}^{r}\mathsf{E}\|\widetilde{\Phi}(t+\omega)-e^{ik\omega}\widetilde{\Phi}(t)\|^2\rho(t)dt=0.$$

which proves that $\widetilde{\Phi}\in\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)$. ■

It is worthwhile to point out that if $k\omega=\pi$ (resp. $k\omega=2\pi$), from Lemmas 3.7 and 3.8, we can get some convolutions results for weighted pseudo S-asymptotically ω-anti-periodic (resp. ω-periodic) stochastic processes.

3.2. Existence of mild solution in $\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)$

We will begin by recollecting some facts about the Weyl fractional integrals and derivatives of order $\alpha>0$, as well as the α-resolvent operators that will be employed to develop the main results. For more details on properties α-resolvent operators, one can make reference to [24]. Suppose that $\mathbb{X}$ is a Banach space. For given function $h:\mathbb{R}\to\mathbb{X}$, the Weyl fractional integral of order $\alpha>0$ is defined by

$$\partial_t^{-\alpha}h(t):=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^{t}(t-s)^{\alpha-1}h(s)ds,\qquad t\in\mathbb{R},$$

when this integral is convergent. The Weyl fractional derivative ∂_t^{α} of order α is defined by

$$\partial_t^{\alpha}h(t)=\frac{d^n}{dt^n}\partial_t^{-(n-\alpha)}h(t),\quad t\in\mathbb{R},$$

where $n=[\alpha]+1$, and the notation $[\alpha]$ represents the integer part of α. Now, Let A be a closed and linear operator with domain $\mathcal{D}(A)$ defined on a Banach space X, and $\alpha>0$. For a given kernel $b(\cdot)\in L^1_{loc}(\mathbb{R}_+)$, it is said that A is the generator of an α-resolvent family if there exists $\xi>0$ and a strongly continuous family $\mathcal{R}_\alpha:\mathbb{R}_+\to\mathcal{B}(\mathbb{X})$ such that

$$\left\{\frac{\lambda^\alpha}{1+\hat{b}(\lambda)}:\mathrm{Re}(\lambda)>\xi\right\}\subseteq\rho(A)$$

and for all $y \in \mathbb{X}$,

$$(\lambda^\alpha - (1+\hat{b}(\lambda))A)^{-1}y = \frac{1}{1+\hat{b}(\lambda)}\left(\frac{\lambda^\alpha}{1+\hat{b}(\lambda)} - A\right)^{-1} y = \int_0^\infty e^{-\lambda t}\mathcal{R}_\alpha(t)y dt, \quad \mathrm{Re}\lambda > \xi.$$

$\{\mathcal{R}_\alpha(t)\}_{t\geq 0}$ is called the α-resolvent family generated by the operator A. Motivated by Ponce [24], we present the concept of mild solutions for Eq.(1.1). For each $t \in \mathbb{R}$, $\mathbb{W}(t)$ is a two-sided and standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$ with $\mathcal{F}_t = \sigma\{\mathbb{W}(r) - \mathbb{W}(s) \,|\, r, s \leq t\}$.

Definition 3.9. *An $\mathcal{F}_t$-progressively measurable process $\{v(t)\}_{t\in\mathbb{R}}$ is called a mild solution of problem* (1.1) *if it satisfies the following stochastic integral equation*

$$v(t) = \int_{-\infty}^t \mathcal{R}_\alpha(t-s)g(s, v(s), \mathbb{P}_{v(s)})ds + \int_{-\infty}^t \mathcal{R}_\alpha(t-s)f(s, v(s), \mathbb{P}_{v(s)})d\mathbb{W}(s)$$

for all $t \in \mathbb{R}$, where $\{\mathcal{R}_\alpha(t)\}_{t\geq 0}$ the resolvent family generated by the operator A.

We establish the existence and uniqueness of weighted square-mean S-asymptotically Bloch type periodic mild solution for Eq.(1.1) under global Lipschitz-type conditions on the second variable of functions.

Theorem 3.10. *Suppose that the operator A generates an α-resolvent operator $\{\mathcal{R}_\alpha(t)\}_{t\geq 0} \subset \mathcal{B}(\mathbb{H})$ such that for $t \geq 0$, $\|\mathcal{R}_\alpha(t)\| \leq \phi_\alpha(t)$ where $\phi_\alpha \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$. Further, assume that $g,\ f \in \mathcal{BC}(\mathbb{R} \times \mathbb{L}^2(\Omega, \mathbb{H}) \times \mathcal{P}_2(\mathbb{H}), \mathbb{L}^2(\Omega, \mathbb{H}))$ satisfy* **(H0)** *and there exists constants $L, L' > 0$ such that for any $v_1, v_2 \in \mathbb{L}^2(\Omega, \mathbb{H})$ and $\mu_1, \mu_2 \in \mathcal{P}_2(\mathbb{H})$,*

$$\mathsf{E}\|g(t, v_1, \mu_1) - g(t, v_2, \mu_2)\|^2 \leq L\left(\mathsf{E}\|v_1 - v_2\|^2 + \mathsf{W}_2^2(\nu_1, \nu_2)\right),$$

$$\mathsf{E}\|f(t, v_1, \mu_1) - f(t, v_2, \mu_2)\|^2 \leq L'\left(\mathsf{E}\|v_1 - v_2\|^2 + \mathsf{W}_2^2(\nu_1, \nu_2)\right),$$

uniformly for all $t \in \mathbb{R}$.
Then Eq.(1.1) *has a unique mild solution $v \in \mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho)$, provided*

$$\|\phi_\alpha\|_{L^1}^2 L + L'\|\phi_\alpha\|_{L^2}^2 < \frac{1}{4}. \tag{3.1}$$

Proof. From Theorem 3.3, for each $v \in \mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho)$, the stochastic processes $s \mapsto f(s, v(s), \mathbb{P}_{v(s)})$, $s \mapsto g(s, v(s), \mathbb{P}_{v(s)})$ belongs to $\mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho)$. From Lemmas 3.7, 3.8 and 2.4-(a), we can define the operator

$$\mathcal{S} : \mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho) \to \mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho)$$

$$\text{by } (\mathcal{S}v)(t) = \int_{-\infty}^t \mathcal{R}_\alpha(t-s)g(s, v(s), \mathbb{P}_{v(s)})ds + \int_{-\infty}^t \mathcal{R}_\alpha(t-s)f(s, v(s), \mathbb{P}_{v(s)})d\mathbb{W}(s)\,.$$

Let $v_1, v_2 \in \mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho)$ and $t \in \mathbb{R}$. Then using Cauchy-Schwartz's inequality and Ito's isometry property of stochastic integral, we have

