MALAYA JOURNAL OF MATEMATIK
Malaya J. Mat. **12(02)**(2024), 186–192.
http://doi.org/10.26637/mjm1202/004

Exploring new proofs for three important trigonometric inequalities

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*R*eceived 25 October 2023; *A*ccepted 24 March 2024

Abstract. In this concise article, we present alternative proofs of three significant inequalities relating to various trigonometric functions. The key ingredients of these proofs are well-known series expansions defined with Bernoulli numbers. We are thus contributing to the development of this technique to establish precise inequalities. In some sense, our results provide a simplified overview of these fundamental mathematical relationships.

AMS Subject Classifications: 33B10, 26D05.

Keywords: Trigonometric functions; Bernoulli's numbers

Contents

1. Introduction

There are different methods for determining the characteristics of an inequality, each offering a valuable strategy for understanding the behavior of mathematical functions. One approach, often used in calculus, is the derivative test. This method involves examining the sign of the derivative of a function to determine whether it is increasing or decreasing over a given interval. By analyzing critical points and concavity intervals, the derivative test provides valuable information about the behavior of functions and their associated inequalities.

Another contemporary method of studying inequality is to use series expansion techniques. By expanding a function into an integer series, one can better understand its behavior and derive inequalities based on the properties of the series. This method is particularly useful for exploring inequalities in the context of complex functions and their convergent properties. For a more comprehensive understanding of these methods and their applications in inequality analysis, we may refer to the following references: [6], [7], and [8]. In them, detailed explanations and examples to aid in the study of inequalities and their associated mathematical concepts are provided.

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In particular, Guo et al. [11] presented an alternative proof for the following double-sided inequalities, contributing to the examination of trigonometric inequalities and their properties:

$$2+\frac{8}{45}t^3\tan t<\left(\frac{\sin t}{t}\right)^2+\frac{\tan t}{t}<2+\left(\frac{2}{\pi}\right)^4 t^3\tan t,\qquad 0<t<\frac{\pi}{2}.$$

Still in the spirit of proposing an alternative proof, Nantomah [12] reestablished the following hyperbolic inequalities:

$$\left(\frac{\sinh t}{t}\right)^{2a}+\left(\frac{\tanh t}{t}\right)^{a}>\left(\frac{t}{\sinh t}\right)^{2a}+\left(\frac{t}{\tanh t}\right)^{a}>2,\qquad t>0\;and\;a\geq 1.$$

On the other hand, Zhu [10] gives the very simple alternative proof of the following inequality:

$$\left(\frac{\sin t}{t}\right)^2+\frac{\tan t}{t}>2,\qquad 0<t<\frac{\pi}{2}.$$

This inequality is known as Wilker's inequality.
Later, Zhu and Zhang [9] gave a new concise proof of the following inequalities with the help of power series expansion of trigonometric functions:

$$\frac{16}{\pi^2}t^3\tan t<\left(\frac{\sin t}{t}\right)^2+\frac{\tan t}{t}-2<\frac{8}{45}t^3\tan t,\qquad 0<t<\frac{\pi}{2}.$$

Thus, the principle of alternative proof is central to understanding all the mathematical facets of inequalities. In order to explain our contribution in this direction, some existing results need to be presented. As remarkable advances in the field, the following three inequalities (in theorem form) are elucidated in [13]:

Theorem 1.1. *For $0<t<\pi/2$, the following inequalities*

$$\left[\sqrt{\frac{1+\cos t}{2}}\right]^{4/3}<\frac{\sin t}{t}<\left[\sqrt{\frac{1+\cos t}{2}}\right]^{\gamma}$$

hold true, where $\gamma=2\ln(\pi/2)/\ln 2\approx 1.30299$.

