MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **12(03)**(2024), 253–261.
http://doi.org/10.26637/mjm1203/003

Study of the inverse continuous Bernoulli distribution

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Received 23 October 2023; Accepted 25 April 2024

Abstract. The continuous Bernoulli distribution, a one-parameter probability distribution defined over the interval $[0,1]$, has recently received increased attention in applied statistics. Numerous studies have highlighted both its merits and limitations, and proposed extended variants. In this article, we present an innovative modification of the continuous Bernoulli distribution through an inverse transformation, introducing the inverse continuous Bernoulli distribution. The main feature of this distribution is that it transfers the properties of the continuous distribution to the interval $[1,+\infty)$ without the need for additional parameters. The first part of this article elucidates the mathematical properties of this novel inverse distribution, including essential probability functions and quantiles. Inference for the associated model is performed using the famous maximum likelihood estimation. A comprehensive simulation study is carried out to evaluate the effectiveness of the estimated model. Its performance is then evaluated in a practical context using data sets from a variety of sources. In particular, our results demonstrate its superior performance to a wide range of analogous models defined over the support interval $[1,+\infty)$, even outperforming the well-established Pareto model.

AMS Subject Classifications: 60E05, 62E99.

Keywords: Continuous Bernoulli distribution, inverse distribution, quantiles, data fitting.

Contents

1. Introduction

In order to explain the mathematical basis of this investigation, we will first examine the continuous Bernoulli (CB) distribution, which was originally introduced in the work of [10]. The definition of this distribution can be succinctly stated as follows:

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Definition 1.1. *The cumulative distribution function (cdf) below defines the CB distribution with the parameter* $\omega \in (0,1)$*:*

$$F_{CB}(x;\omega)=\begin{cases}0, & x<0,\\ x, & \omega=\dfrac{1}{2} \text{ and } x\in[0,1],\\ \dfrac{\omega^{x}(1-\omega)^{1-x}+\omega-1}{2\omega-1}, & \omega\in(0,1)/\left\{\dfrac{1}{2}\right\} \text{ and } x\in[0,1],\\ 1, & x>1.\end{cases} \tag{1.1}$$

Thus, the CB distribution has a support restricted to the interval $[0, 1]$, with a single parameter similar to the conventional power distribution. This distribution finds applications in a wide variety of fields, with particular emphasis on machine learning, probability theory and statistics. In the context of variational autoencoders, it proves to be highly effective in replicating the pixel intensities of real-world images. For a more in-depth exploration of these topics, see the following references: [10], [8], [12] and [9].

Recently, there have been notable developments in the scientific literature in the area of CB distribution extensions. Of particular importance is the power CB (PCB) distribution, as elucidated by the authors in [2]. The PCB distribution introduces a notable extension to the cdf, as shown in Equation (1.1). This extension is achieved by the inclusion of a shape parameter, which serves to increase the modeling versatility of the CB distribution. In [2], the authors proposed a statistical methodology to explore the fundamental mathematical properties of the PCB distribution. Maximum likelihood estimation was also used to explore the nuances of parameter estimation. In order to illustrate the practical utility of the PCB distribution, an in-depth investigation was carried out using two different data sets. These data sets include a collection of trade share data and a comprehensive set of polyester fiber tensile strength data. Through these real-world data fitting exercises, the flexibility and applicability of the PCB distribution was rigorously assessed. The accuracy of the PCB distribution is underlined by the consideration of fair competitors. Conventional statistical standards show that the PCB distribution provides superior results. In addition, the transmuted CB (TCB) distribution introduced in [3] stands out as a highly effective extension. A notable feature of the CB distribution is its parameter, which orchestrates a linear trade-off between the minimum and maximum values of two continuous random variables. Using a statistical approach, the authors derive the fundamental mathematical properties of the TCB distribution. To illustrate the suitability of the model, they examined three proportional data sets: the time to infection of kidney dialysis patients, records of flood peaks, and waiting times for service in a bank. The empirical results highlight the superior fit of the TCB distribution to these data sets compared to well-established competitors. In a related context, [12, Chapter 9] introduced a two-dimensional CB distribution along with some of its key attributes. In addition, the authors in [9] elaborated an exponentiated variant of the CB distribution to construct a fractile (quantile) regression model for responses in the range $[0, 1]$. More recently, the authors in [4] used the CB distribution to give rise to the Op family, considering its cdf as a distribution generator. In particular, the OpTL distribution, rooted in the Topp-Leone distribution, emerged as a novel two-parameter distribution with support in the interval $[0, 1]$. Notably, the OpTL distribution demonstrated superior fit performance compared to contemporary models, including the CB distribution itself. In summary, it is evident that research around the CB distribution will continue to flourish in future, both from a theoretical and practical perspective.

