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New subfamilies of univalent functions defined by Opoola differential operator and connected with modified Sigmoid function

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This paper is dedicated to the occasion of Professor Gaston M. N'Guérékata's 70th birthday

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Abstract. In this exploration, by making use of the Hadamard product of Opoola differential operator and modified sigmoid function, we define new subclasses of analytical and univalent functions $\mathcal{T}_n S^k(\phi, \beta, \xi, \lambda, \delta, L, M, \mu, \rho, \omega)$ and $\mathcal{T}_n \mathcal{V}^k(\phi, \beta, \xi, \lambda, \delta, L, M, \mu, \rho, \omega)$ and discussed some properties of the classes; such as the coefficient estimates, Growth and Closure theorems.
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Contents

1. Introduction and Background

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk and $\mathcal{B}$ denote the class of functions $f(z)$ which are analytic in the open unit disk and of the form

$$f(z) = z + \sum_{t=2}^{\infty} a_t z^t. \tag{1.1}$$

Also, let

$$\gamma(s) = \frac{2}{(1+e^{-s})}; \qquad s \geq 0 \tag{1.2}$$

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with $\gamma(s)=1$ for $s=0$ be the modified Sigmoid function. (See details in [6], [7], [10], [5], [11], [15]).

Additionally, let $T\in\mathcal{A}$ be the class of functions of the form

$$f(z)=z-\sum_{t=2}^{\infty}a_tz^t,\qquad a_t\geq 0 \tag{1.3}$$

For $f_\gamma(z)\in\mathcal{T}_\gamma$, [11] gave the following definition:

$$f_\gamma(z)=z-\sum_{t=2}^{\infty}\gamma(s)\,a_tz^t,\qquad a_t\geq 0 \tag{1.4}$$

as a consequence of (1.3).

Note that $\gamma(s)=1+\frac{1}{2}s-\frac{1}{24}s^3+\frac{1}{240}s^5-\frac{17}{40320}s^7+\cdots$ defined by (1.2). Furthermore, we define identity function for $\mathcal{T}_\gamma$ as

$$e_\gamma(z)=z. \tag{1.5}$$

For the purpose of defining the new differential operator of interest, the following definitions are required:

Definition 1.1. *[12] For $f(z)\in\mathcal{A}$, where $k\geq 0, 1\leq\mu\leq\rho, n\in\mathbb{N}_0$ and $z\in U$, the Opoola differential operator $D_k^n(\mu,\rho)f:\mathcal{A}\to\mathcal{A}$ is defined in [12] as*

$$\begin{aligned}&D_0^k(\mu,\rho)f(z)=f(z)\\&D_1^k(\mu,\rho)f(z)=tzf'(z)-z(\rho-\mu)k+(1+(\rho-\mu-1)k)f(z)\\&D_n^k(\mu,\rho)f(z)=(D(D_k^{n-1}f(z))).\end{aligned}$$

The $f(z)$ given in above (1.1) we get,

$$D_k^n(\mu,\rho,)f(z)=z+\sum_{t=2}^{\infty}[1+(t+\rho-\mu-1)k]^n\,a_tz^t. \tag{1.6}$$

It is evident that (1.6) reduces to Al- Oboudi differential operator [1, 3] and Salagean differential operator [22] by varying the involving parameters appropriately. We further note that, other works on (1.16) can be found in [4, 14, 16–18, 20–24, 26].

Definition 1.2. *[11] introduced the generalized differential operator $D_{\lambda,\omega}^nf_\gamma(z)$ involving sigmoid function which is a special case of (1.6):*

$$D_{\lambda,\omega}^nf_\gamma(z)=\gamma^n(s)z-\sum_{t=2}^{\infty}\gamma^{n+1}(s)[(t-1)(\lambda-\omega)+t]^n\,a_tz^t \tag{1.7}$$

for $\lambda,\omega\geq 0$. For more information on this, interested reader may refer to [8].

Definition 1.3 (Hadamard product or convolution). *The Hadamard (or convolution) of two analytic functions $f(z)$ given by (1.1) and $g(z)=z+\sum_{t=2}^{\infty}b_tz^t$ is given by*

$$f(z)*g(z)=(f*g)(z)=z+\sum_{t=2}^{\infty}a_tb_t\,z^t,\qquad z\in U. \tag{1.8}$$

Following (1.8) for (1.6) and (1.7), a certain new differential operator associated with Sigmoid function involving convolution is defined as follows:

$$\begin{aligned}&D_{\lambda,\omega}^n(\mu,\rho,k)f_\gamma(z)=(D_{\lambda,\omega}^nf_\gamma(z))*(D^n(\mu,\rho,k)f_\gamma(z))\\&=\gamma^n(s)z+\sum_{t=2}^{\infty}\gamma^{n+1}(s)[1+(t+\rho-\mu-1)k)]^n[(t-1)(\lambda-\omega)+t]^na_tz^t\end{aligned} \tag{1.9}$$

Remark 1.1. *When* $\gamma^n(s) = 1,\ \lambda = 1,\ \omega = 2$ *we have*

$$D^n_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z) = \gamma^n(s)z + \sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^n a_t z^t. \qquad (1.10)$$