Theorem 1.2. *For $0<t<\pi/2$, we have*

$$t\tan\frac{t}{2}<\ln\left(\frac{1}{\cos t}\right).$$

Theorem 1.3. *For $0<t<\pi/2$, we have*

$$\ln\left(\frac{t}{\sin t}\right)<\frac{\sin t-t\cos t}{2\sin t}.$$

In [13], Bhayo, Ali, and Sándor established the validity of these inequalities using the concept of monotonicity, thus demonstrating their importance in mathematical analysis. The main goal of this concise article is to provide alternative proofs for Theorems 1.1, 1.2 and 1.3. Our approach relies on power series expansions to demonstrate their validity, which remain new in the literature to the best of our knowledge. We hope that this approach sheds more light on the fundamental principles underlying these inequalities and will be inspirational in future proofs.

The remainder of the paper is organized into three distinct sections. Section 2 provides essential preliminaries and introduces a pivotal lemma, while Section 3 details the alternative proofs. A conclusion is given in Section 4.

2. Preliminaries and Key Lemma

The Bernoulli numbers represent a crucial sequence of rational numbers. Several researchers are actively studying them to solve various mathematical problems. These numbers have found applications in various areas of mathematics, including number theory, combinatorics, and mathematical analysis. In this context, researchers explore the properties and relationships of Bernoulli numbers to discover deeper insights into mathematical structures and phenomena. A notable application of Bernoulli numbers is in their role in integer series expansions for various trigonometric functions. Through the study of Bernoulli numbers, researchers have discovered elegant expressions and relationships that facilitate the derivation of such expansions. These expansions play a fundamental role in mathematical analysis, allowing the representation of trigonometric functions as infinite series. The importance of these results is underlined in several references. In particular, [1], [2], [4], and [5] provide comprehensive information on integer series expansions derived from Bernoulli numbers. These works offer detailed explanations and mathematical proofs to support their claims.

In our current work, we exploit these established results as essential tools in our main proofs. Using integer series expansions derived from Bernoulli numbers, we aim to elucidate key mathematical relationships and advance our understanding of the underlying mathematical structures. Especially, the following inequalities will be used in our main proofs:

$$\cot t = \frac{1}{t} - \sum_{n=1}^{\infty} \frac{2^{2n}|B_{2n}|t^{2n-1}}{(2n)!}, \qquad 0 < |t| < \pi, \tag{2.1}$$

$$\ln\left(\frac{\sin t}{t}\right) = -\sum_{n=1}^{\infty} \frac{2^{2n-1}|B_{2n}|t^{2n}}{n(2n)!}, \qquad 0 < |t| < \pi, \tag{2.2}$$

$$\ln \cos t = -\sum_{n=1}^{\infty} \frac{2^{2n-1}(2^{2n}-1)|B_{2n}|t^{2n}}{n(2n)!}, \qquad |t| < \frac{\pi}{2}, \tag{2.3}$$

$$\frac{1}{\sin^2 t} = \frac{1}{t^2} + \sum_{n=1}^{\infty} \frac{(2n-1)2^{2n}|B_{2n}|t^{2n-2}}{(2n)!}, \qquad 0 < |t| < \pi, \tag{2.4}$$

$$\tan t = \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-1)|B_{2n}|t^{2n-1}}{(2n)!}, \qquad |t| < \frac{\pi}{2} \tag{2.5}$$

and

$$\sec^2 t = \sum_{n=1}^{\infty} \frac{(2n-1)2^{2n}(2^{2n}-1)|B_{2n}|t^{2n-2}}{(2n)!}, \qquad |t| < \frac{\pi}{2}. \tag{2.6}$$

In addition, the following technical lemma will play an important role in one of our proofs.

Lemma 2.1. *[3] For $0 < R \leq \infty$, let $A(t) = \sum_{n=0}^{\infty} a_n t^n$ and $B(t) = \sum_{n=0}^{\infty} b_n t^n$ be two real power series converging on the interval $(-R, R)$. If the sequence $(a_n/b_n)_n$ is increasing(decreasing) and $b_n > 0$ for all n, then the ratio function $A(t)/B(t)$ is also increasing(decreasing) on $(0, R)$.*

We are now in a position to prove Theorems 1.1, 1.2 and 1.3 in alternative manners through the use of derivatives, Bernoulli's series expansions, and Lemma 2.1 when appropriate.