$$\begin{aligned}
&\mathsf{E}\|(\mathcal{S}v_1)(t)-(\mathcal{S}v_2)(t)\|^2\\
&\le 2\mathsf{E}\|\int_{-\infty}^{t}\mathcal{R}_\alpha(t-s)[g(s,v_1(s),\mathbb{P}_{v_1(s)})-g(s,v_2(s),\mathbb{P}_{v_2(s)})]ds\|^2\\
&\quad+2\mathsf{E}\|\int_{-\infty}^{t}\mathcal{R}_\alpha(t-s)[f(s,v_1(s),\mathbb{P}_{v_1(s)})-f(s,v_2(s),\mathbb{P}_{v_2(s)})]d\mathbb{W}(s)\|^2\\
&\le 2\int_{-\infty}^{t}\phi_\alpha(t-s)ds\left(\int_{-\infty}^{t}\phi_\alpha(t-s)\mathsf{E}\|[g(s,v_1(s),\mathbb{P}_{v_1(s)})-g(s,v_2(s),\mathbb{P}_{v_2(s)})]\|^2\,ds\right)\\
&\quad+2\left(\int_{-\infty}^{t}\phi_\alpha^2(t-s)\mathsf{E}\|[f(s,v_1(s),\mathbb{P}_{v_1(s)})-f(s,v_2(s),\mathbb{P}_{v_2(s)})]\|^2\,ds\right)\\
&\le 2L\|\phi_\alpha\|_{L^1}\left(\int_{-\infty}^{t}\phi_\alpha(t-s)\Big[\mathsf{E}\|v_1(s)-v_2(s)\|^2+\mathsf{W}_2^2\big(v_1(s),v_2(s)\big)\Big]ds\right)\\
&\quad+2L'\left(\int_{-\infty}^{t}\phi_\alpha^2(t-s)\Big[\mathsf{E}\|v_1(s)-v_2(s)\|^2+\mathsf{W}_2^2\big(v_1(s),v_2(s)\big)\Big]ds\right)\\
&\le 4L\,\|\phi_\alpha\|_{L^1}^2\sup_{s\in\mathbb{R}}\mathsf{E}\|v_1(s)-v_2(s)\|^2+4L'\|\phi_\alpha\|_{L^2}^2\sup_{s\in\mathbb{R}}\mathsf{E}\|v_1(s)-v_2(s)\|^2\\
&\le 4\bigg(\|\phi_\alpha\|_{L^1}^2L+L'\|\phi_\alpha\|_{L^2}^2\bigg)\|v_1-v_2\|_\infty^2.
\end{aligned}$$

Therefore we have

$$\|\mathcal{S}v_1-\mathcal{S}v_2\|_\infty\le 2\sqrt{\|\phi_\alpha\|_{L^1}^2L+L'\|\phi_\alpha\|_{L^2}^2}\,\|v_1-v_2\|_\infty\,.$$

The conclusion follows from the Banach fixed point theorem.

Remark 3.11. *By taking $k\omega=\pi$, we can derive some existence results for square-mean weighted pseudo S-asymptotically ω-antiperiodic mild solutions to problem* (1.1) *from Theorems 3.10. Moreover, choosing $k\omega=2\pi$, we can derive some existence results for square-mean weighted pseudo S-asymptotically ω-periodic mild solutions to problem* (1.1) *from Theorems 3.10.*

■

In the rest of this section, we prove the existence of the weighted pseudo S-asymptotic Bloch type periodic mild solution for Eq.(1.1) under sublinear growth conditions on g and global Lipschitz assumption on f. First, suppose that $\Psi:\mathbb{R}\to\mathbb{R}$ is a continuous functions which satisfies $\Psi(t)\ge 1$ for all $t\in\mathbb{R}$ and $\lim_{|t|\to\infty}\Psi(t)=\infty$. We define the space

$$\mathcal{C}_\Psi(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})):=\left\{v\in\mathcal{BC}\left(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})\right)\ :\ \lim_{|t|\to\infty}\frac{\mathsf{E}\|v(t)\|^2}{\Psi(t)}=0\right\}.$$

$\mathcal{C}_\Psi(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}))$ is a Banach space equipped with the norm $\|v\|_\Psi=\left(\sup_{t\in\mathbb{R}}\frac{\mathsf{E}\|v(t)\|^2}{\Psi(t)}\right)^{1/2}$.

Lemma 3.12 ([12])**.** *A set $\mathcal{U}\subset\mathcal{C}_\Psi(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}))$ is relatively compact if the following conditions hold*

(a) $\lim_{|t|\to\infty}\frac{\mathsf{E}\|u(t)\|^2}{\Psi(t)}=0$ *uniformly for any $u\in\mathcal{U}$;*

(b) *$\mathcal{U}$ is equicontinuous.*

(c) *The set $\mathcal{U}(t)=\{u(t)\ :\ u\in\mathcal{U}\}$ is relatively compact in $\mathbb{L}^2(\Omega,\mathbb{H})$ for each $t\in\mathbb{R}$.*

In order to accomplish that, we will need the following conditions:

(H2) The functions $g,\ f \in \mathcal{BC}(\mathbb{R}\times\mathbb{L}^2(\Omega,\mathbb{H})\times\mathcal{P}_2(\mathbb{H}),\mathbb{L}^2(\Omega,\mathbb{H}))$ satisfy

1. $f(t,z,\mu)$ and $g(t,z,\mu)$ are uniformly continuous in any bounded subset $D\subseteq\mathbb{L}^2(\Omega,\mathbb{H})$ uniformly for $t\in\mathbb{R}$ and $\mu\in\mathcal{P}_2(\mathbb{H})$
2. For all $\mu\in\mathcal{P}_2(\mathbb{H})$, there is a continuous nondecreasing function $\mathcal{X}_g:[0,+\infty)\to[0,+\infty)$ and positive constant $M_g := M_g(\mu)$ such that
$$\mathsf{E}\|g(t,z,\mu)\|^2 \le M_g\,\mathcal{X}_g(\mathsf{E}\|z\|^2) \text{ for all } t\in\mathbb{R},\ z\in\mathbb{L}^2(\Omega,\mathbb{H}).$$
3. For each $\epsilon>0$ there exist $\delta>0$ such that for every $y,z\in C_\Psi(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}))$, $\|y-z\|_\Psi\le\delta$ implies that
$$\int_{-\infty}^t \phi_\alpha(t-s)\mathbb{E}\|g(t,y,\mathbb{P}_y)-g(t,z,\mathbb{P}_z)\|^2ds \le \frac{\epsilon}{4(\|\phi_\alpha\|_{L^1}+1)} \text{ for all } t\in\mathbb{R},$$
$$\int_{-\infty}^t \phi_\alpha^2(t-s)\mathbb{E}\|f(t,y,\mathbb{P}_y)-f(t,z,\mathbb{P}_z)\|^2ds \le \frac{\epsilon}{4} \text{ for all } t\in\mathbb{R} \text{ and}$$
$$J := \sup_{t\in\mathbb{R}}\Big(\int_{-\infty}^t \phi_\alpha^2(t-s)\Psi(s)ds\Big)<\infty.$$

Our strategy is based on the following Krasnoselskii's fixed point theorem.

Lemma 3.13 ([1])**.** *Suppose* $\mathbf{B}$ *is a closed convex and nonempty subset of a Banach space* $\mathbb{Y}$ *and* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *are two operators verifying*

1. *If* $y,z\in\mathbf{B}$*; then* $\mathcal{S}_1y+\mathcal{S}_2z\in\mathbf{B}$*;*
2. $\mathcal{S}_1$ *is compact and continuous;*
3. $\mathcal{S}_2$ *is a mapping contraction.*

Then, there exists $y\in\mathbf{B}$ *such that* $y=\mathcal{S}_1y+\mathcal{S}_2y$*.*

We have the following existence result.