Remark 1.2. *When* $\gamma^n(s) = 1,\ \ for\ s = 0,\ \lambda = 1,\ \omega = 2,\ \mu = \rho,\ k = 1$ *we have the Salagean differential operator, see [15] and [8].*

Remark 1.3. *When* $\gamma^n(s) = 1,\ \lambda = 1,\ \omega = 2,\ \mu = \rho,$ *we have the Al-Oboudi differential operator, see [3] and [25].*

For the purpose of the main result, we rewrite equation (1.9) as follows for convenience

$$D^\beta_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z) = \gamma^n(s)z + \sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^\beta a_t z^t \qquad (1.11)$$

where $\beta,\lambda,\omega \in \mathbb{R},\ \beta \geq\ 0,\ \lambda \geq 0,\ \omega \geq 0$ and it is denoted by $D^\beta_{\lambda,\omega}(\mu,\rho,k) : \mathcal{A} \to \mathcal{A}$.

Also, if $f \in C,\ f(z) = z - \sum_{t=2}^{\infty} a_t z^t, \quad a_t \geq 0,\ t = 2,\ z \in U.$

Then,

$$D^\beta_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z) = z - \sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^\beta a_t z^t \qquad (1.12)$$

For convenience upon (1.11) we have the following definition

Definition 1.4. *We say that class* $\mathcal{T}_n\, S^k_p(\phi,\beta,\xi,\lambda,\delta,L,M,\mu,\rho,\gamma,\omega)$ *contain function* $f(z) \in \mathcal{T}$ *if and only if*

$$\left| \frac{\frac{D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)}{D^{\beta}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)} - 1}{(M-L)\xi\Big(\frac{D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)}{D^{\beta}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)} - \phi\Big) + L\lambda\Big(\frac{D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)}{D^{\beta}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)} - 1\Big)} \right| < \delta \qquad (1.13)$$

where $|z| < 1,\ 0 < \delta \leq 1,\ \frac{1}{2} \leq \xi \leq 1,\ k \geq 0,\ \mu \geq 0,\ \rho \geq 0, 0 \leq \phi \leq \frac{1}{2}\xi,\ \frac{1}{2} \leq \lambda \leq 1,\ \beta \geq 0,\ 0 < M \leq 1,\ -1 \leq L < M \leq 1,\ \omega \geq 0,\ \gamma^{n+1}(s) = 1,\ s = 0.$

Definition 1.5. *The class* $\mathcal{T}_n\, \mathcal{V}^k(\phi,\beta,\xi,\lambda,\delta,L,M,\mu,\rho,\gamma,\omega)$ *contain function* $f(z) \in \mathcal{T}$ *if and only if*

$$\left| \frac{\frac{D^{\beta+2}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)}{D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)} - 1}{(M-L)\xi\Big(\frac{D^{\beta+2}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)}{D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)} - \phi\Big) + L\lambda\Big(\frac{D^{\beta+2}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)}{D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)} - 1\Big)} \right| < \delta \qquad (1.14)$$

where $|z| < 1,\ 0 < \delta \leq 1,\ \frac{1}{2} \leq \xi \leq 1,\ k \geq 0,\ \mu \geq 0,\ \rho \geq 0, 0 \leq \phi \leq \frac{1}{2}\xi,\ \frac{1}{2} \leq \lambda \leq 1,\ \beta \geq 0,\ 0 < M \leq 1,\ -1 \leq L < M \leq 1,\ \omega \geq 0,\ \gamma^{n+1}(s) = 1,\ s = 0.$

Remark 1.4. *After putting* $\mu = \rho = 1,\ \gamma^n(s) = 1,\ \lambda = 1,\ \omega = 2,$ *we obtain the corresponding results of [25].*

Remark 1.5. *After putting* $\beta = 0,\ \mu = \rho = 1,\ \gamma^n(s) = 1,\ \lambda = 1,\ \omega = 2,$ *we get the corresponding sequal obtained by [9].*

Remark 1.6. *After putting* $\beta = 0,\ \mu = \rho = 1,\ k = 1,\ \lambda = 1,\ \omega = 2,$ *we get the corresponding sequal obtained by [2].*

Remark 1.7. *After putting* $\beta = 0,\ \mu = \rho = 1,\ k = 1,\ \xi = 1,\ \lambda = 1,\ \omega = 2,$ *we get the corresponding sequal obtained by [13].*

Let equation (1.11) be $\left|\frac{A}{B}\right| < \delta$

$$\left|\frac{A}{B}\right| < \delta = \frac{|A|}{|B|} < \delta \Rightarrow |A| < \delta|B| \tag{1.15}$$

when

$$A = \frac{D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_{\gamma}(z)}{D^{\beta}_{\lambda,\omega}(\mu,\rho,k)f_{\gamma}(z)} - 1 \tag{1.16}$$

$$A = \frac{D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_{\gamma}(z) - D^{\beta}_{\lambda,\omega}(\mu,\rho,k)f_{\gamma}(z)}{D^{\beta}_{\lambda,\omega}(\mu,\rho,k)f_{\gamma}(z)} \tag{1.17}$$