In this article, we use the CB distribution to formulate a novel one-parameter distribution defined over the interval $[1, +\infty)$. To achieve this goal, we focus on the conventional inverse scheme, a well-established methodology known for its invaluable contributions to the modeling and analysis of data in various domains. The inverse scheme not only facilitates the exploration of reciprocal relationships, but also provides valuable insights into the characteristics of skewed and heavy-tailed distributions. Moreover, its practical utility extends to important domains such as finance, reliability and extreme value analysis. In the specific context of this study, we briefly present our approach: Starting with a random variable X following the CB distribution, we introduce the concept of an inverse CB (ICB) distribution, characterized by the distribution of the inverse random variable

$Y = 1/X$. The cdf of the ICB distribution, denoted as $F_{ICB}(x;\omega)$, is expressed as a function of $F_{CB}(x;\omega)$, represented as $F_{ICB}(x;\omega) = 1 - F_{CB}(1/x;\omega)$. The exact definition is explained below.

Definition 1.2. *The cdf given below defines the ICB distribution with parameter $\omega \in (0,1)$:*

$$F_{ICB}(x;\omega) = \begin{cases} 0, & x < 1, \\ 1 - \dfrac{1}{x}, & \omega = \dfrac{1}{2} \text{ and } x \geq 1, \\ 1 - \dfrac{\omega^{1/x}(1-\omega)^{1-1/x} + \omega - 1}{2\omega - 1}, & \omega \in (0,1)/\left\{\dfrac{1}{2}\right\} \text{ and } x \geq 1. \end{cases}$$

This cdf is at the core of the theory and inference of the ICB distribution, which will be explained in detail in this article. In particular, the support of the ICB distribution lies within the interval $[1, +\infty)$, making it a direct rival to the Pareto distribution. The creation of such distributions, i.e., with support $[1, +\infty)$, is crucial for modeling various real-world phenomena, such as reliability analysis and survival data, where non-negative outcomes beyond a lower bound are of primary interest. Such distributions provide a robust framework for handling situations where values cannot be less than one, thus ensuring a more accurate representation of the underlying processes. In our research, we empirically demonstrate that the ICB distribution can provide a more robust and accurate fit to real-world data compared to the Pareto distribution, thus supporting its importance in statistical modeling.

The subsequent organization of the article unfolds as follows: Section 2 deals with the basic probability functions that govern the ICB distribution. Section 3 is dedicated to the parameter estimation, simulations and real-world applications. Finally, our study ends in Section 4, where we present our final results.

2. Probability Functions

This section deals with the analysis of the probability density function (pdf), hazard rate function (hrf) and quantile function (qf) associated with the ICB distribution.