Theorem 3.14. *Suppose that the operator* A *generates a compact* α*-resolvent operator* $\{\mathcal{R}_\alpha(t)\}_{t\ge0}\subset\mathcal{B}(\mathbb{H})$ *for* $t>0$ *such that* $\|\mathcal{R}_\alpha(t)\|\le\phi_\alpha(t)$ *where* $\phi_\alpha\in L^1(\mathbb{R}_+)\cap L^2(\mathbb{R}_+)$ *for* $t\ge0$*. Assume that* $g,\ f\in\mathcal{BC}(\mathbb{R}\times\mathbb{L}^2(\Omega,\mathbb{H})\times\mathcal{P}_2(\mathbb{H}),\mathbb{L}^2(\Omega,\mathbb{H}))$ *verify assumptions* **(H0)** *and* **(H2)***. Moreover, suppose that* g *satisfies condition* **(H*1)** *and there exists constants* $L'>0$ *such that for any* $v_1,v_2\in\mathbb{L}^2(\Omega,\mathbb{H})$ *and* $\mu_1,\mu_2\in\mathcal{P}_2(\mathbb{H})$*,*
$$\mathsf{E}\|f(t,v_1,\mu_1)-f(t,v_2,\mu_2)\|^2\le L'\Bigg(\mathsf{E}\|v_1-v_2\|^2+\mathsf{W}_2^2(\nu_1,\nu_2)\Bigg).$$

Then the problem (1.1) *has at least one mild solution in* $\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)$ *provided that*
$$2\,L'J<1.$$

Proof. Define the operator $\mathcal{S}:C_\Psi(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}))\to C_\Psi(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}))$ by
$$\begin{aligned}(\mathcal{S}v)(t)=&\int_{-\infty}^t\mathcal{R}_\alpha(t-s)g(s,v(s),\mathbb{P}_{v(s)})ds+\int_{-\infty}^t\mathcal{R}_\alpha(t-s)f(s,v(s),\mathbb{P}_{v(s)})d\mathbb{W}(s)\\ =&(\mathcal{S}_1v)(t)+(\mathcal{S}_2v)(t).\end{aligned}$$

where

$$(\mathcal{S}_1 v)(t) = \int_{-\infty}^{t} \mathcal{R}_\alpha(t-s) g(s, v(s), \mathbb{P}_{v(s)}) ds \text{ and}$$
$$(\mathcal{S}_2 v)(t) = \int_{-\infty}^{t} \mathcal{R}_\alpha(t-s) f(s, v(s), \mathbb{P}_{v(s)}) d\mathbb{W}(s).$$

In order to show that $\mathcal{S}$ has at least one fixed point in $\mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho)$ through the Krasnoselskii's fixed point theorem, we will divide the proof in several steps.

Step 1. We claim that $\mathcal{S} : C_\Psi(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H})) \to C_\Psi(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}))$.

Let $v \in C_\Psi(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}))$. First, from Lemmas 3.7 and 3.8, $\mathcal{S}(\Phi) \in \mathcal{C}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}))$.

Next, we have that

$$\begin{aligned}
&\mathsf{E}\|(\mathcal{S}v)(t)\|^2 \\
\leq& 2\mathsf{E}\|\int_{-\infty}^{t} \mathcal{R}_\alpha(t-s) g(s, v(s), \mathbb{P}_{v(s)}) ds\|^2 + 2\mathsf{E}\|\int_{-\infty}^{t} \mathcal{R}_\alpha(t-s) f(s, v(s), \mathbb{P}_{v(s)}) d\mathbb{W}(s)\|^2 \\
\leq& 2\int_{-\infty}^{t} \phi_\alpha(t-s) ds \left(\int_{-\infty}^{t} \phi_\alpha(t-s) \mathsf{E}\|g(s, v(s), \mathbb{P}_{v(s)})\|^2\, ds\right) \\
&+ 2\left(\int_{-\infty}^{t} \phi_\alpha^2(t-s) \mathsf{E}\|f(s, v(s), \mathbb{P}_{v(s)}) - f(s, 0, \mathbb{P}_{v(s)}) + f(s, 0, \mathbb{P}_{v(s)})\|^2\, ds\right) \\
\leq& 2\|\phi_\alpha\|_{L^1} \left(\int_{-\infty}^{t} \phi_\alpha(t-s) \left[M_g \mathcal{X}_g(\|v\|_\infty^2)\right]\, ds\right) \\
&+ 4\left(\int_{-\infty}^{t} \phi_\alpha^2(t-s)) \left[L' \mathsf{E}\|v(s)\|^2 + \mathsf{E}\|f(s, 0, \mathbb{P}_{v(s)})\|^2\right]\, ds\right) \\
\leq& 2M_g \mathcal{X}_g(\|v\|_\infty^2)\|\phi_\alpha\|_{L^1}^2 + 4\|\phi_\alpha\|_{L^2}^2 \Big(L'\|v\|_\infty^2 + M_f\Big),
\end{aligned}$$

where $M_f \equiv M_f(\mu) = \sup_{t\in\mathbb{R}} \mathsf{E}\|f(t, 0, \mu)\|^2$ for all $\mu \in \mathcal{P}_2(\mathbb{H})$.

Since $v \in C_\Psi(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}))$ and by invoking condition **(H2)-(3)**, we derive that

$$\lim_{|t|\to\infty} \frac{\mathsf{E}\|(\mathcal{S}v)(t)\|^2}{\Psi(t)} \leq \lim_{|t|\to\infty} \frac{2M_g \mathcal{X}_g(\|v\|_\infty^2)\|\phi_\alpha\|_{L^1}^2 + 4\|\phi_\alpha\|_{L^2}^2 \Big(L'\|v\|_\infty^2 + M_f\Big)}{\Psi(t)} = 0.$$

This prove that $\mathcal{S} : C_\Psi(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H})) \to C_\Psi(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}))$.

Step 2. We claim that $\mathcal{S}$ is continuous on $C_\Psi(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}))$

Let $\epsilon > 0$. From condition **(H1)-(3)**, there exist a real positive constant $\delta > 0$ such that for each $y, z \in C_\Psi(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}))$ with $\|y - z\|_\Psi \leq \delta$, we have

$$\int_{-\infty}^{t} \phi_\alpha(t-s) \mathbb{E}\|g(t, y, \mathbb{P}_y) - g(t, z, \mathbb{P}_z)\|^2 ds \leq \frac{\epsilon}{4(\|\phi_\alpha\|_{L^1} + 1)} \text{ for all } t \in \mathbb{R},$$
$$\int_{-\infty}^{t} \phi_\alpha^2(t-s) \mathbb{E}\|f(t, y, \mathbb{P}_y) - f(t, z, \mathbb{P}_z)\|^2 ds \leq \frac{\epsilon}{4} \text{ for all } t \in \mathbb{R}.$$

Now, we obtain that

$$\begin{aligned}
\mathsf{E}\|(\mathcal{S}y)(t)-&(\mathcal{S}z)(t)\|^2\\
\leq & 2\mathsf{E}\|\int_{-\infty}^{t}\mathcal{R}_\alpha(t-s)[g(s,y(s),\mathbb{P}_{y(s)})-g(s,z(s),\mathbb{P}_{z(s)})]ds\|^2\\
&+2\mathsf{E}\|\int_{-\infty}^{t}\mathcal{R}_\alpha(t-s)[f(s,y(s),\mathbb{P}_{y(s)})-f(s,z(s),\mathbb{P}_{z(s)})]d\mathbb{W}(s)\|^2\\
\leq & 2\int_{-\infty}^{t}\phi_\alpha(t-s)ds\left(\int_{-\infty}^{t}\phi_\alpha(t-s)\mathsf{E}\|[g(s,y(s),\mathbb{P}_{y(s)})-g(s,z(s),\mathbb{P}_{z(s)})]\|^2\,ds\right)\\
&+2\left(\int_{-\infty}^{t}\phi_\alpha^2(t-s)\mathsf{E}\|[f(s,y(s),\mathbb{P}_{y(s)})-f(s,z(s),\mathbb{P}_{z(s)})]\|^2\,ds\right)\\
\leq & 2\|\phi_\alpha\|_{L^1}\left(\int_{-\infty}^{t}\phi_\alpha(t-s)\mathsf{E}\|[g(s,y(s),\mathbb{P}_{y(s)})-g(s,z(s),\mathbb{P}_{z(s)})]\|^2\,ds\right)\\
&+2\left(\int_{-\infty}^{t}\phi_\alpha^2(t-s)\mathsf{E}\|[f(s,y(s),\mathbb{P}_{y(s)})-f(s,z(s),\mathbb{P}_{z(s)})]\|^2\,ds\right)\\
\leq & (2\|\phi_\alpha\|_{L^1})\times\left(\frac{\epsilon}{4(\|\phi_\alpha\|_{L^1}+1)}\right)+2\times\left(\frac{\epsilon}{4}\right)\\
\leq & \epsilon.
\end{aligned}$$

Which implies that

$$\|(\mathcal{S}y)-(\mathcal{S}z)\|_\Psi=\left(\sup_{t\in\mathbb{R}}\frac{1}{\Psi(t)}\mathsf{E}\|(\mathcal{S}y)(t)-(\mathcal{S}z)(t)\|^2\right)^{1/2}\longrightarrow 0 \text{ as } y\to z.$$

This show that $\mathcal{S}$ is continuous on $C_\Psi(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}))$.