$$\begin{aligned} B = (M-L)\xi &\left(\frac{D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_{\gamma}(z) - \phi D^{\beta}_{\lambda,\omega}(\mu,\rho,k)f_{\gamma}(z)}{D^{\beta}_{\lambda,\omega}(\mu,\rho,k)f_{\gamma}(z)}\right) + \\ L\lambda &\left(\frac{D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_{\gamma}(z) - D^{\beta}_{\lambda,\omega}(\mu,\rho,k)f_{\gamma}(z)}{D^{\beta}_{\lambda,\omega}(\mu,\rho,k)f_{\gamma}(z)}\right) \end{aligned} \tag{1.18}$$

$$\left|\frac{A}{B}\right| = \frac{D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_{\gamma}(z) - D^{\beta}_{\lambda,\omega}(\mu,\rho,k)f_{\gamma}(z)}{\begin{aligned}&\Big|(M-L)\xi[D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_{\gamma}(z) - \phi D^{\beta}_{\lambda,\omega}(\mu,\rho,k)f_{\gamma}(z)] \\ &+L\lambda[D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_{\gamma}(z) - D^{\beta}_{\lambda,\omega}(\mu,\rho,k)f_{\gamma}(z)]\Big|\end{aligned}} \tag{1.19}$$

2. Main Results

2.1. Coefficient Estimates

Theorem 2.1. *The class $\mathcal{T}_n\, S^k_p(\phi,\beta,\xi,\lambda,\delta,L,M,\mu,\rho,\omega)$ contains a function $f(z)$ defined by (1.3) if and only if*

$$\begin{aligned} &\sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}\Big\{(t+\rho-\mu-1)k \\ &\qquad [(t-1)(\lambda-\omega)+t]\,[1+L\lambda\delta+(M-L)\delta\xi]+(M-L)\delta\xi(1-\phi)\Big\}\,a_t \\ &\qquad \le (M-L)\delta\xi(1-\phi) \end{aligned} \tag{2.1}$$

Proof. Suppose,

$$\begin{aligned} &\sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}\Big\{(t+\rho-\mu-1)k \\ &\qquad [(t-1)(\lambda-\omega)+t]\,[1+L\lambda\delta+(M-L)\delta\xi]+(M-L)\delta\xi(1-\phi)\Big\}\,a_t \\ &\qquad \le (M-L)\delta\xi(1-\phi) \end{aligned}$$

We have,

$$\begin{aligned} &\Big|D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_{\gamma}(z) - D^{\beta}_{\lambda,\omega}(\mu,\rho,k)f_{\gamma}(z)\Big| \\ &-\ \delta\Big|(M-L)\xi[D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_{\gamma}(z) - \phi D^{\beta}_{\lambda,\omega}(\mu,\rho,k)f_{\gamma}(z)] \\ &+L\lambda[D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_{\gamma}(z) - D^{\beta}_{\lambda,\omega}(\mu,\rho,k)f_{\gamma}(z)]\Big| < 0 \end{aligned} \tag{2.2}$$

with the provision

$$\begin{aligned}
&\Big| z-\sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta+1}a_t\,z^t \\
&-\ z\ +\sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}a_t\,z^t\Big| \\
&-\ \delta\Big|(M-L)\xi\Big[z-\sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta+1}a_t\,z^t \\
&-\ \phi z+\ \phi\sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}a_t\,z^t\Big] \\
&+\ L\lambda\ \Big[z-\sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta+1}a_t\,z^t \\
&-\ z+\sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}a_t\,z^t\Big]\ \Big|<0
\end{aligned} \tag{2.3}$$

Let $S_1=\sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta+1}a_t\,z^t$ and $S_0=\sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}a_t\,z^t$.

The simplified expression becomes,

$$\Big|S_0-S_1\Big|-\delta\Big|(M-L)\xi[z-S_1-\phi z+\phi S_0]+L\lambda[S_0-S_1]\Big|<0 \tag{2.4}$$

S_0-S_1 would have a negative sign. Thus, it would be more convenient to work with S_1-S_0 since $|S_0-S_1|=|S_1-S_0|$. Hence, we have

$$\Big|S_1-S_0\Big|<\delta\Big|(M-L)\xi[z-S_1-\phi z+\phi S_0]-L\lambda[S_1-S_0]\Big| \tag{2.5}$$

We know that: $|A-B|\geq|A|-|B|$

$$\begin{aligned}
\Big|S_1-S_0\Big|<&\ \delta\Big|(M-L)\xi[z-S_1-\phi z+\phi S_0]-L\lambda[S_1-S_0]\Big| \\
&\geq\delta\Big|(M-L)\xi[z-S_1-\phi z+\phi S_0]\Big|-\delta\Big|L\lambda[S_1-S_0]\Big|
\end{aligned} \tag{2.6}$$

$$\therefore\Big|S_1-S_0\Big|\leq\delta\Big|(M-L)\xi[z-S_1-\phi z+\phi S_0]\Big|-\delta\Big|L\lambda[S_1-S_0]\Big| \tag{2.7}$$

Assuming $L\lambda$ is positive, we have

$$\Big|S_1-S_0\Big|(1+\delta L\lambda)\leq\delta\Big|(M-L)\xi[z-S_1-\phi z+\phi S_0]\Big| \tag{2.8}$$

