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Ricci solitons on concircularly flat and W_i-flat Sasakian manifolds admitting a general connection

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Abstract. The prime goal of this work is to investigate Ricci solitons on concircularly flat and W_i-flat $(i = 2, 3, 4)$ Sasakian manifolds. Sasakian manifolds are considered as admitting a connection which generalize the well-known connections so called the Zamkovoy, quarter symmetric metric, Tanaka-Webster and Schouten-Van Kampen connections.

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1. Introduction

Given a contact metric manifold (L, g, η, ξ, ϕ). In [1], [2], [3], there exists a general connection $\bar{\nabla}$ defined by

$$\bar{\nabla}_{F_1} F_2 = \nabla_{F_1} F_2 + c_1[(\nabla_{F_1}\eta)(F_2)\xi - \eta(F_2)\nabla_{F_1}\xi] + c_2\eta(F_1)\phi F_2, \tag{1.1}$$

for all $F_1, F_2 \in \chi(L)$ and $c_1, c_2 \in \mathbb{R}$, where $\chi(L)$ labels the set of all smooth vector fields on L. The connection $\bar{\nabla}$ is a general connection, since

(i) if $c_1 = 0, c_2 = -1$, we occur the quarter symmetric metric connection [4],
(ii) if $c_1 = 1, c_2 = 0$, we occur the Schouten-Van Kampen connection [10],
(iii) if $c_1 = 1, c_2 = -1$, we occur the Tanaka-Webster connection [11],

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(iv) if $c_1 = 1, c_2 = 1$, we occur the Zamkovoy connection [13].

Sasakian manifolds were defined and studied by Sasaki [8], [9] and they have many applications in differential geometry. They expose new samples of differentiable manifolds with particular geometric constructions such as complete Kähler manifolds, manifolds with holonomy groups and Einstein metrics. They also have an essential place in the investigation of orbifolds.

In [12], Yano explored the concircular curvature tensor on an $n-$dimensional Riemannian manifold as follows:

$$W(F_1, F_2)F_3 = R(F_1, F_2)F_3 - \frac{r}{n(n-1)}[g(F_2, F_3)F_1 - g(F_1, F_3)F_2]. \tag{1.2}$$

for all $F_1, F_2, F_3 \in \chi(L)$, where R is the Riemannian curvature tensor and r is the scalar curvature. Pokhariyal and Mishra explored the $W_2,\ W_3$ and the W_4 curvature tensors as follows:

$$W_2(F_1, F_2)F_3 = R(F_1, F_2)F_3 - \frac{1}{n-1}[g(F_2, F_3)F_1 - g(F_1, F_3)QF_2] \tag{1.3}$$

$$W_3(F_1, F_2)F_3 = R(F_1, F_2)F_3 - \frac{1}{n-1}[S(F_1, F_3)F_2 - g(F_2, F_3)QF_1] \tag{1.4}$$

$$W_4(F_1, F_2)F_3 = R(F_1, F_2)F_3 + \frac{1}{n-1}[g(F_1, F_3)QF_2 - g(F_1, F_2)QF_3] \tag{1.5}$$

for all $F_1, F_2, F_3 \in \chi(L)$, where S is the Ricci tensor, Q is the Ricci operator, r is the scalar curvature and R is the Riemannian curvature tensor, respectively [6],[7]. If $W_i = 0$, $i = 1, 2, 3, 4$ for a Riemannian manifold L, we announce that L is a W_i-flat manifold.

The definintion of a Ricci soliton on the manifold L is expressed as

$$(\mathfrak{L}_T g)(F_1, F_2) + 2S(F_1, F_2) + 2\lambda g(F_1, F_2) = 0, \tag{1.6}$$

for all $F_1, F_2 \in \chi(L)$, where T is a smooth vector field on L, $\mathfrak{L}_T g$ is the Lie derivative and λ is a real constant. If λ is smaller than zero the Ricci soliton is shrinking, if λ is larger than zero the Ricci soliton is expanding and if λ is equals to zero the Ricci soliton is steady.