Step 3. We show that there is $\mathbf{k}>0$ such that $\mathcal{S}(\mathbf{B_k})\subseteq\mathbf{B_k}$, where

$$\mathbf{B_k}\equiv\mathbf{B_k}\left(C_\Psi(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}))\right):=\{z\in C_\Psi(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}))\text{ such that }\|z\|_\Psi\leq\mathbf{k}\}$$

represents the closed ball with center at 0 and radius $\mathbf{k}$ in the space $C_\Psi(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}))$. Arguing by contradiction, suppose that for each $\mathbf{k}>0$ there exist $z_\mathbf{k}\in\mathbf{B_k}$ such that $\|\mathcal{S}z_\mathbf{k}\|_\Psi>\mathbf{k}$.
We have

$$\mathsf{E}\|(\mathcal{S}z_\mathbf{k})(t)\|^2\leq 2M_g\mathcal{X}_g(\|v\|_\infty^2)\|\phi_\alpha\|_{L^1}^2+4\|\phi_\alpha\|_{L^2}^2\Big(L'\|v\|_\infty^2+M_f\Big).$$

For all $t\in\mathbb{R}$, we get

$$\frac{\mathsf{E}\|(\mathcal{S}z_\mathbf{k})(t)\|^2}{\Psi(t)}\leq\frac{1}{\Psi(t)}\left(2M_g\mathcal{X}_g(\|v\|_\infty^2)\|\phi_\alpha\|_{L^1}^2+4\|\phi_\alpha\|_{L^2}^2\Big(L'\|v\|_\infty^2+M_f\Big)\right)$$

We get that

$$\begin{aligned}
\mathbf{k}<\|(\mathcal{S}z_\mathbf{k})\|_\Psi&=\sup_{t\in\mathbb{R}}\frac{\mathsf{E}\|(\mathcal{S}z_\mathbf{k})(t)\|^2}{\Psi(t)}\\
&\leq\sup_{t\in\mathbb{R}}\frac{1}{\Psi(t)}\left(2M_g\mathcal{X}_g(\|v\|_\infty^2)\|\phi_\alpha\|_{L^1}^2+4\|\phi_\alpha\|_{L^2}^2\Big(L'\|v\|_\infty^2+M_f\Big)\right)=0.
\end{aligned}$$

Which is a contradiction.

Step 4. We show that $\mathcal{S}_1$ is completely continuous and $\mathcal{S}_2$ is a contraction.
By similar computations as in the proof of theorem 3.10, it's easy to see that $\mathcal{S}_2$ is a contraction provided $4\,L'\|v_\alpha\|^2_{L^2} < 1$. On the other hand, it easy to see that $\mathcal{S}_1 : \mathbf{B_k} \to \mathbf{B_k}$ is continuous.
Let $\mathcal{U} = \mathcal{S}_1(\mathbf{B_k})$ and $u(t) = (\mathcal{S}_1 v)(t)$ for $v \in \mathbf{B_k}$ and $t \in \mathbb{R}$. We aim to prove that $\mathcal{U}$ is relatively compact with the aid of lemma 3.12. For more clarity, we split this step in three claims.

Claim 1. $\mathcal{U}(t)$ **is a relatively compact subset of** $\mathbb{L}^2(\Omega, \mathbb{H})$ **for each** $t \in \mathbb{R}$**.**
We know that $s \longmapsto \phi_\alpha(s)$ is integrable on $\mathbb{R}_+$. Hence, for $\epsilon > 0$, we can choose $b > 0$ such that

$$\int_b^\infty \phi_\alpha(s)ds \le \frac{\epsilon}{\|\phi_\alpha\|_{L^1} M_g \mathcal{X}_g(\|v\|^2_\infty) + 1}.$$

Since for any $0 < a < b < \infty$, let

$$u_a(t) = \int_a^b \mathcal{R}_\alpha(s) g(t-s, v(t-s), \mathbb{P}_{v(t-s)})ds + \int_b^\infty \mathcal{R}_\alpha(s) g(t-s, v(t-s), \mathbb{P}_{v(t-s)})ds,$$

and

$$\mathcal{U}_a(t) := \{u_a(t) \ : \ 0 < a < b < \infty\}.$$

We derive that

$$\mathsf{E}\left\|\int_b^\infty \mathcal{R}_\alpha(s) g(t-s, v(t-s), \mathbb{P}_{v(t-s)})ds\right\|^2 \le \|\phi_\alpha\|_{L^1} M_g \mathcal{X}_g(\|v\|^2_\infty) \int_b^\infty \phi_\alpha(s)ds < \epsilon.$$

and by invoking the mean value theorem for Bochner integral, we get

$$u_a(t) \in (b-a)\overline{co(\mathcal{O})} + \mathbf{B}_\epsilon(\mathbb{L}^2(\Omega, \mathbb{H})),$$

where $co(\mathcal{O})$ represents the convex hull of $\mathcal{O}$ and

$$\mathcal{O} := \{\mathcal{R}_\alpha(s) g(\xi, v, \mu) \ : \ a \le s \le b, \ t-b \le \xi \le t-a, \ \|v\|_\Psi \le \mathbf{k}, \ \mu \in \mathcal{P}_2(\mathbb{H})\}$$

Furthermore, by the compactness of $\mathcal{R}_\alpha(t)$ for $t > 0$, it follows that $\mathcal{O}$ is relatively compact. Hence, we deduce that $\mathcal{U}_a(t) \subseteq (b-a)\overline{co(\mathcal{O})} + \mathbf{B}_\epsilon(\mathbb{L}^2(\Omega, \mathbb{H}))$ is also relatively compact for any $a > 0$.
By Lebesgue dominated convergence theorem, we have that

$$\begin{aligned}\mathbb{E}\|u(t) - u_a(t)\|^2 =& \mathsf{E}\left\|\int_0^\infty \mathcal{R}_\alpha(s) g(t-s, v(t-s), \mathbb{P}_{v(t-s)})ds - \int_a^b \mathcal{R}_\alpha(s) g(t-s, v(t-s), \mathbb{P}_{v(t-s)})ds\right. \\ &\left. - \int_b^\infty \mathcal{R}_\alpha(s) g(t-s, v(t-s), \mathbb{P}_{v(t-s)})ds,\right\|^2 \\ =& \mathsf{E}\left\|\int_0^a \mathcal{R}_\alpha(s) g(t-s, v(t-s), \mathbb{P}_{v(t-s)})ds\right\|^2 \\ \le& \|\phi_\alpha\|_{L^1} M_g \mathcal{X}_g(\|v\|^2_\infty) \int_0^a \phi_\alpha(s)ds \to 0 \text{ as } a \to 0.\end{aligned}$$

Thus there exists relatively compact sets arbitrarily close to the set $\mathcal{U}(t)$. This proves that $\mathcal{U}(t)$ is relatively compact.