Rewriting the expression

$$z-S_1-\phi z+\phi S_0=z-S_1-\phi z+\phi S_0+\phi S_1-\phi S_1+S_0-S_0 \tag{2.9}$$

Collect the like terms

$$z-S_1-\phi z+\phi S_0=(1-\phi)(z-S_0)-(S_1-S_0) \tag{2.10}$$

The inequality (2.8) now becomes

$$\Big|S_1 - S_0\Big|(1+\delta L\lambda) \le \delta\Big|(M-L)\xi(1-\phi)(z-S_0) - (S_1-S_0)(M-L)\xi\Big| \tag{2.11}$$

Apply $|A-B| \ge |A| - |B|$, we have

$$\Big|S_1 - S_0\Big|(1+\delta L\lambda) \le \delta\Big|(M-L)\xi(1-\phi)(z-S_0)\Big| - \delta\Big|(M-L)\xi(S_1-S_0)\Big| \tag{2.12}$$

$$\therefore \Big|S_1 - S_0\Big|(1+\delta L\lambda) + \delta(M-L)\xi\Big|(S_1-S_0)\Big| \le \delta\Big|(M-L)\xi(1-\phi)(z-S_0)\Big| \tag{2.13}$$

Expand the right hand side and apply $|A-B| \ge |A| - |B|$

$$\Big|S_1 - S_0\Big|\Big((1+\delta L\lambda) + \delta(M-L)\xi\Big) \le \delta\Big|(M-L)\xi(1-\phi)z\Big| - \delta\Big|(M-L)\xi(1-\phi)S_0)\Big| \tag{2.14}$$

Add $\delta\Big|(M-L)\xi(1-\phi)S_0)\Big|$ to both sides

$$\Big|S_1 - S_0\Big|\Big((1+\delta L\lambda) + \delta(M-L)\xi\Big) + \delta\Big|(M-L)\xi(1-\phi)S_0)\Big| \le \delta\Big|(M-L)\xi(1-\phi)z\Big| \tag{2.15}$$

Therefore, substituting for $|S_1 - S_0|$ Also, remember $|z| = r < 1$

$$\begin{aligned}&\Big|S_1 - S_0\Big| = \sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}\\ &\Bigg(\Big[(t+\rho-\mu-1)k\,(t-1)(\lambda-\omega)+t\Big]\Big((1+\delta L\lambda)+\delta(M-L)\xi\Big)\\ &+\ \delta(M-L)\xi(1-\phi)a_t r^t \le (M-L)\delta\xi(1-\phi)\Bigg)\end{aligned} \tag{2.16}$$

The proof is obtained. Hence, $f(z) \in \mathcal{T}_n\, S_p^k(\phi,\beta,\xi,\lambda,\delta,L,M,\mu,\rho)$. ■

Theorem 2.2 (Second Class)**.** *The class $\mathcal{T}_n\,\mathcal{V}^k(\phi,\beta,\xi,\lambda,\delta,L,M,\mu,\rho,\omega)$ contains a function $f(z) \in \mathcal{T}$ if and only if ,*

$$\left|\frac{\frac{D^{\beta+2}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)}{D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)} - 1}{(M-L)\xi\Big(\frac{D^{\beta+2}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)}{D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)} - \phi\Big) + L\lambda\Big(\frac{D^{\beta+2}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)}{D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)} - 1\Big)}\right| < \delta$$

Proof. Let

$$\left|\frac{\frac{D^{\beta+2}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)}{D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)} - 1}{(M-L)\xi\Big(\frac{D^{\beta+2}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)}{D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)} - \phi\Big) + L\lambda\Big(\frac{D^{\beta+2}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)}{D^{\beta+1}_{\lambda,\omega}(\mu,\rho,k)f_\gamma(z)} - 1\Big)}\right| < \delta \tag{2.17}$$

$$= \left| \frac{\frac{z-\sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta+1}a_t\,z^t}{z-\sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}a_t\,z^t}-1}{(M-L)\xi\Big(\frac{z-\sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta+1}a_t\,z^t}{z-\sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}a_t\,z^t}-\phi\Big) + L\lambda\Big(\frac{z-\sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta+1}a_t\,z^t}{z-\sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}a_t\,z^t}-1\Big)} \right| < \delta \tag{2.18}$$

$$= \left| \frac{\sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta} \times \Big\{(t+\rho-\mu-1)k[(t-1)(\lambda-\omega)+t]\Big\}a_t\,z^t}{(M-L)\delta\xi(1-\phi)-\sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta} \times \Big((M-L)\xi(1-\phi)+(M-L)\xi+L\lambda\Big)\Big\{(t+\rho-\mu-1)k[(t-1)(\lambda-\omega)+t]\Big\}^{\beta}a_t\,z^t} \right| < \delta \tag{2.19}$$

As $|Ref(z)| \leq |z|$ for all z, we have

$$Re \left| \frac{\sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta} \times \Big\{(t+\rho-\mu-1)k[(t-1)(\lambda-\omega)+t]\Big\}a_t\,z^t}{(M-L)\delta\xi(1-\phi)-\sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta} \times \Big((M-L)\xi(1-\phi)+(M-L)\xi+L\lambda\Big)\Big\{(t+\rho-\mu-1)k[(t-1)(\lambda-\omega)+t]\Big\}^{\beta}a_t\,z^t} \right| < \delta \tag{2.20}$$