First, the pdf relevant to the ICB distribution is obtained by differentiating $F_{ICB}(x;\omega)$ as follows:

$$f_{ICB}(x;\omega) = \begin{cases} 0, & x < 1, \\ \dfrac{1}{x^2}, & \omega = \dfrac{1}{2} \text{ and } x \geq 1, \\ c_\omega \dfrac{1}{x^2}\omega^{1/x}(1-\omega)^{1-1/x}, & \omega \in (0,1)/\left\{\dfrac{1}{2}\right\} \text{ and } x \geq 1, \end{cases} \tag{2.1}$$

where

$$c_\omega = \frac{2\,\mathrm{arctanh}(1-2\omega)}{1-2\omega} \quad \left(\text{or, equivalently, } c_\omega = \frac{\ln(1-\omega) - \ln(\omega)}{1-2\omega}\right).$$

Such a pdf with support $[1, +\infty)$ is essential for modeling and analyzing random variables that are restricted to positive values, such as durations, lifetimes and various natural phenomena. It provides insight into the probability of observing certain outcomes within this constrained range, making it a valuable tool in fields such as reliability analysis and survival studies.

By manipulating $f_{ICB}(x;\omega)$ and $F_{ICB}(x;\omega)$, we obtain the hrf corresponding to the ICB distribution. More precisely, we have

$$\begin{aligned} h_{ICB}(x;\omega) &= \frac{f_{ICB}(x;\omega)}{1-F_{ICB}(x;\omega)} \\ &= \begin{cases} 0, & x<1, \\ \dfrac{1}{x}, & \omega=\dfrac{1}{2} \text{ and } x\geq 1, \\ c_\omega^* \dfrac{\omega^{1/x}(1-\omega)^{1-1/x}}{x^2\left[1-\omega-\omega^{1/x}(1-\omega)^{1-1/x}\right]}, & \omega\in(0,1)/\left\{\dfrac{1}{2}\right\} \text{ and } x\geq 1, \end{cases} \end{aligned}$$

where

$$c_\omega^* = 2\,\mathrm{arctanh}(1-2\omega) \quad (\mathrm{or, equivalently,}\ c_\omega^* = \ln(1-\omega)-\ln(\omega)).$$

The qf corresponding to the ICB distribution is obtained as

$$\begin{aligned} Q_{ICB}(x;\omega) &= F_{ICB}^{-1}(x;\omega) \\ &= \begin{cases} \dfrac{1}{1-x}, & \omega=\dfrac{1}{2} \text{ and } x\in[0,1], \\ \dfrac{c_\omega^*}{\ln(1-\omega)-\ln[(2\omega-1)(1-x)+1-\omega]}, & \omega\in(0,1)/\left\{\dfrac{1}{2}\right\} \text{ and } x\in[0,1]. \end{cases} \end{aligned} \tag{2.2}$$

Having an analytical expression for the qf in distributions with support $[1,+\infty)$ is important for efficient risk assessment and decision making, especially in scenarios involving reliability and tail risk analysis where accurate modeling of extreme events is paramount.

In addition, the qf in Equation (2.2) is used to generate random samples of size n from the ICB distribution. Analytically, for a fixed value of the parameter ω, we obtain the median, lower, and upper quartiles of the ICB distribution when x takes the values $1/2$, $1/4$, and $3/4$, respectively, in this qf.
Furthermore, the Galton skewness and Moor kurtosis as proposed by [7] and [11], respectively, can be obtained by utilizing Equation (2.2) as follows:

$$S_G = \frac{Q(6/8;\omega)-2Q(4/8;\omega)+Q(2/8;\omega)}{Q(6/8;\omega)-Q(2/8;\omega)}$$

and

$$K_M = \frac{Q(7/8;\omega)-Q(5/8;\omega)+Q(3/8;\omega)-Q(1/8;\omega)}{Q(6/8;\omega)-Q(2/8;\omega)}.$$

Table 1 shows the summary statistics of the ICB distribution for different choice of the parameter value ω.