Claim 2. $\mathcal{U}$ is equicontinuous.
Simple computations yield that

$$
\begin{aligned}
& u(t+r)-u(t) \\
=& \int_0^{\infty} \mathcal{R}_\alpha(s) g(t+r-s, v(t+r-s), \mathbb{P}_{v(t+r-s)}) ds-\int_0^{\infty} \mathcal{R}_\alpha(s) g(t-s, v(t-s), \mathbb{P}_{v(t-s)}) ds \\
=& \left[\int_0^r \mathcal{R}_\alpha(s) g(t+r-s, v(t+r-s), \mathbb{P}_{v(t+r-s)}) ds\right. \\
& \qquad +\int_0^a \mathcal{R}_\alpha(s+r) g(t-s, v(t-s), \mathbb{P}_{v(t-s)}) ds \\
& \qquad \quad \left.+\int_a^{\infty} \mathcal{R}_\alpha(s+r) g(t-s, v(t-s), \mathbb{P}_{v(t-s)}) ds\right] \\
& -\left[\int_0^a \mathcal{R}_\alpha(s) g(t-s, v(t-s), \mathbb{P}_{v(t-s)}) ds+\int_a^{\infty} \mathcal{R}_\alpha(s) g(t-s, v(t-s), \mathbb{P}_{v(t-s)}) ds\right] \\
=& \int_0^r \mathcal{R}_\alpha(s) g(t+r-s, v(t+r-s), \mathbb{P}_{v(t+r-s)}) ds \\
& \qquad +\int_0^a [\mathcal{R}_\alpha(s+r)-\mathcal{R}_\alpha(s)] g(t-s, v(t-s), \mathbb{P}_{v(t-s)}) ds \\
& \qquad +\int_a^{\infty} [\mathcal{R}_\alpha(s+r)-\mathcal{R}_\alpha(s)] g(t-s, v(t-s), \mathbb{P}_{v(t-s)}) ds \\
=& F_1(a, v, t, r)+F_2(a, v, t, r),
\end{aligned}
$$

where

$$
\begin{aligned}
F_1(a, v, t, r)= & \int_0^r \mathcal{R}_\alpha(s) g(t+r-s, v(t+r-s), \mathbb{P}_{v(t+r-s)}) ds \\
& +\int_0^a [\mathcal{R}_\alpha(s+r)-\mathcal{R}_\alpha(s)] g(t-s, v(t-s), \mathbb{P}_{v(t-s)}) ds \\
F_2(a, v, t, r)= & \int_a^{\infty} [\mathcal{R}_\alpha(s+r)-\mathcal{R}_\alpha(s)] g(t-s, v(t-s), \mathbb{P}_{v(t-s)}) ds .
\end{aligned}
$$

It follows that

$$
\mathsf{E}\|u(t+r)-u(t)\|^2 \leq 2\left(\mathsf{E}\|F_1(a, v, t, r)\|^2+\mathsf{E}\|F_2(a, v, t, r)\|^2\right)
$$

By using Cauchy-Schwartz's inequality and condition **(H2)-(2)**, we obtain the following estimations

$$
\begin{aligned}
& \mathsf{E}\|F_1(a, v, t, r)\|^2 \\
& \leq 2 \mathsf{E}\left\|\int_0^r \mathcal{R}_\alpha(s) g(t+r-s, v(t+r-s), \mathbb{P}_{v(t+r-s)}) ds\right\|^2 \\
& \quad +2 \mathsf{E}\left\|\int_0^a [\mathcal{R}_\alpha(s+r)-\mathcal{R}_\alpha(s)] g(t-s, v(t-s), \mathbb{P}_{v(t-s)}) ds\right\|^2 \\
& \leq 2\|\phi_\alpha\|_{L^1} \int_0^r \phi_\alpha(s) \mathsf{E}\|g(t+r-s, v(t+r-s), \mathbb{P}_{v(t+r-s)})\|^2 ds \\
& \quad +\left(\int_0^a\|\mathcal{R}_\alpha(s+r)-\mathcal{R}_\alpha(s)\| ds\right)\left(\int_0^a\|\mathcal{R}_\alpha(s+r)-\mathcal{R}_\alpha(s)\| \mathsf{E}\|g(t-s, v(t-s), \mathbb{P}_{v(t-s)})\|^2 ds\right) \\
& \leq 2\|\phi_\alpha\|_{L^1} M_g \mathcal{X}_g(\|v\|_{\infty}^2) \int_0^r \phi_\alpha(s) ds+M_g \mathcal{X}_g(\|v\|_{\infty}^2)\left(\int_0^a\|\mathcal{R}_\alpha(s+r)-\mathcal{R}_\alpha(s)\| ds\right)^2,
\end{aligned}
$$

and

$$\begin{aligned}
&\mathsf{E}\|F_2(a,t,r)\|^2\\
&=\mathsf{E}\|\int_a^\infty [\mathcal{R}_\alpha(s+r)-\mathcal{R}_\alpha(s)]g(t-s,v(t-s),\mathbb{P}_{v(t-s)})ds\|^2\\
&\leq \int_a^\infty \|\mathcal{R}_\alpha(s+r)-\mathcal{R}_\alpha(s)\|ds \int_a^\infty \|\mathcal{R}_\alpha(s+r)-\mathcal{R}_\alpha(s)\| \ \mathsf{E}\|g(t-s,v(t-s),\mathbb{P}_{v(t-s)})\|^2\, ds\\
&\leq M_g\mathcal{X}_g(\|v\|_\infty^2)\left(\int_a^\infty \|\mathcal{R}_\alpha(s+r)-\mathcal{R}_\alpha(s)\|ds\right)^2.
\end{aligned}$$

By the continuity of $(\mathcal{R}_\alpha(t))_{t\geq 0}$ for $t>0$ in the operator norm topology, the right side of the above two inequalities tends to zero as $r\to 0$. Consequently,

$$\lim_{r\to 0}\mathsf{E}\|F_2(a,v,t,r)\|^2=\lim_{r\to 0}\mathsf{E}\|F_2(a,v,t,r)\|^2=0.$$

We deduce that

$$\begin{aligned}
\mathsf{E}\|(\mathcal{S}_1\Phi)(t+r)-(\mathcal{S}_1\Phi)(t)\|^2 =&\mathsf{E}\|u(t+r)-u(t)\|^2\\
\leq& 2\mathsf{E}\|F_1(a,v,t,r)\|^2+2\mathsf{E}\|F_2(a,v,t,r)\|^2\\
&\longrightarrow 0 \text{ as } r\to 0.
\end{aligned}$$

This prove that $\mathcal{U}$ is equicontinuous.

Claim 3. We show that $\lim_{|t|\to\infty}\frac{\mathsf{E}\|u(t)\|^2}{\Psi(t)}=0$**.**

We have that

$$\frac{\mathsf{E}\|u(t)\|^2}{\Psi(t)}\leq\frac{M_g\mathcal{X}_g(\|v\|_\infty^2)\|\phi_\alpha\|_{L^1}^2}{\Psi(t)}\longrightarrow 0 \text{ as } |t|\to\infty.$$

and this convergence in independent of $v\in\mathbf{B_k}(C_\Psi(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H})))$.