We choose z on real axis so that $\frac{D^{\beta+1}_{\lambda,\omega}}{D^{\beta}_{\lambda,\omega}}$ is real and clearing the denominator in the above relation and letting

$z \to 1$ over real values, we will get

$$\begin{aligned}
&\sum_{t=2}^{\infty} \gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta} \\
&\Big\{(t+\rho-\mu-1)k[(t-1)(\lambda-\omega)+t]\Big\}\, a_t\, z^t \\
&<\ (M-L)\xi\delta z(1-\phi) - \sum_{t=2}^{\infty} \gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta} \times \\
&(M-L)\delta\xi z(1-\phi) + \Big((M-L)\delta\xi + L\lambda\delta\Big)\Big\{(t+\rho-\mu-1)k[(t-1)(\lambda-\omega)+t]\Big\}\, a_t\, z^t
\end{aligned} \tag{2.21}$$

$$\begin{aligned}
\Rightarrow &\sum_{t=2}^{\infty} \gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta} \\
&\Big\{(t+\rho-\mu-1)k[(t-1)(\lambda-\omega)+t]\Big\}\, a_t\, z^t + \\
&\sum_{t=2}^{\infty} \gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta} \\
&(M-L)\delta\xi z(1-\phi) + \Big((M-L)\delta\xi + L\lambda\delta\Big)\Big\{(t+\rho-\mu-1)k[(t-1)(\lambda-\omega)+t]\Big\}\, a_t\, z^t \\
&<\ (M-L)\xi\delta z(1-\phi)
\end{aligned} \tag{2.22}$$

As $z \to 1$

$$\begin{aligned}
&= \sum_{t=2}^{\infty} \gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta} \\
&\Big[(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t] \\
&[1+L\lambda\delta+(M-L)\delta\xi] + (M-L)\delta\xi(1-\phi)\Big]\, a_t z^t - (M-L)\xi(1-\phi) < 0
\end{aligned} \tag{2.23}$$

Hence, the proof is obtained. ■

2.2. Growth and Distortion Theorem

Theorem 2.3. *If* $f(z) \in \mathcal{T}_n\, S_p^k(\phi,\beta,\xi,\lambda,\delta,L,M,\mu,\rho,\omega)$ *then*

$$r - r^2 \left\{ \frac{(M-L)\delta\xi(1-\phi)}{\begin{array}{l}(\Big\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta} \\ \times\ \Big[\Big\{[(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\} \\ \times\ [1+L\lambda\delta+(M-L)\delta\xi]+(M-L)\delta\xi(1-\phi)\Big]\end{array}} \right\} \le |f(z)|$$

$$\leq r+r^2\left\{\frac{(M-L)\delta\xi(1-\phi)}{\begin{array}{l}(\Big\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}\\ \times\ \Big[\Big\{[(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}\\ \times\ [1+L\lambda\delta+(M-L)\delta\xi]+(M-L)\delta\xi(1-\phi)\Big]\end{array}}\right\}$$

Equality holds for

$$f(z)=z-\left\{\frac{(M-L)\delta\xi(1-\phi)}{\begin{array}{l}(\Big\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}\\ \times\ \Big[\Big\{[(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}\\ \times\ [1+L\lambda\delta+(M-L)\delta\xi]+(M-L)\delta\xi(1-\phi)\Big]\end{array}}\right\}$$

Proof. From Theorem 2.1 we get $f(z)\in\mathcal{T}_n\,S_p^k(\phi,\beta,\xi,\lambda,\delta,L,M,\mu,\rho,\omega)$ if and only if

$$\begin{aligned}&\sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}\times\\&\Big\{\Big[(1+\rho-\mu)k[(t-1)(\lambda-\omega)+t]\Big]\Big\{1+L\lambda\delta+(M-L)\delta\xi\Big\}+(M-L)\delta\xi(1-\phi)\Big\}a_t\\&\leq(M-L)\delta\xi(1-\phi)\end{aligned}\tag{2.24}$$

Let $h=1-\dfrac{(M-L)\delta\xi(1-\phi)}{\Big\{(1+\rho-\mu)k[(t-1)(\lambda-\omega)+t]\Big\}[1+L\lambda\delta+(M-L)\delta\xi]}$

$\therefore f(z)\in\mathcal{T}_n\,S_p^k(\phi,\beta,\xi,\lambda,\delta,L,M,\mu,\rho,\omega)$ if and only if

$$\begin{aligned}&\sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}\times\\&\Big\{(t+\rho-\mu-h)[(t-1)(\lambda-\omega)+t]\Big\}a_t\leq(1-h)\end{aligned}\tag{2.25}$$

when $t=2$,

$$\begin{aligned}&\Big\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}(2+\rho-\mu-h)\sum_{t=2}^{\infty}\gamma^{n+1}(s)\,a_t\leq\\&\sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu-1)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}\times\\&\Big\{(t+\rho-\mu-h)[(t-1)(\lambda-\omega)+t]\Big\}a_t\leq(1-h)\end{aligned}\tag{2.26}$$

$$
\begin{aligned}
|f(z)| &\leq r+\sum_{t=2}^{\infty}\gamma^{n+1}(s)\,a_t r^t \leq r+r^2\sum_{t=2}^{\infty}\gamma^{n+1}(s)\,a_t \\
&\leq r+r^2\Big[\frac{1-h}{\Big\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}(2+\rho-\mu-h)}\Big]
\end{aligned}
\tag{2.27}
$$