Table 1: Summary statistics of the ICB distribution for varying values of ω

ω	$Q(1/2;\omega)$	$Q(1/4;\omega)$	$Q(3/4;\omega)$	S_G	K_M
0.1	3.7381	2.0000	8.7429	0.4845	2.1481
0.2	2.9495	1.6769	6.6765	0.4909	2.1558
0.3	2.5182	1.5141	5.4966	0.4957	2.1633
0.4	2.2239	1.4094	4.6599	0.4989	2.1690
0.51	1.9802	1.3267	3.9405	0.4999	2.1714
0.6	1.8171	1.2732	3.4425	0.4986	2.1678
0.7	1.6587	1.2224	2.9453	0.4935	2.1527
0.8	1.5129	1.1762	2.4772	0.4823	2.1139
0.9	1.3652	1.1292	2.0000	0.4579	2.0148
0.99	1.1746	1.0659	1.4188	0.3841	1.6875

This table shows that the median, lower and upper quartiles of the ICB distribution are monotonically decreasing functions of the parameter ω, while the skewness and kurtosis are monotonically increasing functions for $\omega \in [0, 0.5)$ and decreasing for $\omega \in (0.5, 1)$.

3. Parameter Estimation, Simulations and Applications

3.1. Parameter Estimation

Here, we adapt the maximum likelihood estimation method in estimating the parameter ω of the ICB distribution, which is supposed to be unknown. Let $x_1, x_2, \ldots, x_n$ be n independent observations of size n from a random variable X following the ICB distribution. Then, the likelihood function associated to Equation (2.1) is specified by

$$
\begin{aligned}
L(\omega; x_1, \ldots, x_n) &= \prod_{i=1}^{n} f_{ICB}(x_i; \omega) \\
&= \begin{cases} \left[\displaystyle\prod_{i=1}^{n} \frac{1}{x_i^2}\right], & \omega = \dfrac{1}{2}, \\ c_\omega^n \left[\displaystyle\prod_{i=1}^{n} \frac{1}{x_i^2}\right] \omega^{\sum_{i=1}^{n} \frac{1}{x_i}} (1-\omega)^{n - \sum_{i=1}^{n} \frac{1}{x_i}}, & \omega \in (0,1)/\left\{\dfrac{1}{2}\right\}, \end{cases}
\end{aligned}
\tag{3.1}
$$

and $\min(x_1, \ldots, x_n) \geq 1$.

By taking the natural logarithm of Equation (3.1), the corresponding log-likelihood function is obtained as

$$
\begin{aligned}
\ell(\omega; x_1, \ldots, x_n) &= \sum_{i=1}^{n} \ln[f_{ICB}(x_i; \omega)] \\
&= \begin{cases} -2\displaystyle\sum_{i=1}^{n} \ln(x_i), & \omega = \dfrac{1}{2}, \\ n\ln(c_\omega) - 2\displaystyle\sum_{i=1}^{n} \ln(x_i) + \ln(\omega)\sum_{i=1}^{n} \frac{1}{x_i} + \ln(1-\omega)\left(n - \sum_{i=1}^{n} \frac{1}{x_i}\right), & \omega \in (0,1)/\left\{\dfrac{1}{2}\right\}, \end{cases}
\end{aligned}
\tag{3.2}
$$

and $\min(x_1, \ldots, x_n) \geq 1$. The maximum likelihood estimate (MLE) of ω, say $\hat{\omega}$, can be obtained by maximizing $\ell(\omega; x_1, \ldots, x_n)$ with respect to ω. In our case, this can be achieved by taking the first derivative of Equation (3.2) with respect to ω and equating the corresponding expression to zero, i.e., $\frac{\partial \ell(\omega; x_1, \ldots, x_n)}{\partial \omega} = 0$.

3.2. Monte Carlo Simulation Study

One of the most important aspects of any statistical model is the performance of its parameter estimate(s). Here we perform a Monte Carlo simulation study to examine the asymptotic behavior of the MLE of ω from the ICB distribution. To achieve this, random samples were generated from the ICB distribution using Equation (2.2). The simulation is repeated 1000 times for different sample sizes ($n = 30,\ 50,\ 100,\ 200$ and 500) and different choices of the parameter value ($\omega = 0.1,\ 0.4,\ 0.6$ and 0.8). The performance of the MLE $\hat{\omega}$ is examined in terms of mean estimate, average bias, mean square error (MSE), and coverage probability. The numerical computation of these quantities is displayed in Tables 2, 3, 4 and 5, respectively.