2. Sasakian manifolds and the general connection $\bar{\nabla}$

An $n = (2k+1)-$dimensional smooth manifold L is announced as an almost contact metric manifold if it appreciates a 1-form η, a (1,1) tensor field ϕ, a Riemannian metric g and a contravariant vector field ξ which fulfill

$$\phi^2(F_1) = -F_1 + \eta(F_1)\xi,\ \eta(\xi) = 1,\ \phi\xi = 0,\ \eta(\phi F_1) = 0,$$

$$\begin{aligned} g(\phi F_1, \phi F_2) &= g(F_1, F_2) - \eta(F_1)\eta(F_2),\ g(\phi F_1, F_2) = -g(F_1, \phi F_2), \\ g(F_1, \xi) &= \eta(F_1),\ \forall F_1, F_2 \in \chi(L). \end{aligned}$$

An almost contact metric manifold (L, g, η, ξ, ϕ) is named a contact metric manifold if

$$g(F_1, \phi F_2) = d\eta(F_1, F_2).$$

Moreover, an almost contact metric manifold (L, g, η, ξ, ϕ) is named normal if

$$2d\eta(F_1, F_2)\xi + [\phi, \phi](F_1, F_2) = 0,$$

where $[\phi, \phi]$ denotes the Nijenhuis tensor.

A normal contact metric manifold is called a Sasakian manifold. For a Sasakian manifold, the following relations are satisfied:

$$\begin{aligned} \nabla_{F_1}\xi &= -\phi F_1, \\ (\nabla_{F_1}\eta)F_2 &= -g(\phi F_1, F_2), \\ R(F_1,F_2)\xi &= \eta(F_2)F_1 - \eta(F_1)F_2, \\ S(F_1,\xi) &= (n-1)\eta(F_1), \\ R(F_1,\xi)F_2 &= \eta(F_2)F_1 - g(F_1,F_2)\xi, \\ Q\xi &= (n-1)\xi, \\ S(\phi F_1,\phi F_2) &= S(F_1,F_2) - (n-1)\eta(F_1)\eta(F_2), \end{aligned} \tag{2.1}$$

where R, S and Q denote the Riemannian curvature tensor, Ricci tensor and the Ricci operator of L, respectively. Furthermore, we aware that a Sasakian manifold is $K-$contact and for a Sasakian manifold the tracking relation is fulfilled

$$(\mathfrak{L}_\xi g)(F_1,F_2) = 0, \tag{2.2}$$

where $\mathfrak{L}_\xi$ is the Lie derivative in the direction ξ which is a Killing vector field.

On the other hand, we write the theorem below for the general connection $\bar{\nabla}$ on a Sasakian manifold from [2].

Theorem 2.1. *Given an $n-$dimensional Sasakian manifold L appreciating the general connection $\bar{\nabla}$ in (1.1). Then*

(i) The connection $\bar{\nabla}$ is expressed as

$$\bar{\nabla}_{F_1}F_2 = \nabla_{F_1}F_2 + c_1[g(F_1,\phi F_2)\xi + \eta(F_2)\phi F_1] + c_2\eta(F_1)\phi F_2. \tag{2.3}$$

(ii) The Ricci tensor $\bar{S}$ of $\bar{\nabla}$ is expressed as

$$\bar{S}(F_1,F_2) = S(F_1,F_2) - Ag(F_1,F_2) + B\eta(F_1)\eta(F_2). \tag{2.4}$$

(iii) The scalar curvature of $\bar{\nabla}$ is expressed as

$$\bar{r} = r - An + B \tag{2.5}$$

for all $F_1, F_2 \in \chi(L)$, where $A = c_1^2 - c_1 - c_2 - c_1c_2$ and $B = c_1^2 + (n-2)c_1c_2 - n(c_1+c_2)$.

The general connection $\bar{\nabla}$ on a Sasakian manifold also satisfies

$$\begin{aligned} \bar{S}(F_2,\xi) &= -(n-1)C\eta(F_2), \\ \overline{Q}F_2 &= QF_2 - AF_2 + B\eta(F_2)\xi, \\ \overline{Q}\xi &= -(n-1)C\xi, \\ \overline{R}(F_1,F_2)\xi &= C[\eta(F_1)F_2 - \eta(F_2)F_1], \\ \overline{R}(\xi,F_2)F_3 &= C[\eta(F_3)F_2 - g(F_2,F_3)\xi], \end{aligned}$$

where $C = c_1 - c_1c_2 + c_2 - 1$.