Similarly,

$$
\begin{aligned}
|f(z)| &\geq r-\sum_{t=2}^{\infty}\gamma^{n+1}(s)\,a_t r^t \geq r-r^2\sum_{t=2}^{\infty}\gamma^{n+1}(s)\,a_t \\
&\geq r-r^2\Big[\frac{1-h}{\Big\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}(2+\rho-\mu-h)}\Big]
\end{aligned}
\tag{2.28}
$$

So

$$
\begin{aligned}
& r-r^2\Big[\frac{1-h}{\Big\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}(2+\rho-\mu-h)}\Big] \leq |f(z)| \\
& \leq r+r^2\Big[\frac{1-h}{\Big\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}(2+\rho-\mu-h)}\Big]
\end{aligned}
\tag{2.29}
$$

Hence the result,

$$
r-r^2\left\{\frac{(M-L)\delta\xi(1-\phi)}{\begin{array}{l}(\Big\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta} \\ \times\ \Big[\Big\{[(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\} \\ \times\ [1+L\lambda\delta+(M-L)\delta\xi]+(M-L)\delta\xi(1-\phi)\Big]\end{array}}\right\} \leq |f(z)|
$$

$$
\leq r+r^2\left\{\frac{(M-L)\delta\xi(1-\phi)}{\begin{array}{l}(\Big\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta} \\ \times\ \Big[\Big\{[(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\} \\ \times\ [1+L\lambda\delta+(M-L)\delta\xi]+(M-L)\delta\xi(1-\phi)\Big]\end{array}}\right\}
$$

■

Corollary 2.4. *If* $f(z) \in \mathcal{T}_n\, S_p^k(\phi,\beta,\xi,\lambda,\delta,L,M,\mu,\rho,\omega)$ *then* $\beta = 0, \xi = 1, \lambda = 1, \delta = 1, L = -1, M = 1$ *and* $\mu = \rho = 1$.

From the expression above, given $\rho = \mu,\ 1+\rho-\mu = 1+\rho-\rho = 1+0 = 1.$

$$= \frac{(1-\phi)}{k[(t-1)(1-\omega)+t](2-\phi)}$$

Hence the result

$$r - r^2\left\{\frac{(1-\phi)}{k[(t-1)(1-\omega)+t](2-\phi)}\right\} \le |f(z)|$$

$$\le r + r^2\left\{\frac{(1-\phi)}{k[(t-1)(1-\omega)+t](2-\phi)}\right\}$$

Theorem 2.5. *If* $f(z) \in \mathcal{T}_n\, \mathcal{V}^k(\phi,\beta,\xi,\lambda,\delta,L,M,\mu,\rho,\omega)$ *then*

$$r - r^2\left\{\frac{(M-L)\delta\xi(1-\phi)}{\begin{array}{l}\left\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\right\}^{\beta+1}\\ \times\ \left\{[(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\right\}\\ \times\ (1+L\lambda\delta+(M-L)\delta\xi)+(M-L)\delta\xi(1-\phi)\end{array}}\right\} \le |f(z)|$$

$$\le r + r^2\left\{\frac{(M-L)\delta\xi(1-\phi)}{\begin{array}{l}\left\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\right\}^{\beta+1}\\ \times\ \left\{[(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\right\}\\ \times\ (1+L\lambda\delta+(M-L)\delta\xi)+(M-L)\delta\xi(1-\phi)\end{array}}\right\}$$

Proof. Similarly, we can prove this theorem as it is relevant to Theorem 2.3. So it is sufficient to substitute $\beta = \beta + 1$ in the above Theorem 2.3 and the subsequent corollary. ■

Theorem 2.6. *For* $f(z) \in \mathcal{T}_n\, S_p^k(\phi, \beta, \xi, \lambda, \delta, L, M, \mu, \rho, \omega)$ *then*

$$1 - r\left\{\frac{(M-L)^2\delta\xi(1-\phi)}{\begin{array}{l}\Big\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta} \\ \times\ \Big\{[(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\} \\ \times\ (1+L\lambda\delta+(M-L)\delta\xi)+(M-L)\delta\xi(1-\phi)\end{array}}\right\} \le |f(z)|$$

$$\le 1 + r\left\{\frac{(M-L)^2\delta\xi(1-\phi)}{\begin{array}{l}\Big\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta} \\ \times\ \Big\{[(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\} \\ \times\ (1+L\lambda\delta+(M-L)\delta\xi)+(M-L)\delta\xi(1-\phi)\end{array}}\right\}$$