3. Proofs

Let us prove Theorems 1.1, 1.2 and 1.3, in turns.

3.1. Proof of Theorem 1.1

To prove this result, let us consider the function

$$F(t)=\frac{\ln\left(\frac{t}{\sin t}\right)}{\ln\left(\frac{1}{\sqrt{(1+\cos t)/2}}\right)}=\frac{\ln\left(\frac{t}{\sin t}\right)}{\ln\left(\frac{1}{\sqrt{\cos^2\frac{t}{2}}}\right)}=\frac{\ln\left(\frac{t}{\sin t}\right)}{\ln\left(\frac{1}{\cos\frac{t}{2}}\right)}=\frac{A(t)}{B(t)}, \tag{3.1}$$

where

$$A(t)=\ln\left(\frac{t}{\sin t}\right)=\sum_{n=1}^{\infty}\frac{2^{2n-1}|B_{2n}|t^{2n}}{n(2n)!}=\sum_{n=1}^{\infty}a_n t^{2n},$$

where

$$a_n=\frac{2^{2n-1}|B_{2n}|}{n(2n)!}$$

and

$$B(t)=\ln\left(\frac{1}{\cos\frac{t}{2}}\right)=\sum_{n=1}^{\infty}\frac{2^{2n-1}(2^{2n}-1)|B_{2n}|t^{2n}}{2^{2n}n(2n)!}=\sum_{n=1}^{\infty}b_n t^{2n}$$

where

$$b_n=\frac{2^{2n-1}(2^{2n}-1)|B_{2n}|}{2^{2n}n(2n)!}.$$

Let us now set

$$\begin{aligned}c_n&=\frac{b_n}{a_n}\\&=\frac{2^{2n-1}(2^{2n}-1)|B_{2n}|}{2^{2n}n(2n)!}\Bigg/\frac{2^{2n-1}|B_{2n}|}{n(2n)!}=\frac{2^{2n}-1}{2^{2n}}.\end{aligned}$$

Clearly c_n is increasing for $n\geq 1$.
Therefore, by Lemma 2.1, $B(t)/A(t)$ is strictly increasing and $A(t)/B(t)$ is strictly decreasing, so $F(t)$ too. This implies that $F(\pi/2)<F(t)<F(0)$. Since $\lim_{t\to 0}F(t)=\frac{4}{3}$ and

$$\lim_{t\to\pi/2}F(t)=\frac{2\ln\left(\frac{\pi}{2}\right)}{\ln 2}=1.30299=\gamma.$$

It follows from Equation (3.1) that

$$\left[\sqrt{\frac{1+\cos t}{2}}\right]^{4/3}<\frac{\sin t}{t}<\left[\sqrt{\frac{1+\cos t}{2}}\right]^{\gamma}.$$

This ends this alternative proof. □

3.2. Proof of Theorem 1.2

Let us set

$$f(t) = \ln\left(\frac{1}{\cos t}\right) - t\tan\frac{t}{2} = -\ln(\cos t) - t\tan\frac{t}{2}.$$