Now, suppose a Ricci soliton (λ,ξ,g) on L. Using (1.6) and (2.2), we occur

$$\begin{aligned} 0 &= (\mathfrak{L}_T g)(F_2,F_3) + 2S(F_2,F_3) + 2\lambda g(F_2,F_3) \\ &= S(F_2,F_3) + \lambda g(F_2,F_3). \end{aligned} \tag{2.6}$$

Putting $F_3 = \xi$ in (2.6), we occur

$$S(F_2,\xi) = -\lambda\eta(F_2). \tag{2.7}$$

Substituting QF_2 to F_2 in (2.7), we get

$$S^2(F_2,\xi) = \lambda^2\eta(F_2) \text{ (see [5])}. \tag{2.8}$$

3. Ricci solitons on concircularly flat Sasakian manifolds appreciating the general connection $\bar{\nabla}$

Theorem 3.1. *Given a concircularly flat Sasakian manifold L appreciating a Ricci soliton (λ, ξ, g) with respect to the general connection $\bar{\nabla}$. In this case, the Ricci soliton is steady, shrinking or expanding if and only if $r = (n-1)B,\ r > (n-1)B$ or $r < (n-1)B$.*

Proof. If $(M, \bar{\nabla})$ is a concircularly flat Sasakian manifold, i.e. $\overline{W} = 0$, from (1.2) we have

$$\bar{R}(F_1, F_2)F_3 = \frac{\bar{r}}{n(n-1)}[g(F_2, F_3)F_1 - g(F_1, F_3)F_2]. \tag{3.1}$$

Inner product in (3.1) with a vector field F_4 gives

$$g(\bar{R}(F_1, F_2)F_3, F_4) = \frac{\bar{r}}{n(n-1)}[g(F_2, F_3)g(F_1, F_4) - g(F_1, F_3)g(F_2, F_4)]. \tag{3.2}$$

Contracting (3.2) over F_1 and F_4, we obtain

$$n\overline{S}(F_2, F_3) = \overline{r}g(F_2, F_3). \tag{3.3}$$

Putting (2.4) and (2.5) in (3.3), we occur

$$nS(F_2, F_3) = (r + B)g(F_2, F_3) - nB\eta(F_2)\eta(F_3). \tag{3.4}$$

Taking $F_3 = \xi$ and using (2.7) in (3.4), we get

$$S(F_2, \xi) = \frac{1}{n}(r + B)\eta(F_2) - B\eta(F_2)$$

or

$$\begin{aligned}\lambda &= -\frac{1}{n}(r + B) + B \\ &= -\frac{r - (n-1)B}{n}.\end{aligned}$$

This completes the proof. ∎

4. Ricci solitons on W_2-flat Sasakian manifolds appreciating the general connection $\bar{\nabla}$

Theorem 4.1. *Given a W_2-flat Sasakian manifold L appreciating a Ricci soliton (λ, ξ, g) with respect to the general connection $\bar{\nabla}$. In this case, the Ricci soliton is steady, shrinking or expanding if and only if $r = (n-1)B,\ r > (n-1)B$ or $r < (n-1)B$.*

Proof. If $(M, \bar{\nabla})$ is a W_2-flat Sasakian manifold, i.e. $\overline{W}_2 = 0$, from (1.3) we have

$$\bar{R}(F_1, F_2)F_3 = \frac{1}{n-1}[g(F_2, F_3)\overline{Q}F_1 - g(F_1, F_3)\overline{Q}F_2]. \tag{4.1}$$

Inner product in (4.1) with a vector field F_4 gives

$$g(\bar{R}(F_1, F_2)F_3, F_4) = \frac{1}{n-1}[g(F_2, F_3)g(\overline{Q}F_1, F_4) - g(F_1, F_3)g(\overline{Q}F_2, F_4)]. \tag{4.2}$$

Contracting (4.2) over F_1 and F_4, we obtain

$$\overline{S}(F_2, F_3) = \frac{1}{n-1}[\overline{r}g(F_2, F_3) - \overline{S}(F_2, F_3)]. \tag{4.3}$$