Hence, by claims 1, 2, 3 and lemma 3.12, we deduce that $\mathcal{U}$ is relatively compact in $C_\Psi(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}))$. Hence $\mathcal{S}_1$ is completely continuous

Step 5. $\mathcal{S}$ **has a fixed point in** $\overline{\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)\cap\mathbf{B_k}}^\Psi$

From Theorem 3.3 and 3.6, for each $v\in\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)$, the stochastic processes $s\mapsto f(s,v(s),\mathbb{P}_{v(s)})$, $s\mapsto g(s,v(s),\mathbb{P}_{v(s)})$ belongs to $\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)$. From Lemmas 3.7, 3.8 and 2.4-(a), we obtain

$$\mathcal{S}(\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho))\subseteq\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho).$$

From step 1, it follows that

$$\mathcal{S}\left(\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)\cap\mathbf{B_k}\right)\subseteq\left(\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)\cap\mathbf{B_k}\right).$$

On the other hand, by the continuity of $\mathcal{S}$, we have that

$$\begin{aligned}
\mathcal{S}\left(\overline{\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)\cap\mathbf{B_k}}^\Psi\right)&\subseteq\overline{\mathcal{S}\left(\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)\cap\mathbf{B_k}\right)}^\Psi\\
&\subseteq\overline{\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)\cap\mathbf{B_k}}^\Psi.
\end{aligned}$$

Thanks to Krasnoselski Theorem 3.13, we deduce that $\mathcal{S}$ admits a fixed point

$$v\in\overline{\mathcal{WSABP}_{\omega,k}(\mathbb{R},\mathbb{L}^2(\Omega,\mathbb{H}),\rho)\cap\mathbf{B_k}}^\Psi.$$

Step 7. We prove that $v \in \mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho)$.
Let $\{v_n\} \subset \mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho) \cap \mathbf{B_k}$ such that $\|v_n - v\|_\Psi \to 0$. We obtain

$$\begin{aligned}\mathsf{E}\|(\mathcal{S}v_n)(t) - v(t)\|^2 &= \mathsf{E}\|(\mathcal{S}v_n)(t) - (\mathcal{S}v)(t)\|^2\\ &\le 2\|\phi_\alpha\|_{L^1}\left(\int_{-\infty}^t \phi_\alpha(t-s)\mathsf{E}\|[g(s, v_n(s), \mathbb{P}_{v_n(s)}) - g(s, v(s), \mathbb{P}_{v(s)})]\|^2\, ds\right)\\ &\quad + 2\left(\int_{-\infty}^t \phi_\alpha^2(t-s)\mathsf{E}\|[f(s, v_n(s), \mathbb{P}_{v_n(s)}) - f(s, v(s), \mathbb{P}_{v(s)})]\|^2\, ds\right).\end{aligned}$$

By using **(H2)-(3)**, we derive that $\{\mathcal{S}v_n\}$ converges to $\{\mathcal{S}v\}$ uniformly in $\mathbb{R}$. Thus, $v = \mathcal{S}v \in \mathcal{WSABP}_{\omega,k}(\mathbb{R}, \mathbb{L}^2(\Omega, \mathbb{H}), \rho)$ is mild solution of the problem (1.1). ■

Remark 3.15. *By taking $k\omega = \pi$, we can derive some existence results for square-mean weighted pseudo S-asymptotically ω-antiperiodic mild solutions to problem* (1.1) *from Theorems 3.10 and 3.14. Moreover, choosing $k\omega = 2\pi$, we can derive some existence results for square-mean weighted pseudo S-asymptotically ω-periodic mild solutions to problem* (1.1) *from Theorems 3.10 and 3.14.*

For example, we have the following results.

Corollary 3.16. *Suppose that the operator A generates a compact α-resolvent operator $\{\mathcal{R}_\alpha(t)\}_{t\ge 0} \subset \mathcal{B}(\mathbb{H})$ for $t > 0$ such that $\|\mathcal{R}_\alpha(t)\| \le \phi_\alpha(t)$ where $\phi_\alpha \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ for $t \ge 0$ and the functions $g,\ f \in \mathcal{BC}(\mathbb{R} \times \mathbb{L}^2(\Omega, \mathbb{H}) \times \mathcal{P}_2(\mathbb{H}), \mathbb{L}^2(\Omega, \mathbb{H}))$ verify assumptions* **(H*0)-(H2)**. *Moreover, suppose that g satisfies condition* **(H*1)**, *and there exists constants $L' > 0$ such that for any $v_1, v_2 \in \mathbb{L}^2(\Omega, \mathbb{H})$ and $\mu_1, \mu_2 \in \mathcal{P}_2(\mathbb{H})$,*

$$\mathsf{E}\|f(t, v_1, \mu_1) - f(t, v_2, \mu_2)\|^2 \le L'\left(\mathsf{E}\|v_1 - v_2\|^2 + \mathsf{W}_2^2(\nu_1, \nu_2)\right).$$

Then the problem (1.1) *has at least one square-mean weighted pseudo S-asymptotically ω-antiperiodic mild solution provided that $2\,L'J < 1$..*

Corollary 3.17. *Suppose that the operator A generates a compact α-resolvent operator $\{\mathcal{R}_\alpha(t)\}_{t\ge 0} \subset \mathcal{B}(\mathbb{H})$ for $t > 0$ such that $\|\mathcal{R}_\alpha(t)\| \le \phi_\alpha(t)$ where $\phi_\alpha \in L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ for $t \ge 0$ and the functions $g,\ f \in \mathcal{BC}(\mathbb{R} \times \mathbb{L}^2(\Omega, \mathbb{H}) \times \mathcal{P}_2(\mathbb{H}), \mathbb{L}^2(\Omega, \mathbb{H}))$ verify assumptions* **(H**0)-(H2)**. *Moreover, suppose that g satisfies condition* **(H*1)**, *and there exists constants $L' > 0$ such that for any $v_1, v_2 \in \mathbb{L}^2(\Omega, \mathbb{H})$ and $\mu_1, \mu_2 \in \mathcal{P}_2(\mathbb{H})$,*

$$\mathsf{E}\|f(t, v_1, \mu_1) - f(t, v_2, \mu_2)\|^2 \le L'\left(\mathsf{E}\|v_1 - v_2\|^2 + \mathsf{W}_2^2(\nu_1, \nu_2)\right).$$

Then the problem (1.1) *has at least one square-mean weighted pseudo S-asymptotically ω-periodic mild solution provided that $2\,L'J < 1$..*

4. Example

To illustrate our theoretical results, we consider

$$\rho(t) = 1 + t^2 \quad \text{for} \quad t \in \mathbb{R}.$$

Then, $\rho \in \Lambda_\infty$ and satisfies $(\mathbf{H}^\rho)$.
Let $\mathbb{H} = L^2[0, \pi]$, $1 < \alpha < 2$, $\nu > 0$ and consider the following problem

$$\begin{cases} \partial_t^\alpha v(t,\xi) = & -\nu\, v(t,\xi) - \dfrac{\nu^2}{4}\displaystyle\int_{-\infty}^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} v(t,\xi) ds \\ & +g\big(t, v(t, \xi, \mathbb{P}_{v(t,\xi)}\big) + f\big(t, v(t,\xi), \mathbb{P}_{v(t,\xi)}\big)\dfrac{\partial \mathbb{W}(t)}{\partial t}, \quad (t,\xi) \in \mathbb{R} \times (0,\pi) \\ & v(t,0) = v(t,\pi) = 0, \end{cases} \tag{4.1}$$

where $\mathbb{W}(t)$ is a two-sided and standard one-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{F}_t)$. The problem (4.1) can be written into the form (1.1) with $v(t)(\xi) = v(t, \xi)$, $b(t) = \dfrac{\nu}{4}\dfrac{t^{\alpha-1}}{\Gamma(\alpha)}$ and $A = -\nu I$, I is the identity operator on the Hilbert space $\mathbb{H}$. It follows from [24, Example 4.17], that A generates a $\alpha-$resolvent family $\{\mathcal{R}_\alpha(t)\}_{t\geq 0}$ with its Laplace transform satisfying