Proof. Since $f(z) \in \mathcal{T}_n\, S_p^k(\phi, \beta, \xi, \lambda, \delta, L, M, \mu, \rho, \omega)$ we have by Theorem 2.3,

$$\begin{aligned}&\Big\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}[(2+\rho-\mu-h)\sum_{t=2}^{\infty}\gamma^{n+1}(s)\,a_t \\ &\le \sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{[1+(t+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}\times \\ &\Big\{(t+\rho-\mu)k[(t-1)(\lambda-\omega)+t]\Big\}a_t \le (1-h)\end{aligned} \tag{2.30}$$

In look of Theorem 2.3 we have

$$\begin{aligned}\sum_{t=2}^{\infty}\gamma^{n+1}(s)\,a_t &= \sum_{t=2}^{\infty}\gamma^{n+1}(s)\Big\{(t+\rho-\mu-1)[(t-1)(\lambda-\omega)+t]\Big\}a_t + t\sum_{t=2}^{\infty}\gamma^{n+1}(s)\,a_t \\ &\le \frac{(M-L)(1-h)}{\Big\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}(2+\rho-\mu-h)}\end{aligned} \tag{2.31}$$

$$\begin{aligned}&|f'(z)| \le 1 + \sum_{t=2}^{\infty}\gamma^{n+1}(s)\,ta_t|z|^{t-1} \le 1 + r\sum_{t=2}^{\infty}\gamma^{n+1}(s)\,ta_t \le \\ &\qquad 1 + r\Big[\frac{(M-L)(1-h)}{\Big\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}(2+\rho-\mu-h)}\Big]\end{aligned} \tag{2.32}$$

Similarly,

$$|f'(z)| \geq 1 - \sum_{t=2}^{\infty} \gamma^{n+1}(s)\, t a_t |z|^{t-1} \geq 1 + r \sum_{t=2}^{\infty} \gamma^{n+1}(s)\, t a_t \geq$$
$$1 - r\Big[\frac{(M-L)(1-h)}{\Big\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}(2+\rho-\mu-h)}\Big] \tag{2.33}$$

So,

$$1 - r\Big[\frac{(M-L)(1-h)}{\Big\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}(2+\rho-\mu-h)}\Big] \leq |f'(z)| \leq$$
$$1 - r\Big[\frac{(M-L)(1-h)}{\Big\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta}(2+\rho-\mu-h)}\Big] \tag{2.34}$$

Substituting the value of h in the above inequality, we have

$$r - r^2 \left\{ \begin{array}{l} \dfrac{(M-L)^2\delta\xi(1-\phi)}{\Big\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta+1}} \\ \times \ \Big\{[(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\} \\ \times \ (1+L\lambda\delta+(M-L)\delta\xi)+(M-L)\delta\xi(1-\phi) \end{array} \right\} \leq |f(z)|$$

$$\leq r + r^2 \left\{ \begin{array}{l} \dfrac{(M-L)^2\delta\xi(1-\phi)}{\Big\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta+1}} \\ \times \ \Big\{[(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\} \\ \times \ (1+L\lambda\delta+(M-L)\delta\xi)+(M-L)\delta\xi(1-\phi) \end{array} \right\}$$

Hence the proof is obtained. ■

Corollary 2.7. *For* $f(z) \in \mathcal{T}_n\, S_p^k(\phi,\beta,\xi,\lambda,\delta,L,M,\mu,\rho,\omega)$.
In particular, if $\rho = \mu = 1$*, then we have* $1+\rho-\mu = 1+\rho-\rho = 1.$

$$= \frac{(M-L)^2\delta\xi(1-\phi)}{\Big\{k[(t-1)(\lambda-\omega)+t]\Big\}[1+L\lambda\delta+(M-L)\delta\xi]} \times \tag{2.35}$$

$$\frac{k[(t-1)(\lambda-\omega)+t](1+L\lambda\delta+(M-L)\delta\xi)}{\left\{k[(t-1)(\lambda-\omega)+t]\right\}^{\beta}\left\{k[(t-1)(\lambda-\omega)+t]\right\}} \times$$
$$[1+L\lambda\delta+(M-L)\delta\xi)+(M-L)\delta\xi(1-\phi)]$$

$$\frac{(M-L)^2\delta\xi(1-\phi)}{\left\{[1+k][(t-1)(\lambda-\omega)+t]\right\}^{\beta}\left\{k[(t-1)(\lambda-\omega)+t]\right\}}$$
$$[1+L\lambda\delta+(M-L)\delta\xi)+(M-L)\delta\xi(1-\phi)]$$

Hence, we obtain

$$1-r\left\{\begin{array}{l}\dfrac{(M-L)^2\delta\xi(1-\phi)}{\left\{[1+k][(t-1)(\lambda-\omega)+t]\right\}^{\beta}\left\{k[(t-1)(\lambda-\omega)+t]\right\}}\\ [1+L\lambda\delta+(M-L)\delta\xi)+(M-L)\delta\xi(1-\phi)]\end{array}\right\} \leq |f(z)|$$

$$\leq 1+r\left\{\begin{array}{l}\dfrac{(M-L)^2\delta\xi(1-\phi)}{\left\{[1+k][(t-1)(\lambda-\omega)+t]\right\}^{\beta}\left\{k[(t-1)(\lambda-\omega)+t]\right\}}\\ [1+L\lambda\delta+(M-L)\delta\xi)+(M-L)\delta\xi(1-\phi)]\end{array}\right\}$$