Table 2: Mean estimate of the MLE $\hat{\omega}$

n	$\omega = 0.1$	$\omega = 0.4$	$\omega = 0.6$	$\omega = 0.8$
30	0.1116	0.4023	0.5984	0.7801
50	0.1084	0.4021	0.5977	0.7908
100	0.1042	0.4014	0.5968	0.7940
200	0.1037	0.4007	0.5946	0.7960
500	0.1013	0.4006	0.6002	0.7999

Table 3: Average bias of the MLE $\hat{\omega}$

n	$\omega = 0.1$	$\omega = 0.4$	$\omega = 0.6$	$\omega = 0.8$
30	0.0116	0.0023	-0.0016	-0.0198
50	0.0084	0.0021	-0.0023	-0.0091
100	0.0042	0.0014	-0.0032	-0.0059
200	0.0037	0.0007	-0.0054	-0.0040
500	0.0013	0.0006	0.0002	-0.0006

Table 4: MSE of the MLE $\hat{\omega}$

n	$\omega = 0.1$	$\omega = 0.4$	$\omega = 0.6$	$\omega = 0.8$
30	0.0045	0.0208	0.0202	0.0119
50	0.0028	0.0126	0.0137	0.0063
100	0.0012	0.0069	0.0069	0.0034
200	0.0007	0.0035	0.0034	0.0017
500	0.0002	0.0013	0.0013	0.0006

Table 5: Coverage probability of the $100(1-\alpha)\%$ confidence interval of the MLE $\hat{\omega}$

$1-\alpha$	n	$\omega = 0.1$	$\omega = 0.4$	$\omega = 0.6$	$\omega = 0.8$
0.95	30	0.910	0.896	0.898	0.903
	50	0.916	0.920	0.909	0.921
	100	0.932	0.933	0.931	0.943
	200	0.947	0.942	0.946	0.951
	500	0.945	0.954	0.955	0.952
0.90	30	0.879	0.856	0.843	0.864
	50	0.870	0.876	0.858	0.884
	100	0.896	0.875	0.871	0.883
	200	0.903	0.885	0.892	0.904
	500	0.904	0.901	0.906	0.906

Remarks:

i.) In Table 2, the data reveal a converging trend as n increases, with the mean estimate $\hat{\omega}$ approaching the true parameter value.

ii.) The findings in Table 3 shed light on the relationship between sample size and bias. As n increases, we observe a corresponding decrease (increase) in average bias. In addition, this table illustrates the existence of both negative and positive biases for the MLE $\hat{\omega}$.

iii.) The results in Table 4 show a notable trend as n increases, with the MSE steadily approaching zero.

iv.) Table 5 provides insights into the coverage probability of two different confidence intervals for the estimator $\hat{\omega}$. As n increases, our observations indicate a convergence of the coverage probability towards the nominal levels associated with the 95% and 90% confidence intervals, respectively.

3.3. Applications

In this part, we assess the suitability of the ICB distribution in the context of a real-life scenario, employing two distinct data sets. Specifically, we examine the fits of the Pareto and New Pareto (NP) distributions. The NP distribution is derived from [1]. Both of them share the same support interval of $[1, +\infty)$ with the ICB distribution. These distributions are evaluated for their capacity to model the data sets alongside the ICB distribution.

Data sets:
Data set I comprises 31 recorded flood peak exceedances (measured in m^3/s) for the Wheaton River in the vicinity of Carcross, located within the Yukon Territory, Canada. The data set spans the years from 1958 to 1984. For this data set, the authors in [5] conducted an investigation employing this specific data set to assess the suitability of the generalized Pareto distribution. In a separate study, the authors in [6] harnessed this same data set to elucidate the adaptability of a generalized Lindley distribution. The data set is organized and presented as follows: 2.8, 14.1, 9.9, 10.4, 10.7, 30.0, 3.6, 5.6, 30.8, 13.3, 4.2, 25.5, 3.4, 11.9, 21.5, 27.6, 36.4, 2.7, 64.0, 1.5, 2.5, 27.4, 1.0, 27.1, 20.2, 16.8, 5.3, 9.7, 27.5, 2.5, 27.