$$\hat{\mathcal{R}}_\alpha(\lambda) = \frac{\lambda^\alpha}{(\lambda^\alpha + \nu/2)^2} = \frac{\lambda^{\alpha-\nu/2}}{(\lambda^\alpha + \nu/2)} \cdot \frac{\lambda^{\alpha-\nu/2}}{(\lambda^\alpha + \nu/2)}$$

and

$$\mathcal{R}_\alpha(t) = (r * r)(t) \text{ where } r(t) = t^{\frac{\alpha}{2}}\mathcal{E}_{\alpha,\alpha/2}\Big(-\frac{\nu}{2}t^\alpha\Big)$$

and $\mathcal{E}_{\alpha,\alpha/2}(\cdot)$ is the Mittag-Leffter function (see [26]). From [23, Theorem 4.12], there exists a constant $C > 0$, depending only on α, such that, for $t \geq 0$

$$\|\mathcal{R}_\alpha(t)\| \leq \frac{C}{1 + \nu t^\alpha} := \phi_\alpha(t).$$

Simple calculations yield that :

$$\|\phi_\alpha\|_{L^1} = \frac{C}{\alpha\nu^{1/\alpha}}\mathsf{B}\left(\frac{1}{\alpha}, 1 - \frac{1}{\alpha}\right) < \infty$$

and

$$\|\phi_\alpha\|^2_{L^2} = \frac{C^2}{\alpha\nu^{(1/\alpha)-1}}\mathsf{B}\left(\frac{1}{\alpha}, 2 - \frac{1}{\alpha}\right) < \infty,$$

where $\mathsf{B}(\cdot,\cdot)$ denotes the Beta function.

First, to illustrate the Theorem 3.10, let take the forcing terms are follows:

$$\begin{aligned} f(t, z, \mathbb{P}_z)(\xi) =& M_1(t, z)(\xi) + \widetilde{M}_1(t, z, \mathbb{P}_z)(\xi) \quad \text{and} \\ g(t, z, \mathbb{P}_z)(\xi) =& M_2(t, z)(\xi) + \widetilde{M}_2(t, z, \mathbb{P}_z)(\xi), \end{aligned}$$

where $M_1(t, z)(\xi) = \gamma(t)\sigma_1(z(t)(\xi))$, $\quad M_2(t, z)(\xi) = \gamma(t)\sigma_2(z(t)(\xi))$

$$\begin{aligned} \widetilde{M}_1(t, z, \mathbb{P}_z)(\xi) &= \frac{\gamma(t)}{1 + t^2}\left[\cos(z(t)(\xi)) + \int_{\mathbb{L}^2(0,\pi)} \ell(x)\mathbb{P}_{z(t,\xi)}(dx)\right] \text{ and} \\ \widetilde{M}_2(t, z, \mathbb{P}_z)(\xi) &= \frac{\gamma(t)}{1 + t^2}\left[\sin(z(t)(\xi)) + \int_{\mathbb{L}^2(0,\pi)} \ell(x)\mathbb{P}_{z(t,\xi)}(dx)\right]. \end{aligned}$$

We suppose that $\gamma(t)$ is bounded continuous function such that $\gamma(t+\omega) = \gamma(t)$ with $\omega \in \mathbb{R}$ and $\ell : \mathbb{L}^2(0, \pi) \to \mathbb{R}$ is a 1-Lipschitz continuous function. Furthermore, the functions σ_i $(i = 1, 2)$ are such that

$$\sigma_i(e^{ik\omega}x) = e^{ik\omega}\sigma_i(x), \text{ and } \quad \mathsf{E}\|\sigma_i(u) - \sigma_i(v)\|^2_{\mathbb{H}} \leq L_i\,\mathsf{E}\|u - v\|^2_{\mathbb{H}},\ L_i \geq 0 \quad \text{for} \quad i = 1, 2.$$

Now, for $(i = 1, 2)$, we have that

$$M_i(t + \omega, z)(\xi) = \gamma(t + \omega)\sigma_i(z)(\xi) = \gamma(t)e^{ik\omega}\sigma_i(e^{-ik\omega}z)(\xi) = e^{ik\omega}M_i(t, e^{-ik\omega}z)(\xi),$$

then we get following estimation for $z \in \mathbb{L}^2(\Omega, \mathbb{H})$ and $r > 0$:

$$\begin{aligned}
&\frac{1}{m(r,\rho)} \int_{-r}^{r} \mathsf{E}\|f(t+\omega, z, \mathbb{P}_z) - e^{ik\omega} f(t, e^{-ik\omega} z, \mathbb{P}_{e^{ik\omega} z})\|_{\mathbb{H}}^2 \rho(t) dt \\
&\quad = \frac{1}{m(r,\rho)} \int_{-r}^{r} \mathsf{E}\|\gamma(t+\omega) M_1(z) + \widetilde{M}_1(t+\omega, z, \mathbb{P}_z) \\
&\qquad\qquad - e^{ik\omega} \left(\gamma(t) M_1(e^{-ik\omega} z) + \widetilde{M}_1(t, e^{-ik\omega} z, \mathbb{P}_{e^{ik\omega} z}) \right) \|_{\mathbb{H}}^2 \, \rho(t) dt \\
&\quad = \frac{1}{m(r,\rho)} \int_{-r}^{r} \mathsf{E}\|\widetilde{M}_1(t+\omega, z, \mathbb{P}_z) - e^{ik\omega} \widetilde{M}_1(t, e^{-ik\omega} z, \mathbb{P}_{e^{ik\omega} z})\|_{\mathbb{H}}^2 \, \rho(t) dt \\
&\quad \le \frac{\|\gamma\|_\infty^2}{m(r,\rho)} \int_{-r}^{r} \mathsf{E} \left\| \frac{1}{\rho(t+\omega)} \left[\cos(z) + \int_{\mathbb{L}^2(0,\pi)} \ell(x) \mathbb{P}_z(dx) \right] \right. \\
&\qquad\qquad \left. - \frac{1}{\rho(t)} e^{ik\omega} \left[\cos(e^{-ik\omega} z) + \int_{\mathbb{L}^2(0,\pi)} \ell(x) \mathbb{P}_{e^{-ik\omega} z}(dx) \right] \right\|_{\mathbb{H}}^2 \rho(t) dt
\end{aligned}$$

$$\begin{aligned}
&\le \frac{2\|\gamma\|_\infty^2}{m(r,\rho)} \int_{-r}^{r} \left(\mathsf{E} \left\| \frac{1}{\rho(t+\omega)} \left[\cos(z) + \int_{\mathbb{L}^2(0,\pi)} \ell(x) \mathbb{P}_z(dx) \right] \right\|_{\mathbb{H}}^2 \right. \\
&\qquad \left. + \mathsf{E} \left\| \frac{1}{\rho(t)} e^{ik\omega} \left[\cos(e^{-ik\omega} z) + \int_{\mathbb{L}^2(0,\pi)} \ell(x) \mathbb{P}_{e^{-ik\omega} z}(dx) \, (\xi) \right] \right\|_{\mathbb{H}}^2 \right) \rho(t) dt.
\end{aligned}$$