Putting (2.4) and (2.5) in (4.3), we occur

$$nS(F_2, F_3) = (r + B)g(F_2, F_3) - nB\eta(F_2)\eta(F_3). \tag{4.4}$$

Taking $F_3 = \xi$ and using (2.7) in (4.4), we get

$$S(F_2, \xi) = \frac{1}{n}(r + B)\eta(F_2) - B\eta(F_2)$$

or

$$\begin{aligned} \lambda &= -\frac{1}{n}(r + B) + B \\ &= -\frac{r - (n-1)B}{n}. \end{aligned}$$

This completes the proof. ■

5. Ricci solitons on W_3-flat Sasakian manifolds appreciating the general connection $\bar{\nabla}$

Theorem 5.1. *Given a W_3-flat Sasakian manifold L appreciating a Ricci soliton (λ, ξ, g) with respect to the general connection $\bar{\nabla}$. In this case, the Ricci soliton is steady, shrinking or expanding if and only if $r = -2(n-1)A + (n-1)B,\ r < -2(n-1)A + (n-1)B$ or $r > -2(n-1)A + (n-1)B$.*

Proof. If $(M, \bar{\nabla})$ is a W_3-flat Sasakian manifold, i.e. $\overline{W}_3 = 0$, from (1.4) we have

$$\bar{R}(F_1, F_2)F_3 = \frac{1}{n-1}[\overline{S}(F_1, F_3)F_2 - g(F_2, F_3)\overline{Q}F_1]. \tag{5.1}$$

Inner product in (5.1) with a vector field F_4 gives

$$g(\bar{R}(F_1, F_2)F_3, F_4) = \frac{1}{n-1}[\overline{S}(F_1, F_3)g(F_2, F_4) - g(F_2, F_3)g(\overline{Q}F_1, F_4)]. \tag{5.2}$$

Contracting (5.2) over F_1 and F_4, we obtain

$$\overline{S}(F_2, F_3) = \frac{1}{n-1}[\overline{S}g(F_2, F_3) - \overline{r}(F_2, F_3)]. \tag{5.3}$$

Putting (2.4) and (2.5) in (5.3), we occur

$$S(F_2, F_3) = \frac{-r + An - B}{n-2}g(F_2, F_3) + Ag(F_2, F_3) - B\eta(F_2)\eta(F_3). \tag{5.4}$$

Taking $F_3 = \xi$ and using (2.7) in (5.4), we get

$$S(F_2, \xi) = (\frac{-r + An - B}{n-2} + A)\eta(F_2) - B\eta(F_2)$$

or

$$\lambda = \frac{r - 2(n-1)A + (n-1)B}{n-2}.$$

This completes the proof. ■

6. Ricci solitons on W_4-flat Sasakian manifolds appreciating the general connection $\bar{\nabla}$

Theorem 6.1. *Given a W_4-flat Sasakian manifold L appreciating a Ricci soliton (λ, ξ, g) with respect to the general connection $\bar{\nabla}$. In this case, the Ricci soliton is steady, shrinking or expanding if and only if $-A+B=0$, $-A+B<0$ or $-A+B>0$.*

Proof. If $(M, \bar{\nabla})$ is a W_4-flat Sasakian manifold, i.e. $\overline{W}_4=0$, from (1.5) we have

$$\bar{R}(F_1,F_2)F_3=-\frac{1}{n-1}[g(F_1,F_3)\overline{Q}F_2-g(F_1,F_2)\overline{Q}F_3] \tag{6.1}$$

Inner product in (6.1) with a vector field F_4 gives

$$g(\bar{R}(F_1,F_2)F_3,F_4)=-\frac{1}{n-1}[g(F_1,F_3)g(\overline{Q}F_2,F_4)-g(F_1,F_2)g(\overline{Q}F_3,F_4)]. \tag{6.2}$$

Contracting (6.2) over F_1 and F_4, we obtain

$$\overline{S}(F_2,F_3)=\frac{1}{n-1}[\overline{S}g(F_2,F_3)-\overline{S}(F_2,F_3)]=0. \tag{6.3}$$

Putting (2.4) in (6.3), we occur

$$S(F_2,F_3)=Ag(F_2,F_3)-B\eta(F_2)\eta(F_3). \tag{6.4}$$

Taking $F_3=\xi$ and using (2.7) in (6.4), we get

$$S(F_2,\xi)=A\eta(F_2)-B\eta(F_2)$$

or

$$\lambda=-A+B.$$

This completes the proof. ∎

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