On the other hand, Data set II consists of the time-to-failure (103 h) of turbocharger of one type of engine given in [13]. The data set is shown as follows: 1.6, 2.0, 2.6, 3.0, 3.5, 3.9, 4.5, 4.6, 4.8, 5.0, 5.1, 5.3, 5.4, 5.6, 5.8, 6.0, 6.0, 6.1, 6.3, 6.5, 6.5, 6.7, 7.0, 7.1, 7.3, 7.3, 7.3, 7.7, 7.7, 7.8, 7.9, 8.0, 8.1, 8.3, 8.4, 8.4, 8.5, 8.7, 8.8, 9.0.

To facilitate model comparison, we employ the following information criteria: the maximized log-likelihood (ℓ^*), the Akaike information criterion (AIC), the corrected Akaike information criterion ($AICc$), the Bayesian information criterion (BIC), and the Hannan-Quinn information criterion ($HQIC$). These criteria are rigorously defined as follows:

$$AIC = -2\ell^* + 2k, \qquad AICc = AIC + \frac{2k\,(k+1)}{n-k-1},$$

$$BIC = -2\ell^* + k\ln(n)\,, \qquad HQIC = -2\ell^* + 2k\ln\left[\ln(n)\right],$$

where n is the sample size and k is the number of parameter(s) in the considered model.

A smaller value of AIC, $AICc$, BIC and $HQIC$ indicates a better fit of the respective distributions to the analyzed data set. Table 6 provides a comprehensive summary of the goodness of fit results for the ICB, Pareto and NP models applied to the two data sets considered.

Table 6: Summary results for Data sets I and II

Models	MLE	ℓ^*	AIC	$AICc$	BIC	$HQIC$
			Data set I			
ICB	$\hat{\omega}$ = 0.004	-120.8408	243.6816	243.6897	247.8962	245.3354
NP	$\hat{\alpha}$ = 0.6305	-125.1094	252.2187	252.2268	256.4333	253.8725
Pareto	$\hat{\theta}$ = 0.4054	-135.4429	272.8858	272.8939	277.1005	274.5397
			Data set II			
ICB	$\hat{\omega}$ = 0.0058	-114.3882	230.7765	230.7845	234.9911	232.4303
NP	$\hat{\alpha}$ = 0.8611	-126.3683	254.7366	254.7447	258.9512	256.3904
Pareto	$\hat{\theta}$ = 0.5165	-143.8719	289.7438	289.7518	293.9384	291.3976

From this table, it can be seen that the AIC, $AICc$, BIC and $HQIC$ values associated with the ICB distribution show a significant reduction compared to those of the Pareto and NP distributions. As a result, the ICB distribution consistently outperforms its competitors in analising the two data sets considered.

4. Conclusion

In conclusion, the introduction of the ICB distribution represents an advance in the field of probability distributions. This considered modification has allowed us to extend the properties of the CB distribution to the interval $[1, +\infty)$ without the need for additional parameters. Through a study of its mathematical properties, including quantiles, we have laid the foundation for understanding and using this novel distribution.

Our article has also shown that the ICB distribution can be effectively used in statistical modeling. The use of maximum likelihood estimation for inference proved to be a robust and practical approach. Through a thorough simulation study, we provided empirical evidence of the model's performance, which was consistently superior to both the Pareto and new Pareto models when applied to diverse real-world data sets. These results show the potential of the ICB distribution as a valuable tool for modeling data in various domains. This article provides the foundations for further exploration and adoption of the ICB distribution in statistical and data analysis contexts.

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