Therefore, we have

$$f'(t) = \tan t - \tan\frac{t}{2} - \frac{t}{2}\sec^2\frac{t}{2}.$$

Owing to Equations (2.5) and (2.6), we have

$$\begin{aligned}
f'(t) &= \sum_{n=1}^{\infty}\frac{2^{2n}(2^{2n}-1)|B_{2n}|t^{2n-1}}{(2n)!} - \sum_{n=1}^{\infty}\frac{2^{2n}(2^{2n}-1)|B_{2n}|t^{2n-1}}{2^{2n-1}(2n)!}\\
&\quad - \frac{t}{2}\sum_{n=1}^{\infty}\frac{(2n-1)2^{2n}(2^{2n}-1)|B_{2n}|t^{2n-2}}{2^{2n-2}(2n)!}\\
&= \sum_{n=1}^{\infty}\frac{2^{2n}(2^{2n}-1)|B_{2n}|t^{2n-1}}{(2n)!} - \sum_{n=1}^{\infty}\frac{2(2^{2n}-1)|B_{2n}|t^{2n-1}}{(2n)!}\\
&\quad - \sum_{n=1}^{\infty}\frac{(2n-1)2(2^{2n}-1)|B_{2n}|t^{2n-1}}{(2n)!}\\
&= \sum_{n=1}^{\infty}\frac{(2^{2n}-1)|B_{2n}|t^{2n-1}}{(2n)!}\left[2^{2n}-2-2(2n-1)\right]\\
&= \sum_{n=1}^{\infty}\frac{(2^{2n}-1)|B_{2n}|t^{2n-1}}{(2n)!}\left(2^{2n}-4n\right).
\end{aligned}$$

For $n \geq 1$, it is clear that $2^{2n} \geq 4n$. This implies that $f'(t) > 0$, so $f(t)$ is strictly increasing. In particular, we have $f(t) > f(0) = 0$, which is equivalent to

$$t\tan\frac{t}{2} < \ln\left(\frac{1}{\cos t}\right).$$

This ends this alternative proof. □

3.3. Proof of Theorem 1.3

Let us set

$$f(t) = \ln\left(\frac{t}{\sin t}\right) - \frac{\sin t - t\cos t}{2\sin t}.$$

Hence, after some developments, we establish that

$$\begin{aligned}
f'(t) &= \frac{\sin t}{t}\left[\frac{\sin t - t\cos t}{\sin^2 t}\right] - \frac{1}{2}\left[\frac{\sin t(\cos t - \cos t + t\sin t) - \cos t(\sin t - t\cos t)}{\sin^2 t}\right]\\
&= \frac{\sin t - t\cos t}{t\sin t} - \frac{1}{2}\left[\frac{t\sin^2 t - \sin t\cos t + t\cos^2 t}{\sin^2 t}\right]\\
&= \frac{1}{t} - \cot t - \frac{1}{2}\left[\frac{t - \sin t\cos t}{\sin^2 t}\right] = \frac{1}{t} - \cot t - \frac{t}{2\sin^2 t} + \frac{\cot t}{2}\\
&= \frac{1}{t} - \frac{\cot t}{2} - \frac{t}{2\sin^2 t}.
\end{aligned}$$

Owing to Equations (2.1) and (2.4), we have

$$f'(t) = \frac{1}{t} - \frac{1}{2}\left[\frac{1}{t} - \sum_{n=1}^{\infty} \frac{2^{2n}|B_{2n}|t^{2n-1}}{(2n)!}\right] - \frac{t}{2}\left[\frac{1}{t^2} + \sum_{n=1}^{\infty} \frac{2^{2n}(2n-1)|B_{2n}|t^{2n-2}}{(2n)!}\right]$$

$$= \sum_{n=2}^{\infty} \frac{2^{2n}|B_{2n}|t^{2n-1}}{2(2n)!} - \sum_{n=2}^{\infty} \frac{2^{2n}(2n-1)|B_{2n}|t^{2n-1}}{2(2n)!}$$
$$= \sum_{n=2}^{\infty} \frac{2^{2n}|B_{2n}|t^{2n-1}}{(2n)!}(1-n).$$

It is clear that $1-n<0$ for $n \geq 2$, implying that $f'(t)<0$. Hence $f(t)$ is a strictly decreasing function and, in particular, $f(t)<f(0)=0$, so

$$\ln\left(\frac{t}{\sin t}\right) < \frac{\sin t - t\cos t}{2\sin t}.$$

This ends this alternative proof. □

4. Conclusion

In this concise article, we have reestablished important existing theorems in the area of trigonometric inequalities, with an approach involving series expansions based on Bernoulli numbers. In some sense, this extends the applicability of such series expansions to explore comprehensive mathematical results beyond conventional methodologies.

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