By lemma 2.2, Hölder's inequality and the representation of Wasserstein distance in terms of random variables, we have

$$\int_{\mathbb{L}^2(0,\pi)} \ell(x) \mathbb{P}_z(dx) \le \mathsf{W}_1(\mathbb{P}_z, 0) \le \mathsf{W}_2(\mathbb{P}_z, 0) \le (\mathbb{E}\|z\|^2)^{1/2} < \infty$$

$$\begin{aligned}
&\frac{1}{m(r,\rho)} \int_{-r}^{r} \mathsf{E}\|f(t+\omega, z, \mathbb{P}_z) - e^{ik\omega} f(t, e^{-ik\omega} z, \mathbb{P}_{e^{ik\omega} z})\|_{\mathbb{H}}^2 \rho(t) dt \\
&\le \frac{2\|\gamma\|_\infty^2}{m(r,\rho)} \int_{-r}^{r} \left(\left| \frac{1}{1+(t+\omega)^2} \right|^2 2(1 + \mathsf{E}\|z\|_{\mathbb{H}}^2) + \left| \frac{1}{1+t^2} \right|^2 2(1 + \mathsf{E}\|e^{ik\omega} z\|_{\mathbb{H}}^2) \right) \rho(t) dt \\
&\le \frac{4\,(1 + \mathsf{E}\|z\|_{\mathbb{H}}^2) \|\gamma\|_\infty^2}{m(r,\rho)} \int_{-r}^{r} \left(\left| \frac{1}{1+(t+\omega)^2} \right|^2 + \left| \frac{1}{1+t^2} \right|^2 \right) \rho(t) dt.
\end{aligned}$$

We have

$$\frac{1}{m(r,\rho)} \int_{-r}^{r} \left| \frac{1}{1+t^2} \right|^2 \rho(t) dt = \frac{1}{m(r,\rho)} \int_{-r}^{r} \frac{dt}{1+t^2} = \frac{\arctan(r)}{r + \frac{r^3}{3}} \to 0 \text{ as } r \to \infty. \tag{4.2}$$

Note that by the assumption $(\mathbf{H}^\rho)$, there exits a constant $b > 0$ such that for a.e $t \in \mathbb{R}$, we have

$$\frac{\rho(t-\omega)}{\rho(t)} \le b, \qquad \frac{m(t-|\omega|,\rho)}{m(t,\rho)} \le b.$$

Then

$$\begin{aligned}
&\frac{1}{m(r,\rho)}\int_{-r}^{r}\left|\frac{1}{1+(t+\omega)^2}\right|^2\rho(t)dt\\
\leq&\frac{m(r+|\omega|,\rho)}{m(r,\rho)}\left(\frac{1}{m(r+|\omega|,\rho)}\int_{-r-|\omega|}^{r+|\omega|}\left|\frac{1}{1+t^2}\right|^2\frac{\rho(t-\omega)}{\rho(t)}\rho(t)dt\right)\\
\leq&\frac{b^2}{m(r+|\omega|,\rho)}\int_{-r-|\omega|}^{r+|\omega|}\left|\frac{1}{1+t^2}\right|^2\rho(t)dt\to 0 \text{ as } r\to\infty \text{ (similarly to (4.2))}.
\end{aligned}$$

Hence

$$\lim_{r\to\infty}\frac{1}{m(r,\rho)}\int_{-r}^{r}\mathsf{E}\|f(t+\omega,z,\mathbb{P}_z))-e^{ik\omega}f(t,e^{-ik\omega}z,\mathbb{P}_{e^{ik\omega}z}))\|_{\mathbb{H}}^2\rho(t)dt=0.$$

Similarly, we have

$$\lim_{r\to\infty}\frac{1}{m(r,\rho)}\int_{-r}^{r}\mathsf{E}\|g(t+\omega,z,\mathbb{P}_z))-e^{ik\omega}g(t,e^{-ik\omega}z,\mathbb{P}_{e^{ik\omega}z}))\|_{\mathbb{H}}^2\rho(t)dt=0.$$

Therefore f,g satisfy **(H0)**. Let $u,v\in\mathbb{L}^2(\Omega,\mathbb{H})$ and $t\in\mathbb{R}$ and $\mu_1,\mu_2\in\mathcal{P}_2(\mathbb{H})$, then we have following estimation:

$$\begin{aligned}
&\mathsf{E}\|f(t,u,\mu_1)-f(t,v,\mu_2)\|_{\mathbb{H}}^2\\
&\quad\leq 3\|\gamma\|_\infty^2\Bigg(\mathsf{E}\|\sigma_1(u)-\sigma_1(v)\|_{\mathbb{H}}^2\\
&\qquad\qquad+\mathsf{E}\|\cos(u)-\cos(v)\|+\left\|\int_{\mathbb{L}^2(0,\pi)}\ell(x)\mu_1(dx)-\int_{\mathbb{L}^2(0,\pi)}\ell(x)\mu_2(dx)\right\|_{\mathbb{H}}^2\Bigg)\\
&\quad\leq 3\|\gamma\|_\infty^2\Bigg(L_1\mathsf{E}\|u-v\|_{\mathbb{H}}^2+\mathsf{E}\|u-v\|_{\mathbb{H}}^2+\|\int_{\mathbb{L}^2(0,\pi)}\ell(x)\mu_1(dx)-\int_{\mathbb{L}^2(0,\pi)}\ell(x)\mu_2(dx)\|_{\mathbb{H}}^2\Bigg)\\
&\quad\leq 3\|\gamma\|_\infty^2\Bigg(L_1\mathsf{E}\|u-v\|_{\mathbb{H}}^2+\mathsf{E}\|u-v\|_{\mathbb{H}}^2+\|\int_{\mathbb{L}^2(0,\pi)}\ell(x)(\mu_1-\mu_2)(dx)\|_{\mathbb{H}}^2\Bigg).
\end{aligned}$$

By Lemma 2.2 and Hölder's inequality, we have that

$$\begin{aligned}
\mathsf{E}\|f(t,u,\mu_1)-f(t,v,\mu_2)\|_{\mathbb{H}}^2&\leq 3\|\gamma\|_\infty^2\Big[(L_1+1)\mathsf{E}\|u-v\|_{\mathbb{H}}^2+\mathsf{W}_1^2(\mu_1,\mu_2)\Big]\\
&\leq 3\|\gamma\|_\infty^2(L_1+1)\Big[\mathsf{E}\|u-v\|_{\mathbb{H}}^2+\mathsf{W}_2^2(\mu_1,\mu_2)\Big].
\end{aligned}$$

Similarly, we obtain

$$\mathsf{E}\|g(t,u,\mu_1)-g(t,v,\mu_2)\|_{\mathbb{H}}^2\leq 3\|\gamma\|_\infty^2(L_2+1)\Big(\mathsf{E}\|u-v\|_{\mathbb{H}}^2+\mathsf{W}_2^2(\mu_1,\mu_2)\Big).$$

Putting $L=3\|\gamma\|_\infty^2(L_2+1)$ and $L'=3\|\gamma\|_\infty^2(L_2+1)$, we obtain that

$$\begin{aligned}
&\|\phi_\alpha\|_{L^1}^2L+L'\|\phi_\alpha\|_{L^2}^2\\
:=&\frac{C}{\alpha\nu^{1/\alpha}}\mathsf{B}\left(\frac{1}{\alpha},1-\frac{1}{\alpha}\right)3\|\gamma\|_\infty^2(L_1+1)+3\|\gamma\|_\infty^2(L_2+1)\frac{C^2}{\alpha\nu^{(1/\alpha)-1}}\mathsf{B}\left(\frac{1}{\alpha},2-\frac{1}{\alpha}\right).
\end{aligned}$$

Hence, condition (3.1) of Theorem 3.10 is fulfilled by choosing $\|\gamma\|_\infty$ is small enough. Therefore, by Theorem 3.10, the problem (4.1) has a unique square-mean weighted pseudo S-asymptotically Bloch type periodic mild solution on $\mathbb{R}$.

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