Theorem 2.8. *If* $f(z)\in\mathcal{T}_n\mathcal{V}^k(\phi,\beta,\xi,\lambda,\delta,L,M,\mu,\rho,\omega)$ *then*

$$1-r\left\{\begin{array}{l}\dfrac{(M-L)^2\delta\xi(1-\phi)}{\left\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\right\}^{\beta+1}}\\ \times\ \left\{[(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\right\}\\ \times\ (1+L\lambda\delta+(M-L)\delta\xi)+(M-L)\delta\xi(1-\phi)\end{array}\right\} \leq |f(z)|$$

$$\leq 1+r\left\{\begin{array}{l}\dfrac{(M-L)^2\delta\xi(1-\phi)}{\left\{[1+(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\right\}^{\beta+1}}\\ \times\ \left\{[(1+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\right\}\\ \times\ (1+L\lambda\delta+(M-L)\delta\xi)+(M-L)\delta\xi(1-\phi)\end{array}\right\}$$

Proof. Similarly, we can prove this theorem as it is analogous to Theorem 2.6. So it is sufficient to substitute $\beta=\beta+1$ in the above Theorem 2.6 and the subsequent corollary. ■

2.3. Closure Theorem

Theorem 2.9. *Let* $f_1(z) = z$ *and*

$$f_t(z) = \frac{(M-L)\delta\xi(1-\phi)}{\begin{array}{l}\Big\{[1+(t+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta} \\ \times\ \Big[\Big\{(t+\rho-\mu-1)k[(t-1)(\lambda-\omega)+t]\Big\} \\ \times\ [1+L\lambda\delta+(M-L)\delta\xi]+(M-L)\delta\xi(1-\phi)\Big]\end{array}} z^t,$$

for $t = 2, 3, 4, \cdots$.

Then the class $\mathcal{T}_n\, S_p^k(\phi, \beta, \xi, \lambda, \delta, L, M, \mu, \rho, \omega)$ *contains a function* $f(z)$ *if and only if* $f(z) = \sum_{t=1}^{\infty} \gamma^{n+1}(s)\, k_t f_t(z),\ \forall\, k_t \geq 0$ *and* $\sum_{t=1}^{\infty} \gamma^{n+1}(s)\, k_t = 1$.

Proof. Let $f(z) = \sum_{t=1}^{\infty} \gamma^{n+1}(s)\, k_t f_t(z),\ \forall\, k_t \geq 0$ and $\sum_{t=1}^{\infty} \gamma^{n+1}(s)\, k_t = 1$. We have,

$$f(z) = \sum_{t=1}^{\infty} \gamma^{n+1}(s)\, k_t f_t(z) = k_1 f_1(z) + \sum_{t=2}^{\infty} \gamma^{n+1}(s)\, k_t f_t(z) \tag{2.36}$$

$$\therefore f(z) = z - \sum_{t=2}^{\infty} \gamma^{n+1}(s)\, k_t \left\{ \frac{(M-L)\delta\xi(1-\phi)}{\begin{array}{l}\Big\{[1+(t+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta} \\ \times\ \Big[\Big\{(t+\rho-\mu-1)k[(t-1)(\lambda-\omega)+t]\Big\} \\ \times\ [1+L\lambda\delta+(M-L)\delta\xi]+(M-L)\delta\xi(1-\phi)\Big]\end{array}} z^t, \right\} \tag{2.37}$$

Then

$$\begin{aligned} &\sum_{t=2}^{\infty} \gamma^{n+1}(s) \frac{(M-L)\delta\xi(1-\phi)}{\begin{array}{l}\Big\{[1+(t+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta} \\ \Big[\Big\{(t+\rho-\mu-1)k[(t-1)(\lambda-\omega)+t]\Big\} \\ [1+L\lambda\delta+(M-L)\delta\xi]+(M-L)\delta\xi(1-\phi)\Big]\end{array}} \\ &\times \frac{\begin{array}{l}\Big\{[1+(t+\rho-\mu)k][(t-1)(\lambda-\omega)+t]\Big\}^{\beta} \\ \Big[\Big\{(t+\rho-\mu-1)k[(t-1)(\lambda-\omega)+t]\Big\} \\ [1+L\lambda\delta+(M-L)\delta\xi]+(M-L)\delta\xi(1-\phi)\Big]\end{array}}{(M-L)\delta\xi(1-\phi)} \\ &= \sum_{t=2}^{\infty} \gamma^{n+1}(s)\, k_t = 1-k \leq 1 \end{aligned} \tag{2.38}$$

Hence $f(z) \in \mathcal{T}_n\, S_p^k(\phi, \beta, \xi, \lambda, \delta, L, M, \mu, \rho, \gamma)$.
Conversely, suppose $f(z) \in \mathcal{T}_n\, S_p^k(\phi, \beta, \xi, \lambda, \delta, L, M, \mu, \rho, \gamma)$ then we have Theorem 2.1 ■

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